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RANDOM PERTUBATIONS, PERIODICITY AND CHAOTIC BEHAVIOR IN NONLINEAR SYSTEMS

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Random perturbations, periodicity and chaotic behavior in nonlinear systems*)

by

J. Grasman

ABSTRACT

This paper deals with chaotic behavior of deterministic systems and with the influence of random perturbations upon nonlinear systems.

KEYWORDS & PHRASES: *nonlinear system, strange attractor, random perturbation*

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1. INTRODUCTION

We consider a process given by n state variables depending continuously upon the time and satisfying a system of differential equations of the type

$$(1) \qquad \begin{aligned} dx_1/dt &= f_1(x_1,\dots,x_n) \\ dx_n/dt &= f_n(x_1,\dots,x_n) \end{aligned} \qquad \text{or } dx/dt = f(x).$$

Many fysiological processes can be described by models of this form. As an example we mention the Hodgkin-Huxley equations, which model the electric pulses in a nerve cell, see [2].

In the state space $\mathbb{R}^n$ the solution of (1) for some initial value $x^{(0)}$ is represented by a curve (trajectory) that starts in $x^{(0)}$ and exhibits some limit behavior for $t \to \infty$. We expect that for a bounded solution the trajectory tends to some limit value (stable stationary solution) or to a closed curve (limit cycle). For systems with $n \geq 3$ there are still other possibilities of bounded limit behavior.

2. THE HORSE SHOE MAPPING

We consider a dynamical system (1) with n=3 and construct in the state space a surface H which is transversal to the trajectories. It is assumed that each trajectory that starts in H returns there within a finite time, see figure 1. The mapping $P:H \to H$, which relates to a starting point $x^{(0)}$ a return point $x^{(1)}$ is called the Poincaré mapping and plays an important role in the theory of dynamical systems. If the Poincaré mapping P is such that the rectangle abcd is mapped as the "horse shoe" a'b'c'd', see figure 2, then it is possible to construct a set of bounded non-periodic solutions. For any sequence of elements A and B, say AABABBA..., we can find a $x^{(0)}$, such that the sequence $x^{(0)}, x^{(1)}, x^{(2)}, \dots$ with $x^{(k)} = P(x^{(k-1)})$ consists of elements $x^{(k)}$ which in the corresponding order belong to either A or B, see figure 2. This mapping constructed by SMALE [15] teaches us that irregular behavior may be present in a well-defined, deterministic system.

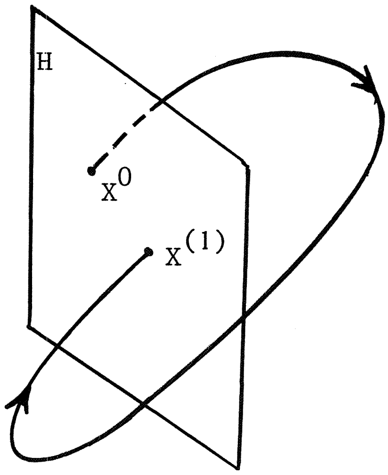

Fig. 1 The Poincaré mapping

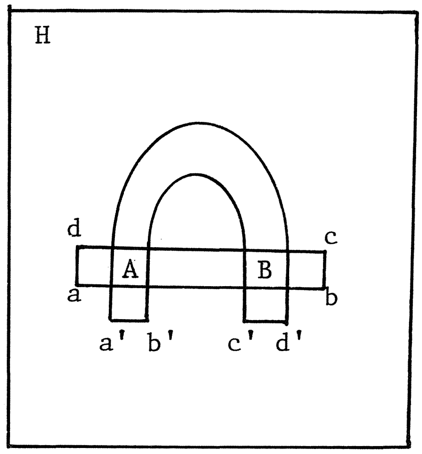

Fig. 2 The horse shoe mapping

3. HOMOCLINIC POINTS

Next we consider a 3-dimensional dynamical system that has invariant 2-dimensional surfaces I in the state space (if $x^{(0)} \in I$, then $x(t) \in I$ for all $t > 0$). We suppose that there are two such surfaces I_1 and I_2 which are intersecting. The common curve is also a trajectory of the system, see figure 3. Moreover, it is supposed that $q_0 \in H \cap I_1 \cap I_2$ is not a fixed point of the Poincaré mapping, so $Pq_0 \neq q_0$. In that case I_1 and I_2 are entwined in a special way, as besides the point q_0 a different point $q_1 = P(q_0) \in H \cap I_1 \cap I_2$ must exist, see figure 4. If $P(q_1) = q_0$ we have a closed curve. If this is not the case, then a third point $q_2 = P(q_1) \in H \cap I_1 \cap I_2$ exists, etc... POINCARÉ [10] constructed two invariant surfaces which besides an infinite sequence of points $q_k \in H$ also have in common a fixed point p of the mapping P, see figure 5. This is a saddle point, so on the one invariant surface the solution approaches p and on the other it leaves p. The points q_k are called homoclinic points because of their double asymptotic relation with the saddle point p.

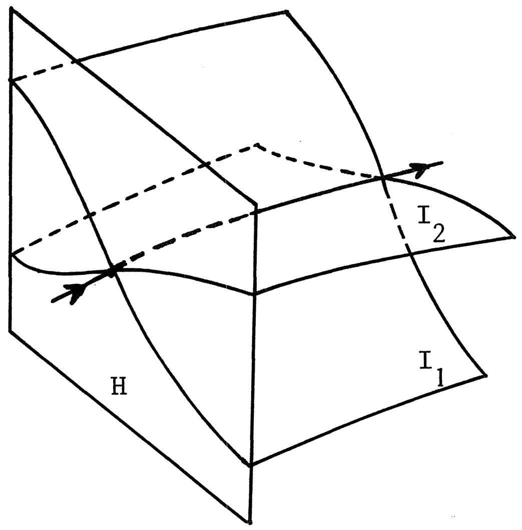

Fig. 3 Two intersecting invariant surfaces

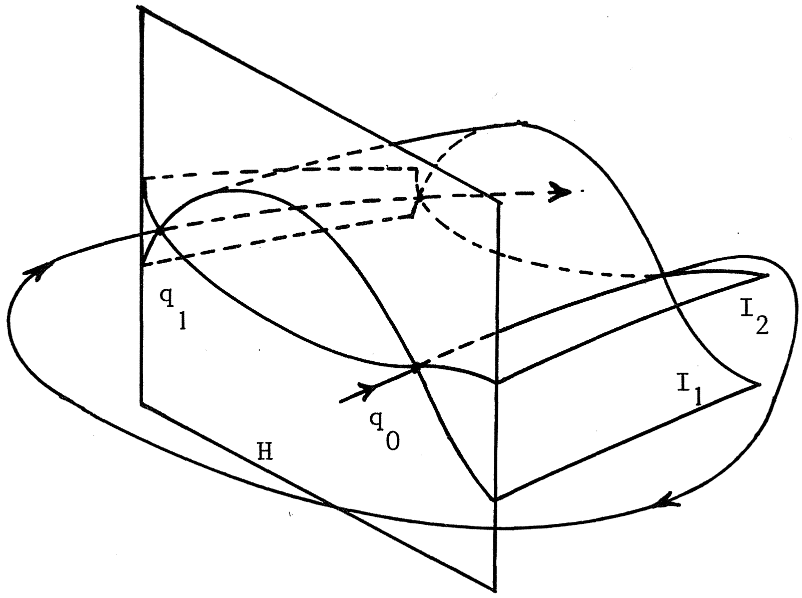

Fig. 4 Repeated intersections of two invariant surfaces

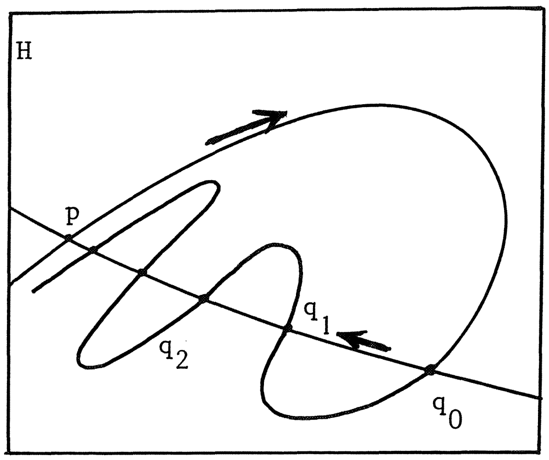

Fig. 5 A saddle point and homoclinic points of the mapping P

Now we will make it plausible that in a neighbourhood of p, bounded non-periodic solutions are possible. In figure 6 the rectangle R is chosen such that after repeated application of the mapping P a horse shoe P^jR arises that intersects R. From section 2 it follows that we may select initial values for which the bounded trajectory does not close.

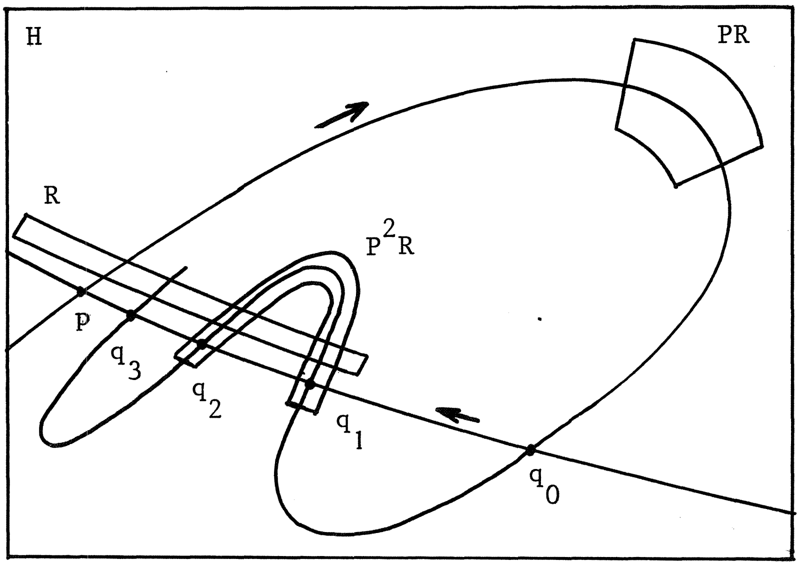

Fig. 6 The horse shoe mapping P^2

4. STRANGE ATTRACTORS

In the foregoing we showed that trajectories of a continuous dynamical system may stay in a bounded region of the state space for $t \to \infty$ without approaching a stationary or periodic orbit. The question arises whether for every initial value such a trajectory wanders around in its own way or that for a set of initial values $U \subset H$ a common limit behavior is possible.

A strange attractor is defined as follows, see [13]. Suppose $q_0 \in U$ and $q_k = P(q_{k-1})$, then a set of points $A \subset U$ is strange attractor if:

a. an arbitrarily small neighborhood of A is contained in U,

b. $q_k \in U$ for $k = 1,2,\ldots,$

c. q_k remains in a arbitrarily small neighborhood of A for k sufficiently large,

d. q_k is sensitive to variations in q_0,

e. for $q_0 \in A$ there exists a k such that q_k comes arbitrarily close to an arbitrarily chosen point of A.

The concept of a strange attractor has been introduced by RUELLE and TAKENS [14] and a well-known example of it is found in the Lorenz equations (see [8]),

$$(2)\qquad \begin{aligned} dx_1/dt &= -cx_1 + cx_2, \\ dx_2/dt &= -x_1x_2 + ax_1 - x_2, \\ dx_3/dt &= x_1x_2 - bx_3 \end{aligned}$$

for certain values of the parameters, e.g. for $a = 28$, $b = 8/3$ and $c = 10$. In [13] H is chosen $x_3 = 27$. In meteorology the above strange attractor is proposed as a model for the onset of turbulence.

5. DIFFERENCE EQUATIONS

If we forget that the mapping $P:H \to H$ in the preceding sections connects a starting point with an end point of a continuous trajectory in the state space, then we have to do with a more general class of difference equations, which gives the state of a system at discrete times $t_0, t_1, \ldots$ by

$$(3)\qquad x^{(n+1)} = P(x^{(n)}).$$

The logistic difference equation for a 1-dimensional state space

$$(4)\qquad x^{(n+1)} = cx^{(n)}(1-x^{(n)}),$$

is a well-known example of such a discrete system. For certain values of c it tends to a stationary state for others to a k-periodic state and also chaotic behavior is found for certain values of c, see [9]. A simple example of a mapping in $\mathbb{R}^2$ with a strange attractor is the Henon attractor, see [13]:

$$(5)\qquad \begin{aligned} P_1 &= x_2 + 1 - ax_1^2, \qquad & a = 1.4, \\ P_2 &= bx_1, \qquad & b = 0.3. \end{aligned}$$

6. THE INFLUENCE OF RANDOM PERTURBATIONS

A non-linear system of type (1) may have more than one stable solution, e.g. two limit cycles. Depending on the initial values the system will tend to one of these solutions as $t \to \infty$. We now introduce random perturbations which change (1) into a stochastic differential equation

$$dX = f(X)dt + \varepsilon\sigma dW, \tag{6}$$

where W is a n-dimensional Wiener process, σ the diffusion matrix and ε a measure for intensity of the perturbations. The expected time of leaving the attraction domain of one of the limit cycles (because of the perturbations) is of the order $O(\exp(c/\varepsilon))$. If during a period of time the solution of (6) switches many times from one domain of attraction to the other, it is difficult to recognize the periodicity in the signal and one may think of the presence of a strange attractor.

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