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ON A CLASS OF SENSITIVE CHEAP CONTROL PROBLEMS

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On a class of sensitive cheap control problems*)

by

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ABSTRACT

For a class of linear singular optimal control problems with a non-
unique singular arc, an asymptotic solution of the corresponding nearly
singular problem is constructed and its validity is proved from the con-
vergence of the power series solution of the Riccati equation.

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1. INTRODUCTION

In this paper we study a class of linear time-invariant, n-dimensional dynamical systems of the type

(1ab) $\dot{x} = Ax + Bv, \quad x(0) = x_0,$

with performance index

(1c) $J = \int_0^\infty x'Qx + \varepsilon^2 v'Rv \, dt, \quad 0 < \varepsilon << 1,$

where Q is a symmetric positive semi-definite matrix and R is symmetric and positive definite. We denote the n-dimensional state space by X. The control vector takes its values in the linear m-dimensional space U and $v(\cdot)$; $\mathbb{R}^+ \to U$ is assumed to be a piece-wise continuous mapping. We will analyse the corresponding algebraic Riccati equation in the limit $\varepsilon \to 0$ for the case

(2ab) $X = K + B \quad \text{and} \quad K \cap B \neq 0,$

where $K = \text{Ker } Q$ and $B = \text{Im } B$. As we pointed out in [2], the singular optimal control problem, that is (1) with $\varepsilon = 0$, may, under certain conditions, have a family of solutions. By the conditions (2ab) we limit ourselves to a representative subclass of problems with a non-unique singular arc. In this paper we sketch the method of proving rigorously the correctness of the asymptotic approximation of (1) satisfying (2). It is assumed that the pair

(A,B) is stabilizable and the pair (C,A) with C'C = Q is observable. In addition to this similar types of hypotheses holding on a subsystem of (1) will be made later on.

2. CONVERGENCE OF A FORMAL POWER SERIES

First we remark that the algebraic Riccati equation that corresponds with (1) has a unique symmetric positive definite solution, as we may conclude from the following theorem by Wonham [6].

THEOREM 1. *The class of symmetric, positive definite matrices contains a unique element that satisfies*

(3) $A'P + PA + PBR^{-1}B'P + C'C = 0$

provided that (A,B) is stabilizable and (C,A) is observable.

For proving the convergence of the formal power series solution of the algebraic Riccati equation we need the following theorem of Hautus [4].

THEOREM 2. *Let the functions* $F_{ij}(Z,\varepsilon)$, $i,j = 1,\ldots,n$ *with* $Z = \{Z_{ij}\}_{i,j=1,\ldots,n}$ *be analytic in a neighborhood of* $(Z,\varepsilon) = (0,0)$ *and let* $Z = \hat{Z}(\varepsilon)$ *be a formal power series solution of the equation* $F(Z;\varepsilon) = 0$. *If* $\det F_Z(\hat{Z}(\varepsilon);\varepsilon) \neq 0$ *in a neighborhood of* $\varepsilon = 0$, *then* $\hat{Z}(\varepsilon)$ *is convergent.*

Thus, if for the algebraic Riccati equation we can prove that def $F_Z(\hat{Z}(\varepsilon);\varepsilon) \neq 0$ and that the symmetric power series expansion for the solution is positive definite, then it represents a symmetric positive definite analytic function satisfying the Riccati equation. According to Theorem 1 this function is the unique gain function that corresponds with the optimal solution.

3. THE OPTIMAL SOLUTION FOR $\varepsilon \to 0$

The system (1) can be transformed into

$$\begin{pmatrix} \dot{x}_s \\ \dot{x}_k \end{pmatrix} = \begin{pmatrix} A_s & 0 \\ A_{ks} & A_k \end{pmatrix} \begin{pmatrix} x_s \\ x_k \end{pmatrix} + \begin{pmatrix} B_s & 0 \\ 0 & B_k \end{pmatrix} \begin{pmatrix} u_s \\ u_k \end{pmatrix} ,$$

$$\begin{pmatrix} x_s(0) \\ x_k(0) \end{pmatrix} = \begin{pmatrix} x_{s0} \\ x_{k0} \end{pmatrix}$$

with performance index

$$(4c) \quad J = \int_0^\infty x_s' Q_s x_s + \varepsilon^2 \{x_k' M_k x_k + 2x_k' N_{ks} U_s + 2x_k' N_k u_k +$$

$$u_s' R_s u_s + 2u_k' R_{ks} u_s + u_k' R_k u_k \} \, dt,$$

where for $x = (x_s, x_k)$ such a n-dimensional basis is taken that $(0, x_k) \in K$ and $(x_s, 0) \notin K$.

It is known that (4) has an optimal solution with

$$(5) \quad u = -\varepsilon^{-2} R^{-1} (B'P + \varepsilon^2 N') x,$$

where

$$R = \begin{pmatrix} R_s & R_{ks} \\ R_{ks} & R_k \end{pmatrix}, \quad B = \begin{pmatrix} B_s & 0 \\ 0 & B_k \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ N_{ks} & N_k \end{pmatrix}$$

and P is the uniquely determined symmetric positive definite matrix satisfying the algebraic Riccati equation

$$(6) \quad P(A-BR^{-1}N') + (A-BR^{-1}N')'P - \varepsilon^{-2} PBR^{-1}B'P + Q +$$

$$+ \varepsilon^2 (M-NR^{-1}N') = 0.$$

Substitution of the formal power series

$$(7ab) \quad P = \varepsilon \sum_{j=0}^\infty P^{(j)} \varepsilon^j, \quad P^{(j)} = \begin{pmatrix} P_{sj} & P_{ksj}' \\ P_{ksj} & P_{kj} \end{pmatrix}$$

into (6) yields after equating terms of equal powers of ε the coefficients $P^{(j)}$; we find that

$$(8a) \quad P_{s0} B_s \{R_s - R_{ks}' R_k^{-1} R_{ks}\} B_s' P_{s0} = Q_s,$$

$$(8bc) \quad P_{ks0} = P_{k0} = 0,$$

$$(8d) \quad P_{k1} \{A_k - B_k R_k^{-1} N_k'\} + \{A_k - B_k R_k^{-1} N_k'\}' P_{k1} +$$

$$-P_{k1} B_k R_k^{-1} B_k' P_{k1} + M_k - N_k R_k^{-1} N_k' = 0.$$

We will assume that (8d) has a unique symmetric positive definite solution. Then we are in the position to prove the following.

THEOREM 3. *The symmetric formal power series solution* (7) *has a positive radius of convergence and is positive definite. Hence it represents an analytic function, defined in a neighborhood of $\varepsilon = 0$, being the unique symmetric, positive definite solution of* (6).

Outline of the proof. Substitution of

$$(9) \quad P = \varepsilon P^{(0)} + \varepsilon^2 P^{(1)} + \varepsilon^2 Z$$

in (6) yields

$$(10a) \quad F(Z;\varepsilon) \equiv ZH(\varepsilon) + H(\varepsilon)'Z - \varepsilon ZBR^{-1}B'Z + \varepsilon G = 0$$

with

$$(10b) \quad H(\varepsilon) = \varepsilon(A-BR^{-1}N') - BR^{-1}B'(P^{(0)} + \varepsilon P^{(1)}).$$

It is shown that $\det(H) \neq 0$, which is equivalent with $\det(\partial F/\partial Z) \neq 0$ at $Z = 0$. From theorem 2 we conclude that (7) has a positive radius of convergence. Furthermore, it can be shown that for

$$(11a) \quad x'Px = \varepsilon Q_1(x;\varepsilon) + O(\varepsilon^3 |x|^2)$$

with

$$(11b) \quad Q_1 = x_s' P_{s0} x_s + \varepsilon x_s' P_{s1} x_s + 2\varepsilon x_s' P_{ks1}' x_k + \varepsilon x_k' P_{k1} x_k,$$

the following inequality holds

$$(12) \quad Q_1(x;\varepsilon) > \delta\varepsilon |x|^2,$$

where δ is some arbitrary small positive number independent of ε. Consequently, P is positive definite and from theorem 1 it follows that the convergent series (7) is a representation of the analytic, unique, positive definite, symmetric solution of (6). □

In the limit for $\varepsilon \to 0$ the optimal solution has the following behavior. (see [3] for the derivation of this result). Initially, it jumps from (x_{s0}, x_{k0}) to the subspace K:

(13a) $(x_s, x_k) =$

$$(0, x_{k0} - B_k R_k^{-1} R_{ks} (R_s - R'_{ks} R_k^{-1} R_{ks})^{-1} B_s^{-1} x_{s0})$$

Then within the subspace K the optimal solution of

(13b) $\dot{x}_k = A_k x_k + B_k u_k$

with initial value (13a) on performance index

(13c) $J = \int_0^\infty x'_k M_k x_k + 2 x'_k N_k u_k + u'_k R_k u_k \, dt$

has to be found, which is equivalent to solving
the Riccati equation (8d).

4. CONCLUDING REMARKS

The main result of this paper concerns the limit
behavior of solutions of nearly singular optimal
control problems (1) with a nontrivial control-
lability subspace contained in the kernel of Q. For
this class of problems the corresponding singular
problem, that is (1) with $\varepsilon = 0$, has a family of
solutions, each of them composed of an initial
pulse and a singular arc. We presented a method for
selecting the correct limit solution and proved the
validity of the result.
The class of control systems we analyzed has var-
ious applications in technology, see [1]. The non-
uniqueness of the singular problem brings about
that in the large the behavior of the optimal solu-
tion within the subspace K follows from $O(\varepsilon^2)$ terms
of the performance index. As a consequence of this
the solution may vary in an indefinite way within
the subspace K as $\varepsilon \to 0$. This fact may for a great
deal explain the observed sensitivity of the sin-
gular arc on the fysical parameters in a model for
optimal heating and cooling by solar energy [5,
p.302], which is of the type we dealt with.

5. LITERATURE

1. P. DORATO, A review of the application of
    modern control theory to solar energy systems,
    Proc. 18th IEEE conference on decision and con-
    trol, Fort Lauderdale, Florida (1979), p.907-
    910.

2. J. GRASMAN, Non-uniqueness in singular optimal
    control, Int. Symp. on Math. Theory of Networks
    and Systems, vol. 3 (1979), p.415-420.

3. J. GRASMAN, On a class of optimal control prob-
    lems with an almost cost free solution, Math.
    Centre Amsterdam, Report TW 204 (1980), to ap-
    pear in IEEE Trans. on Automatic Control.

4. M.L.J. HAUTUS, Formal and convergent solutions
    of ordinary differential equations, Proc. Kon.
    Ned. Akad. v. Wetensch., Amsterdam, Series A 81
    (1978), p.216-229.

5. H.J. OBERLE, Numerical computation of singular
    control problems with application to optimal
    heating and cooling by solar energy, Appl. Math.
    Optim. 5 (1979), p. 297-314.

6. W.M. WONHAM, On a matrix Riccati equation of
    stochastic control, SIAM J. Control 6 (1968),
    p. 681-697.