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# THE MOTION OF A HALF-PLANE SEA UNDER INFLUENCE OF A NON-STATIONARY WIND

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## § 1. Fundamental equations

We consider a half-plane sea $y > 0$. If the components of the total stream parallel and normal to the coast are $w_x$ and $w_y$, and if $\zeta$ is the elevation of the disturbed surface their Laplace transform satisfy the equations

$$
\begin{aligned}
(p+\lambda)\bar{w}_x - \Omega\bar{w}_y + c^2\frac{\partial\bar{\zeta}}{\partial x} &= \frac{1}{\varrho}\bar{W}_x \\
(p+\lambda)\bar{w}_y + \Omega\bar{w}_x + c^2\frac{\partial\bar{\zeta}}{\partial y} &= \frac{1}{\varrho}\bar{W}_y \qquad \text{1-1}\\
\frac{\partial\bar{w}_x}{\partial x} + \frac{\partial w_y}{\partial y} + p\,\bar{\zeta} &= 0 \,,
\end{aligned}
$$

and the boundary condition

$$y = 0 \qquad \bar{w}_y = 0 \,. \qquad \text{1-2}$$

$W_x$ and $W_y$ are the components of the wind and throughout this report it is assumed that $W_x$ and $W_y$ do not depend upon **x**.

As a consequence of this the equations (1-1) have a solution for which $w_x$, $w_y$ and $\zeta$ are also independent of $x$ and hence 1-1 reduces to

$$
\begin{aligned}
(p+\lambda)\bar{w}_x - \Omega\bar{w}_y &= \frac{1}{\varrho}\bar{W}_x \\
(p+\lambda)\bar{w}_y + \Omega w_x + c^2\frac{\partial\bar{\zeta}}{\partial y} &= \frac{1}{\varrho}\bar{W}_y \qquad \text{1-3}\\
\frac{\partial\bar{w}_y}{\partial y} + p\,\bar{\zeta} &= 0.
\end{aligned}
$$

Elimination of $\bar{w}_x$ and $\bar{\zeta}$ gives

$$\frac{\partial^2\bar{w}_y}{\partial y^2} - k^2\bar{w}_y = \bar{F} \,, \qquad \text{1-4}$$

where

$$k^2 = \frac{p\{(p+\lambda)^2 + \Omega^2\}}{c^2(p+\lambda)} \,, \qquad \text{1-5}$$

and

$$\bar{F} = \frac{p}{\varrho c^2(p+\lambda)}\left\{\Omega\,\bar{W}_x - (p+\lambda)\bar{W}_y\right\} \,. \qquad \text{1-6}$$

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Research carried out under the direction of Prof.Dr D. van Dantzig

The solution of 1-4 with the boundary condition of y = 0 and which also vanishes at infinity is

$$\bar{w}_y = \int_0^{\infty} \frac{e^{-k(y+\eta)} - e^{-k|y-\eta|}}{2k} \bar{F}(\eta)\, d\eta \quad . \qquad 1\text{-}7$$

Next, $\bar{w}_x$ and $\bar{\zeta}$ can be found from

$$\bar{w}_x = \frac{\Omega}{p+\lambda}\bar{w}_y + \frac{\bar{W}_x}{p+\lambda} \quad , \qquad 1\text{-}8$$

$$\bar{\zeta} = -\frac{1}{\varrho}\frac{\partial \bar{w}_y}{\partial y} \quad . \qquad 1\text{-}9$$

If $W_x$ and $W_y$ do not depend of y, so that we have a uniform windfield, 1-7 may be simplified since also $\bar{F}(\eta)$ does not depend of $\eta$. Without difficulty we obtain

$$\bar{w}_y = -\frac{1-e^{-ky}}{k^2}\bar{F} \quad , \qquad 1\text{-}10$$

and

$$\bar{\zeta} = \frac{e^{-ky}}{k}\frac{\bar{F}}{p} \quad . \qquad 1\text{-}11$$

In particular we shall consider a wind with a constant direction

$$\begin{cases} W_x = -\varrho c\; w(t)\cos\alpha \;, \\ W_y = -\varrho c\; w(t)\sin\alpha \;, \end{cases} \qquad 1\text{-}12$$

where $\alpha$ is the direction of the wind with respect to the coast, so that $0 < \alpha < \pi$ corresponds to seawind, and where w(t) is a positive function representing the intensity of the wind.
If we take 1-12 the expression 1-6 for $\bar{F}$ becomes

$$\bar{F} = \frac{p\bar{w}}{c(p+\lambda)}\left\{-\Omega\cos\alpha + (p+\lambda)\sin\alpha\right\} \;, \qquad 1\text{-}13$$

and the expression 1-11 for $\bar{\zeta}$ becomes

$$\bar{\zeta} = \frac{(p+\lambda)^{\frac{1}{2}}}{p^{\frac{1}{2}}\left\{(p+\lambda)^2 + \Omega^2\right\}^{\frac{1}{2}}}\left(\sin\alpha - \frac{\Omega}{p+\lambda}\cos\alpha\right)\bar{w}\, e^{-ky} . \qquad 1\text{-}14$$

If $\Omega = 0$ 1-14 becomes

$$\bar{\zeta} = \frac{\bar{w}\sin\alpha\; e^{-ky}}{p^{\frac{1}{2}}(p+\lambda)^{\frac{1}{2}}} \quad . \qquad 1\text{-}15$$

The following generalization might be of interest. We consider a windfield which is uniform in the strip $0 < y < b$ but which vanishes outside this strip.

The expressions for $\bar{w}_y$ and $\bar{\zeta}$ become in this case

$$\left\{\begin{array}{llll} \bar{w}_y = -\dfrac{(1-e^{-ky}) + e^{-ky}\operatorname{shky}}{k^2}\bar{F}, & & 0 < y < b, & \text{1-16} \\ \bar{\zeta} = \dfrac{e^{-ky} - e^{-kb}\operatorname{chky}}{k}\dfrac{\bar{F}}{p}, & & 0 < y < b. & \text{1-17} \end{array}\right.$$

If the windfield 1-12 is chosen the expression for $\bar{\zeta}$ may be written as

$$\bar{\zeta} = \frac{(p+\lambda)^{\frac{1}{2}}}{p^{\frac{1}{2}}\{(p+\lambda)^2 + \Omega^2\}^{\frac{1}{2}}}\left(\sin\alpha - \frac{\Omega}{p+\lambda}\cos\alpha\right)\bar{w}(e^{-ky} - e^{-kb}\operatorname{chky}). \quad \text{1-18}$$

## § 2. A uniform wind of constant direction

According to 1-14 the Laplace transform of the elevation due to a uniform wind of constant direction in a half-plane sea can be represented by

$$\bar{\zeta}(y) = \bar{\varphi}\cdot\bar{\psi}, \qquad y \geq 0 \qquad \text{2-1}$$

where

$$\bar{\varphi} = \left(\sin\alpha - \frac{\Omega}{p+\lambda}\cos\alpha\right)\bar{w}, \qquad \text{2-2}$$

and

$$\bar{\psi}(y) = \frac{(p+\lambda)^{\frac{1}{2}}e^{-ky}}{p^{\frac{1}{2}}\{(p+\lambda)^2 + \Omega^2\}^{\frac{1}{2}}}. \qquad \text{2-3}$$

According to the convolution theorem of Laplace transforms we have

$$\zeta(y,t) = \int_{-\infty}^{\infty} \varphi(\tau)\psi(y,t-\tau)\,d\tau. \qquad \text{2-4}$$

The original of $\bar{\varphi}$ is

$$\varphi(t) = \sin\alpha\, w(t) - \Omega\cos\alpha\, e^{-\lambda t}\int_{-\infty}^{t} e^{\lambda t} w(\tau)\,d\tau. \qquad \text{2-5}$$

The original of $\bar{\psi}$ can be determined by elementary methods only in the case $y = 0$. We have

$$\bar{\psi}(0) = \frac{(p+\lambda)^{\frac{1}{2}}}{p^{\frac{1}{2}}\{(p+\lambda)^2 + \Omega^2\}} = \frac{1}{\sqrt{p^2 + \lambda p}} - \frac{1}{\sqrt{p^2 + \lambda p}}\left\{1 - \frac{p+\lambda}{\sqrt{(p+\lambda)^2 + \Omega^2}}\right\}.$$

It is well-known that

$$\frac{1}{\sqrt{p^2 - a^2}} \doteq I_o(at) \quad ,$$

and

$$\frac{1}{\sqrt{p^2 + a^2}} \doteq J_o(at) \quad .$$

From this we may derive by means of the elementary rules of the Laplace transformation

$$1 - \frac{p + \lambda}{\sqrt{(p+\lambda)^2 + \Omega^2}} \doteq e^{-\lambda t}\Omega J_1(\Omega t) \quad ,$$

and

$$\frac{1}{\sqrt{p^2 + \lambda p}} = e^{-\frac{\lambda}{2}t} I_o(\tfrac{\lambda}{2}t) \quad .$$

Thus we have for $\psi(0,t)$ the following expression

$$\psi(0,t) = e^{-\frac{\lambda}{2}t} I_o(\tfrac{\lambda}{2}t) - \Omega\{e^{-\lambda t}J_1(\Omega t)\} * \{e^{-\frac{\lambda}{2}t} I_o(\tfrac{\lambda}{2}t)\} \quad ,$$

or

$$\psi(0,t) = e^{-\frac{\lambda}{2}t}\left[I_o(\tfrac{\lambda}{2}t) - \Omega\int_0^t e^{-\frac{\lambda}{2}t} J_1(\Omega\tau) I_o\{\tfrac{\lambda}{2}(t-\tau)\}\, d\tau\right]. \qquad 2\text{-}6$$

The elevation at the coast $\zeta(0,t)$ may be found from 2-4, 2-5 and 2-6.

## §3. The subcase $\Omega = 0$

The Laplace transform of the elevation of the sea is given by 1-15 or

$$\bar{\zeta}(y) = \bar{w}\sin\alpha \,.\, \frac{e^{-\frac{y}{c}\sqrt{p^2 + \lambda p}}}{\sqrt{p^2 + \lambda p}} \quad . \qquad 3\text{-}1$$

By means of the Laplace pair

$$\frac{e^{-b\sqrt{p^2 - a^2}}}{\sqrt{p^2 - a^2}} \doteq I_o(a\sqrt{t^2 - b^2})U(t-b) \qquad 3\text{-}2$$

it follows that

$$\Omega = 0: \qquad \zeta(y,t) = \left\{e^{-\frac{\lambda}{2}t} I_o(\tfrac{\lambda}{2}\sqrt{t^2 - \tfrac{y^2}{c^2}})U(t - \tfrac{y}{c})\right\} * w\sin\alpha \quad , \qquad 3\text{-}3$$

and in particular for $y = 0$

$$\Omega = 0: \qquad \zeta(0,t) = \left\{ e^{-\frac{\lambda}{2}t} I_0\left(\frac{\lambda t}{2}\right) \right\} * w \sin\alpha \ . \tag{3-4}$$

It is clear that in this case the elevation is only influenced by the normal component of the wind. The tangential component does not affect the level of the sea but causes a stream along the coast which is determined by 1-3

$$w_x = \frac{1}{\rho} \int_0^{\infty} e^{-\lambda\tau} W_x(t-\tau)\, d\tau \ . \tag{3-5}$$

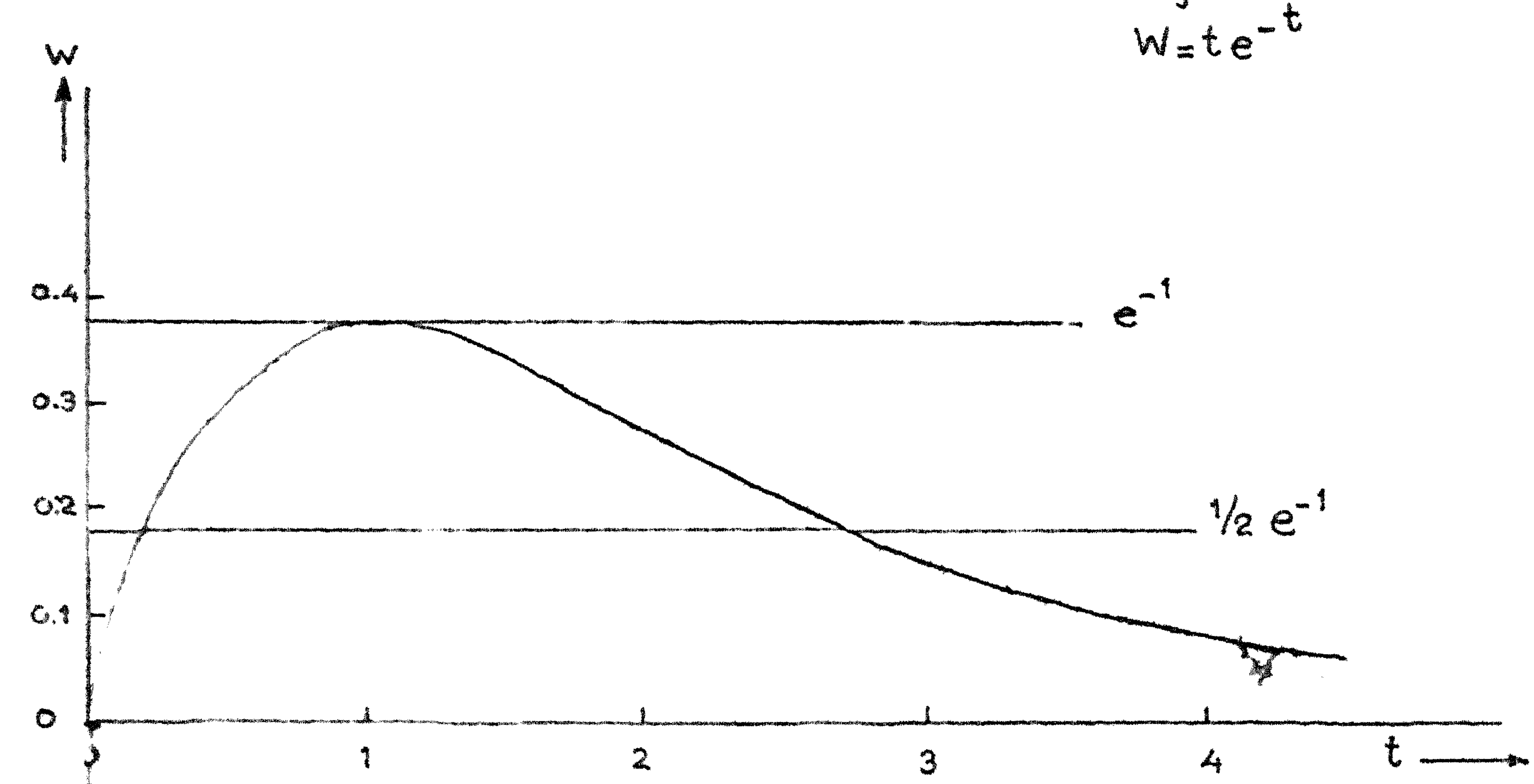
4. A particular windfield

We consider a windfield of the form 1-12, where

$$w(t) = m^2 t e^{-mt} \ . \tag{4-1}$$

A graph of $w(t)$ for $m = 1$ is given in figure 1.

Fig. 1
$W = t e^{-t}$



The maximum value of $w(t)$ is reached at $t = \frac{1}{m}$ and $w_{max} = me^{-1}$. We shall sometimes write $\frac{1}{m} = T$ since T has the physical meaning of being proportional to the duration of the storm. The time during which $w(t)$ is above half its maximum is $2.45T$ $(0.23 < \frac{t}{T} < 2.68)$.

The "energy" of the storm is proportional to $\int_0^\infty w(t)dt$ which is constant, so that 4-1 represents storms of the same "energy" and variable duration.

For this particular windfield the factor $\varphi(t)$ in §2 becomes

$$\begin{cases} \varphi(t) = \dfrac{\Omega\cos\alpha + (m-\lambda)\sin\alpha}{m-\lambda}\, mte^{-mt} - \dfrac{\Omega\cos\alpha}{(m-\lambda)^2}\, m(e^{-\lambda t} - e^{-mt}), & m \neq \lambda, \\ \varphi(t) = \lambda te^{-\lambda t}\sin\alpha - \frac{1}{2}\Omega\lambda t^2 e^{-\lambda t}\cos\alpha\ , & m = \lambda. \end{cases} \qquad 4\text{-}2$$

The elevation at the coast for $\Omega = 0$ becomes in this case, cf. 3-4

$$\Omega = 0: \qquad \zeta(0,t) = m^2 e^{-mt}\int_0^t (t-\tau)e^{-(\frac{\lambda}{2}-m)\tau} I_0\left(\frac{\lambda\tau}{2}\right)d\tau \,.\, \sin\alpha\ . \qquad 4\text{-}3$$

The elevation at the coast for the limit case $m \to \infty$ and for arbitrary $\Omega$ might be of theoretical interest.
For $m \to \infty$ we have $\bar{w} \to 1$, so that by means of 2-1 and 2-2

$$\bar{\zeta}(0) = (\sin\alpha - \frac{\Omega}{p+\lambda}\cos\alpha)\,\bar{\psi}(0),$$

so that

$$m \to \infty : \qquad \zeta(0,t) = \sin\alpha\,\psi(0,t) - \Omega\cos\alpha\, e^{-\lambda t}\int_0^t e^{\lambda\tau}\psi(0,\tau)d\tau\ . \qquad 4\text{-}4$$

## §5. Expansions

By means of the expansion theorems of the Laplace transformation which state that an expansion of the image for small p (large p) corresponds to an expansion of the original for large t (small t) we may derive expressions for $\zeta$ at the coast in the form of series which can be used for large t and small t respectively.

Generally spoken we may expect a convergent series in ascending powers of t which converges everywhere, which has practical use only for small t such as the expansion of $e^{-t}$ into a power series, and next we may expect an asymptotic series for large t.

If the windfield of the preceding section is taken we obtain from 1-18 for a half-plane sea

$$\bar{\zeta}(0,p) = \left(\sin\alpha - \frac{\Omega}{p+\lambda}\cos\alpha\right)\frac{m^2}{(p+m)^2}\;\frac{(p+\lambda)^{\frac{1}{2}}}{p^{\frac{1}{2}}\left\{(p+\lambda)^2+\Omega^2\right\}^{\frac{1}{2}}} \quad . \qquad 5\text{-}1$$

There singularities at $p = -\lambda$, $p = -m$ and $p = -\lambda \pm \Omega i$. In the following section we shall take the numerical values $\lambda = 0.08$, $\Omega = 0.44$ and $0.05 \leq m \leq 0.25$. The practical use of the asymptotic expansion is restricted by the singularity which is nearest to the origin. This is practically the singularity at $p = -\lambda$ due to the coefficient of friction. Thus an asymptotic series for large t is obtained which can be used only when the friction already has an appreciable effect i.e. when the storm is over and when the after-effect of the storm is damped out by the friction. The practical use of the convergent series for small t is restricted by the singularity which is most distant from the origin. This is at $p = -\lambda \pm \Omega i$. Since $|\lambda \pm \Omega i|$ exceeds $\lambda$ by a factor 5.6 there is an intermediate region where both expansions in the p-plane are bad. This region corresponds roughly to the time interval during which the Coriolis effect is already important and the friction has not yet damped out the elevation $\zeta$. However, the maximum height of the elevation $\zeta(0,t)$ just occurs at this interval.

Thus we may expect that generally the maximum height of the elevation at the coast cannot be derived from the asymptotic expansion for large t, but that it may be obtained from the convergent expansion for small t if a sufficient number of terms is taken into account. In a particular case this number proved to be of the order of twenty!

It is sufficient to consider only the cases $\alpha = 0^\circ$ and $\alpha = 90^\circ$ since

$$\zeta(\alpha^\circ) = \sin\alpha\,\zeta(90^\circ) + \cos\alpha\,\zeta(0^\circ) \quad . \qquad 5\text{-}2$$

Next we shall derive an expansion for large p. It is a little more convenient to derive an expansion in powers of $p+\lambda$ and we shall put

$$p + \lambda = \Omega s, \qquad m - \lambda = \Omega\nu, \qquad \lambda = \Omega\mu \quad . \qquad 5\text{-}3$$

We may write

$$\begin{aligned}\bar{\zeta}(0^\circ) &= -\Omega^{-3}s^{-4}\left(1+\frac{\nu}{s}\right)^{-2}\left(1-\frac{\mu}{s}\right)^{-\frac{1}{2}}\left(1+\frac{1}{s^2}\right)^{-\frac{1}{2}} = \\ &= -\Omega^{-3}s^{-4}\sum_0^\infty a_n s^{-n} \quad ,\end{aligned} \qquad 5\text{-}4$$

where $a_o = 1$,

and $$a_n = \sum_{k+\ell+2m=n} \binom{-2}{k}\binom{-\frac{1}{2}}{\ell}\binom{-\frac{1}{2}}{m} \kappa^k (-\mu)^{\ell} \quad . \qquad 5\text{-}5$$

The computation of the coefficients $a_n$ is not difficult since $|\kappa|$ and $\mu$ are small and consequently higher powers of those quantities may be neglected.

The original of 5-4 is

$$\zeta(0^o,t) = -\Omega t^3 e^{-\lambda t} \sum_0^{\infty} \frac{a_n \Omega^n t^n}{(n+3)!} \quad . \qquad 5\text{-}6$$

For $\zeta(90^o)$ we obtain in a similar way

$$\zeta(90^o,t) = t^2 e^{-\lambda t} \sum_0^{\infty} \frac{a_n \Omega^n t^n}{(n+2)!} \quad . \qquad 5\text{-}7$$

Both expansions are everywhere convergent but convergence is slow for large t, say $t \gg 6$.

## §6. A numerical case

Computations have been carried out for the case of a homogeneous windfield of the following form

$$w(t) = m^2 t e^{-mt} \; , \qquad 6\text{-}1$$

where $T = \frac{1}{m} = 4, 8, 12\frac{1}{2}, 16$ and 20 (hours).
The coefficients of friction and of Coriolis are

$\lambda = 0.08 \qquad \Omega = 0.44 \qquad (\text{hours}^{-1}).$

The elevation at the coast $\zeta(0^o,t)$ has been calculated for $\alpha = 0^o$ and $\alpha = 90^o$ by means of 4-2, 2-6 and 2-4. In some cases the expansions 5-6 and 5-7 have been used for small t-values.

The function 6-1 is given in figure 1. It appears that a storm of this type increases rapidly to its extremum but decreases rather slowly afterwards.

In fig. 2-7 graphs of $\zeta(t)$ have been given for the various T-values including also T = 0, the case of a sudden outburst of wind at t = 0 in the sense of a Dirac deltafunction. For each T value $\zeta$ has been plotted versus time for a n mber of $\alpha$-values

$\alpha = 0^o \; 40^o \; 80^o \; 90^o \; 110^o \; 130^o \; 170^o$ .

From this the following points may be observed:

i $\zeta(\alpha)$ attains the maximum positive elevation for $\alpha = 170^\circ$ approximately so that the elevation at the coast is much more influenced by a wind which is tangential to the coast than by a wind which is normal to the coast.

ii after $\zeta(t)$ has reached its peak, the elevation decreases gradually and in a slightly oscillatory way. For small T the oscillations become more pronounced as we may see from fig. 2 where T = 0. These oscillations have a period of about $2\pi/\Omega$. They correspond in the analytical expression of the Laplace transform of $\zeta(t)$ to the singularities at $p = -\lambda \pm \Omega i$.

iii the elevation reaches its peak value some time after the wind maximum. For the case T = 4 the so-called time-lag has bee plotted versus $\alpha$ in fig. 11. For $\alpha = 0^\circ$ there is a time-lag of about 8 hrs, for $\alpha = 40^\circ$ it is even more. If $\alpha$ is about $70^\circ$ $\zeta$ has a positive extremum at about 5 hrs and a negative extremum at about 14 hrs which has nearly the same absolute value. If $\alpha = 80^\circ$ the first extremum is already predominant and as $\alpha$ increases the second extremum disappears into an oscillation.

In fig. 8 $\zeta(t)$ has been plotted for T = 4, $\Omega = 0$ and the $\alpha$-values $10^\circ, 40^\circ, 50^\circ, 70^\circ, 90^\circ$. We may compare these graphs with fig. 3 where the case T = 4, $\Omega = 0.44$ has been considered. The positive extrema have been plotted versus $\alpha$ for $\Omega = 0$ and $\Omega = 0.44$ respectively in figure 9. We observe the remarkable fact that the influence of $\Omega$ practically results into a shift in the $\alpha$-values of roughly $80^\circ$. The absolute value of the maximum elevation appears to be hardly affected.

Analytically this follows from the formulae

$$\Omega = 0 \qquad \bar{\zeta} = \sin\alpha \frac{\bar{w}}{\sqrt{p(p+\lambda)}} ,$$

$$\text{and } \Omega \neq 0 \qquad \bar{\zeta} = \sin(\alpha - \theta) \frac{\bar{w}}{\sqrt{p(p+\lambda)}} ,$$

$$\text{where} \qquad \mathrm{tg}\, \theta = \frac{\Omega}{p+\lambda} .$$

Finally in figure 10 we have plotted the total maximum of $\zeta$, i.e. for variable $\alpha$ end time, versus the duration of the storm. Since storms of the same "energy" are considered, a short storm gives a higher value of $\zeta_{max}$ than a long storm since for a long storm the influence of the friction $\lambda$ becomes more important.

## § 7. Generalizations

### A The influence of a windfield upon a canal

We consider the strip $0 < y < b$ and a windfield $W_x, W_y$ which does not depend on x. The differential equations are the same as those of § 1 but we have a second boundary condition

$$y = b, \; \bar{w}_y = 0$$

The solution becomes

$$\bar{w}_y = - \frac{shky}{kshkb} \int_0^b shk(b-\eta)\bar{F}d\eta + \frac{1}{k} \int_0^y shk(y-\eta)\bar{F}d\eta \, , \qquad 7\text{-}1$$

or

$$\bar{w}_y = \frac{1}{2kshkb} \int_0^b \left\{ chk(y+\eta-b) - chk(|y-\eta|-b) \right\} \bar{F}d\eta \, . \qquad 7\text{-}2$$

For $b \to \infty$ 7-2 gives again the solution 1-7 of the halfplane.
In the case of a uniform windfield

$$W_x = - \rho c w(t) \cos \alpha$$

$$W_y = - \rho c w(t) \sin \alpha$$

we find from 7-1

$$\bar{w}_y = \left\{ \frac{shky}{shkb} (1-chkb) - (1-chky) \right\} \frac{p}{k^2 c} (\sin \alpha - \frac{\Omega}{p+\lambda} \cos \alpha) \bar{w} \, , \qquad 7\text{-}3$$

and next

$$\bar{\zeta}(y,p) = \frac{(p+\lambda)^{\frac{1}{2}}}{p^{\frac{1}{2}} \left\{ (p+\lambda)^2 + \Omega^2 \right\}^{\frac{1}{2}}} (\sin \alpha - \frac{\Omega}{p+\lambda} \cos \alpha) \bar{w} \, \frac{chk(b-y) - chky}{shkb} \, , \qquad 7\text{-}4$$

and at the coast $y = 0$

$$\bar{\zeta}(0,p) = \frac{(p+\lambda)^{\frac{1}{2}}}{p^{\frac{1}{2}} \left\{ (p+\lambda)^2 + \Omega^2 \right\}^{\frac{1}{2}}} (\sin \alpha - \frac{\Omega}{p+\lambda} \cos \alpha) \bar{w} \, \frac{e^{kb} - 1}{e^{kb} + 1} \, . \qquad 7\text{-}5$$

We may develop $\frac{e^{kb} - 1}{e^{kb} + 1}$ into a series

$$1 - 2e^{-kb} + 2e^{-2kb} - 2e^{-3kb} \ldots \, .$$

The first term represents the disturbance due to the first coast $y = 0$; the second term represents the disturbance due to the second coast $y = b$, the third term represents the reflection of the disturbance at $y = 0$ with respect to the coast $y = b$, and generally the positive terms are the repeated reflection due to the disturbance of the first coast and the negative terms those of the second coast.

For $\epsilon^{-mkb}$ we may write

$$\epsilon^{-\frac{mb}{c}(p+\frac{\lambda}{2})}\left\{1-\left(\frac{\Omega^2}{2}-\frac{\lambda^2}{8}\right)\frac{mb}{c}\frac{1}{p+\lambda}+\dots\right\},$$

so that the influence of this term becomes noticable after a delay of $t=\frac{mb}{c}$ .

## B A half-plane sea with an exponentially increasing depth

We shall only consider the case of a uniform windfield of constant direction working upon the whole sea.
We put

$$h=h_o e^{\beta y} \quad . \qquad 7\text{-}6$$

If $c^2$ now means $gh_o$ the equations of motion become

$$\begin{aligned}(p+\lambda)\bar{w}_x - \Omega\bar{w}_y &= \frac{1}{\rho}\bar{W}_x\\ (p+\lambda)\bar{w}_y + \Omega\bar{w}_x + c^2e^{\beta y}\frac{\partial\bar{\zeta}}{\partial y} &= \frac{1}{\rho}\bar{W}_y \qquad 7\text{-}7\\ \frac{\partial\bar{w}_y}{\partial y}+p\bar{\zeta} &= 0 \quad .\end{aligned}$$

Elimination of $\bar{w}_x$ and $\bar{w}_y$ gives

$$\frac{\partial^2\bar{\zeta}}{\partial y^2}+\beta\frac{\partial\bar{\zeta}}{\partial y}-e^{-\beta y}k^2\bar{\zeta}=0 \ , \qquad 7\text{-}8$$

with the boundary condition

$$y=0\ , \qquad \frac{\partial\bar{\zeta}}{\partial y}=\frac{1}{\rho c^2}\left(\bar{W}_y-\frac{\Omega}{p+\lambda}\bar{W}_x\right). \qquad 7\text{-}9$$

If we introduce the new variable $u=e^{-\beta y}$, 7-8 becomes

$$\frac{\partial^2\bar{\zeta}}{\partial u^2}=\frac{k^2}{\beta^2}\frac{\bar{\zeta}}{u} \ , \qquad 7\text{-}10$$

which has the general solution

$$\bar{\zeta}=\frac{2k}{\beta}u^{\frac{1}{2}}\left\{AI_1\left(\frac{2k}{\beta}u^{\frac{1}{2}}\right)+BK_1\left(\frac{2k}{\beta}u^{\frac{1}{2}}\right)\right\} \quad . \qquad 7\text{-}11$$

For $y\to\infty$ we have $\bar{\zeta}\to 0$ so that $B=0$.

For y = 0 7-9 becomes

$$\frac{2k^2A}{\beta} I_o\left(\frac{2k}{\beta}\right) = \frac{1}{\rho c^2}\left(-\bar{W}_y + \frac{\Omega}{p+\lambda}\bar{W}_x\right),$$

so that finally

$$\bar{\zeta}(y,p) = \frac{(p+\lambda)^{\frac{1}{2}}}{p^{\frac{1}{2}}\left\{(p+\lambda)^2 + \Omega^2\right\}^{\frac{1}{2}}}\left(\sin\alpha - \frac{\Omega}{p+\lambda}\cos\alpha\right)\bar{w}\,\frac{u^{\frac{1}{2}} I_1\left(\frac{2k}{\beta}u^{\frac{1}{2}}\right)}{I_o\left(\frac{2k}{\beta}\right)} . \qquad 7\text{-}12$$

For $\beta \to 0$ we have

$$\frac{u^{\frac{1}{2}} I_1\left(\frac{2k}{\beta}u^{\frac{1}{2}}\right)}{I_o\left(\frac{2k}{\beta}\right)} \approx \exp\left\{\frac{2k}{\beta}\left(e^{-\frac{\beta}{2}y} - 1\right)\right\} \to e^{-ky} ,$$

so that the result 1-15 of the first section is obtained.

If we consider the elevation at the coast and if $\beta$ is small the last factor of 7-12 is approximately

$$\frac{I_1\left(\frac{2k}{\beta}\right)}{I_o\left(\frac{2k}{\beta}\right)} \approx \frac{1 - \frac{3\beta}{16k} \cdots}{1 + \frac{\beta}{16k} \cdots} \approx 1 - \frac{\beta}{4k} \qquad 7\text{-}13$$

If t is not large we may replace k by $\frac{1}{c}\left(p+\frac{\lambda}{2}\right)$ and we may compare the elevation at y = 0 for $\beta \neq 0$ to that for $\beta = 0$

$$\zeta_\beta(0,t) \approx \zeta_o(0,t) - \frac{\beta c}{4} e^{-\frac{\lambda}{2}t}\int_o^t e^{\frac{\lambda}{2}\tau}\zeta_o(\tau)\,d\tau . \qquad 7\text{-}14$$

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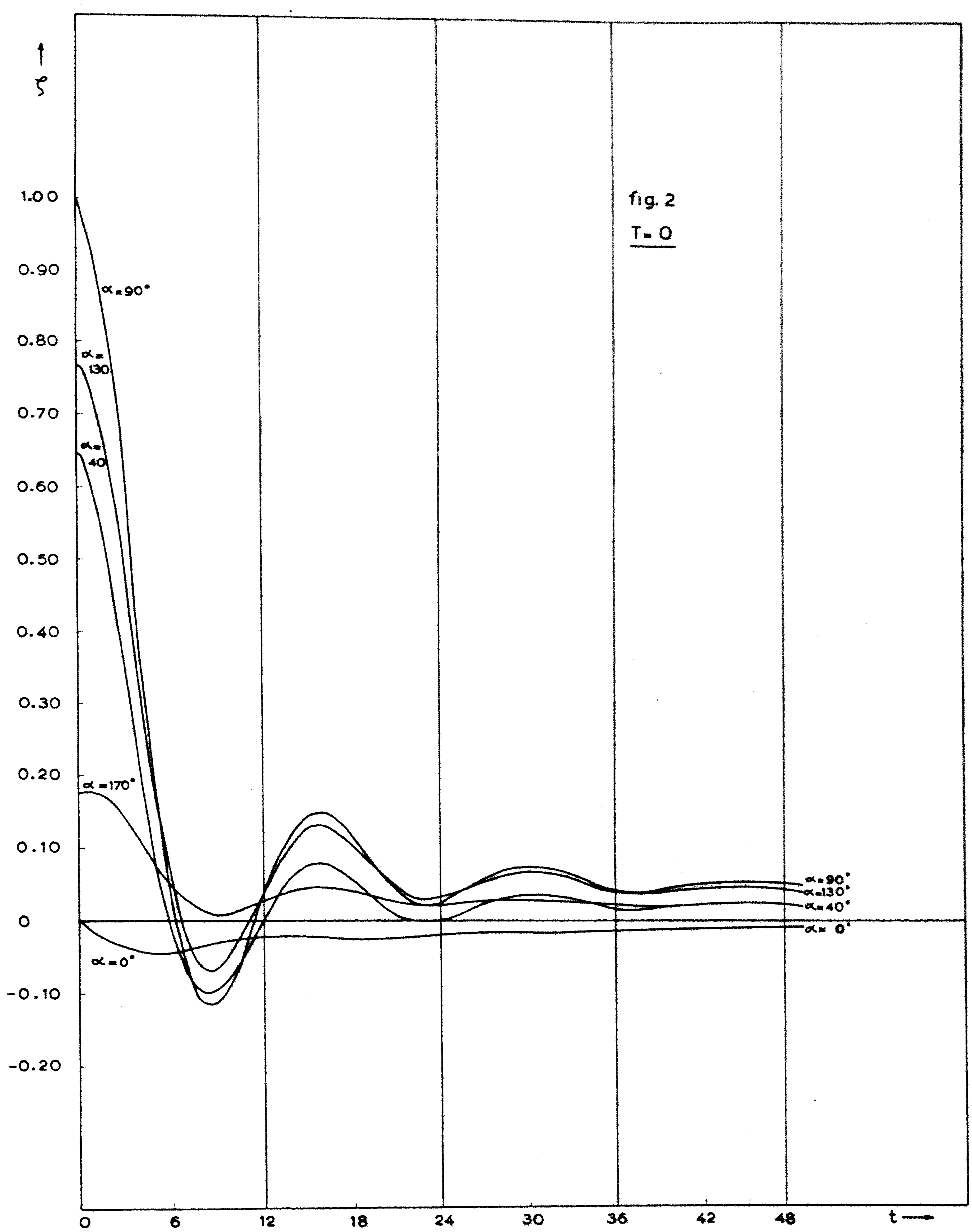

ζ
fig. 2
T= O
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0
-0.10
-0.20
α=90°
α= 130
α= 40
α=170°
α=0°
α=90°
α=130°
α=40°
α= 0°
0
6
12
18
24
30
36
42
48
t

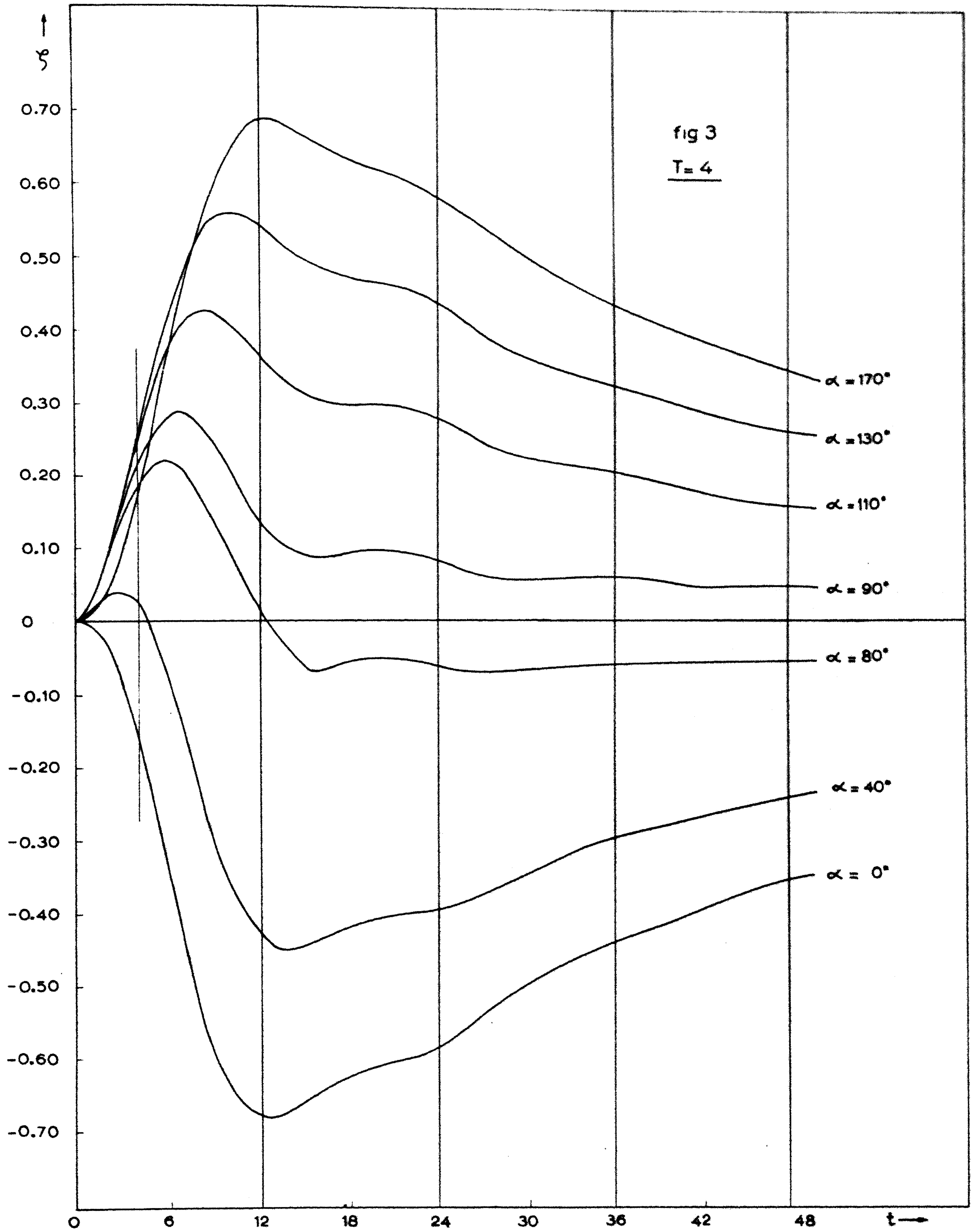
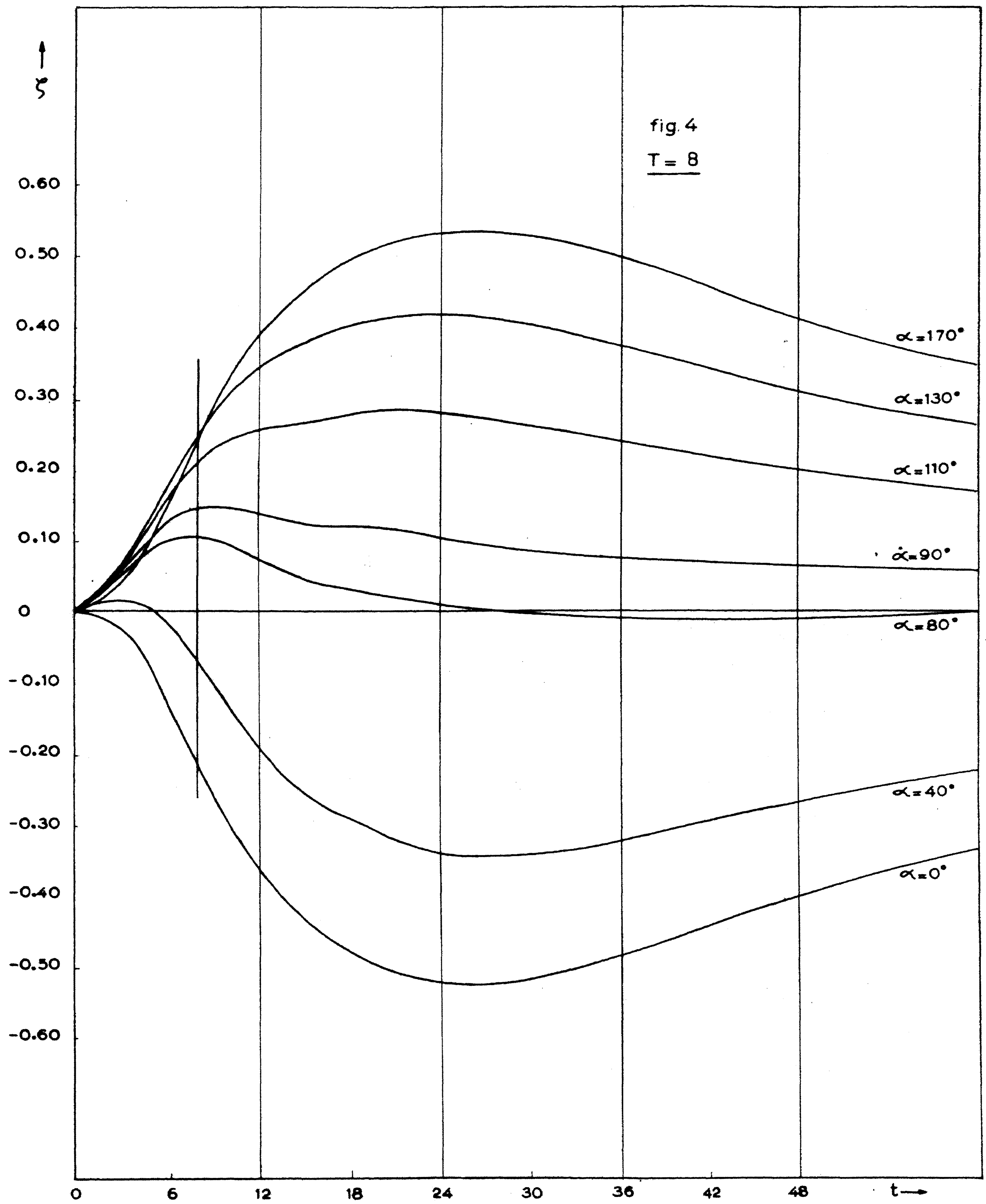
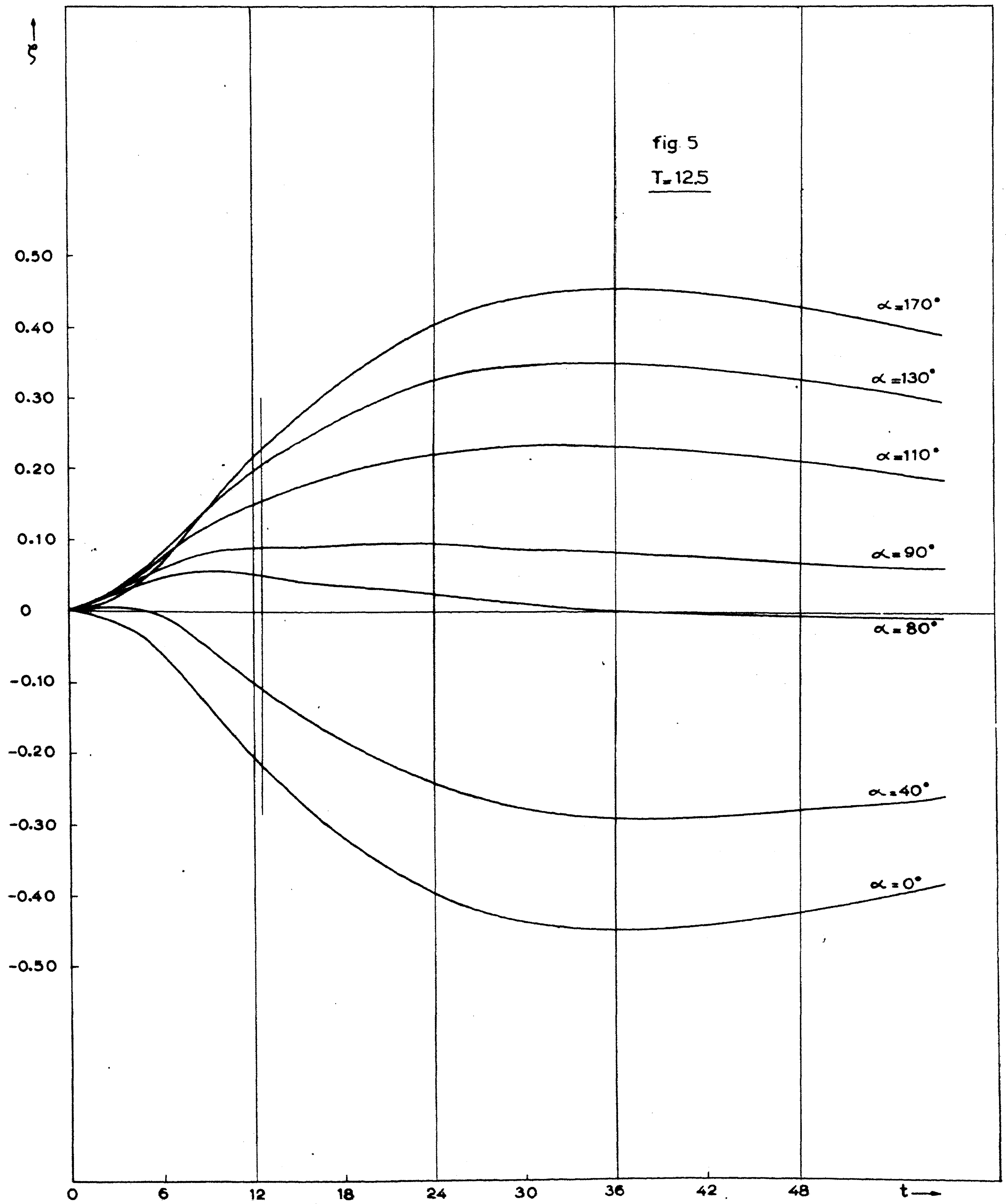


fig 3
T= 4
ξ
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0
-0.10
-0.20
-0.30
-0.40
-0.50
-0.60
-0.70
0
6
12
18
24
30
36
42
48
t
α = 170°
α = 130°
α = 110°
α = 90°
α = 80°
α = 40°
α = 0°



ζ
fig. 4
T = 8
0.60
0.50
0.40
0.30
0.20
0.10
0
-0.10
-0.20
-0.30
-0.40
-0.50
-0.60
α=170°
α=130°
α=110°
α=90°
α=80°
α=40°
α=0°
0
6
12
18
24
30
36
42
48
t


ξ
fig. 5
T= 12.5
0.50
0.40
0.30
0.20
0.10
0
-0.10
-0.20
-0.30
-0.40
-0.50
α=170°
α=130°
α=110°
α=90°
α=80°
α=40°
α=0°
0
6
12
18
24
30
36
42
48
t

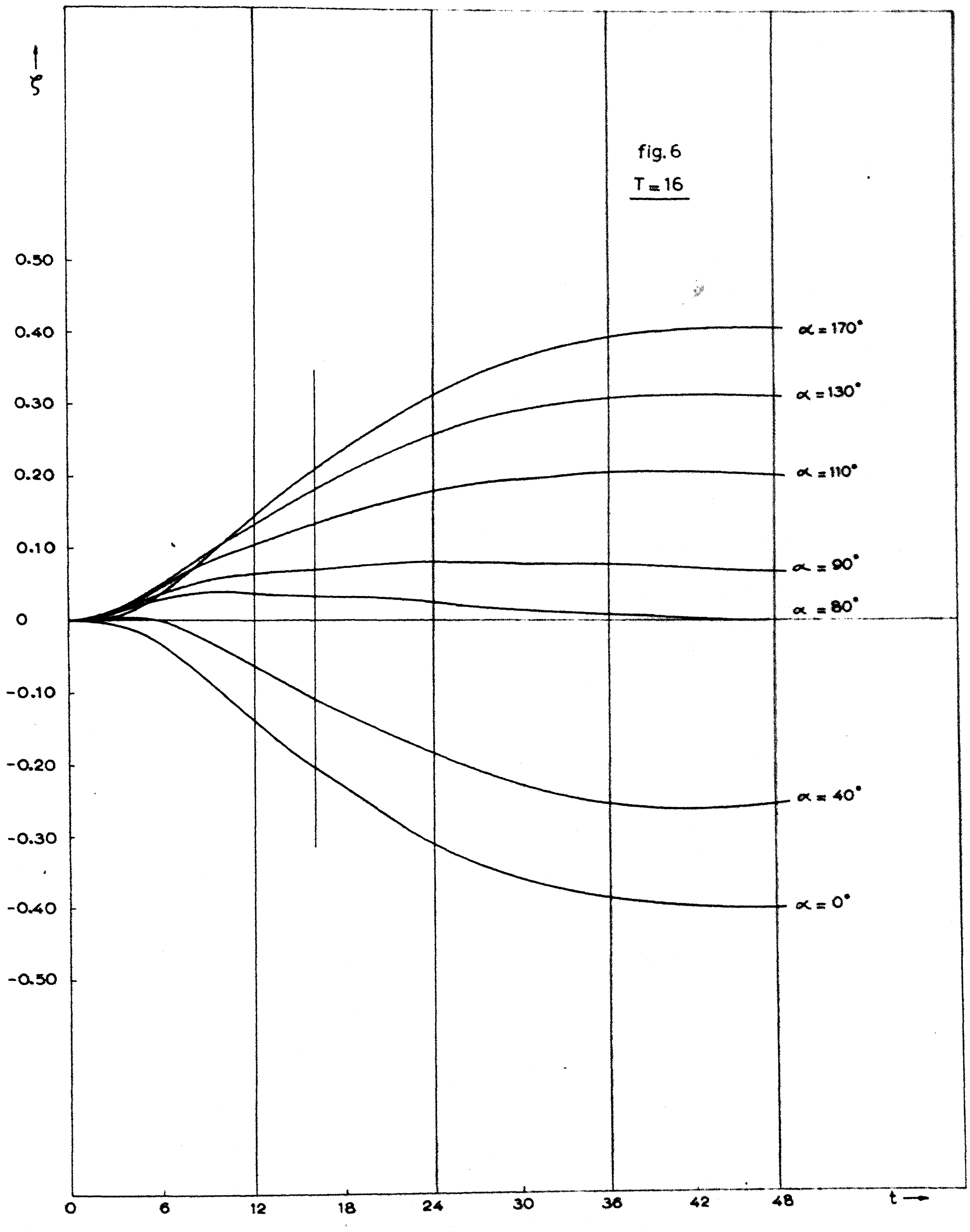


fig. 6
T = 16
ζ
0.50
0.40
0.30
0.20
0.10
0
-0.10
-0.20
-0.30
-0.40
-0.50
α = 170°
α = 130°
α = 110°
α = 90°
α = 80°
α = 40°
α = 0°
0
6
12
18
24
30
36
42
48
t

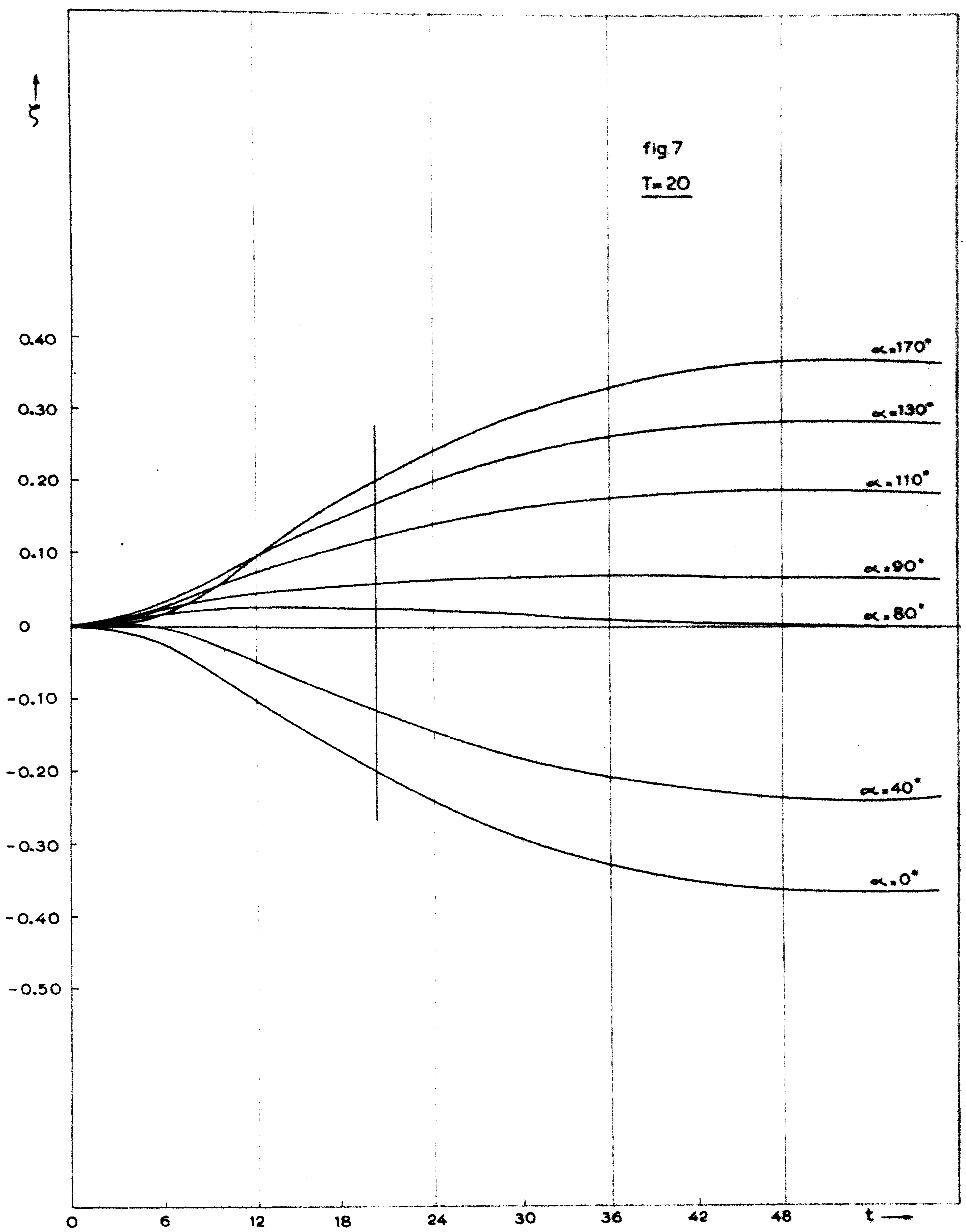


fig.7
T=20
ζ
0.40
0.30
0.20
0.10
0
-0.10
-0.20
-0.30
-0.40
-0.50
α=170°
α=130°
α=110°
α=90°
α=80°
α=40°
α=0°
0
6
12
18
24
30
36
42
48
t

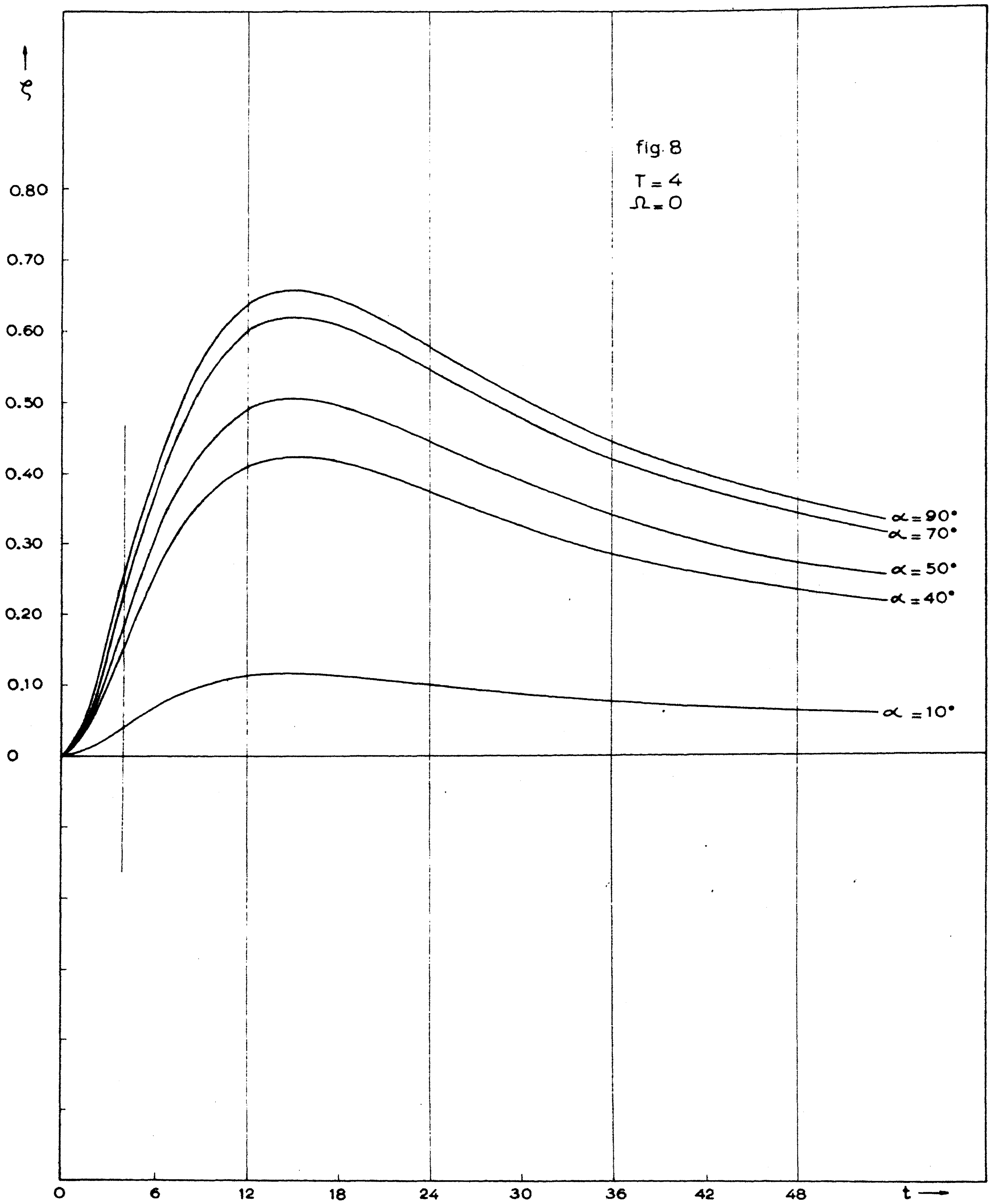


fig. 8
T = 4
Ω = 0
ξ
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0
α = 90°
α = 70°
α = 50°
α = 40°
α = 10°
0
6
12
18
24
30
36
42
48
t

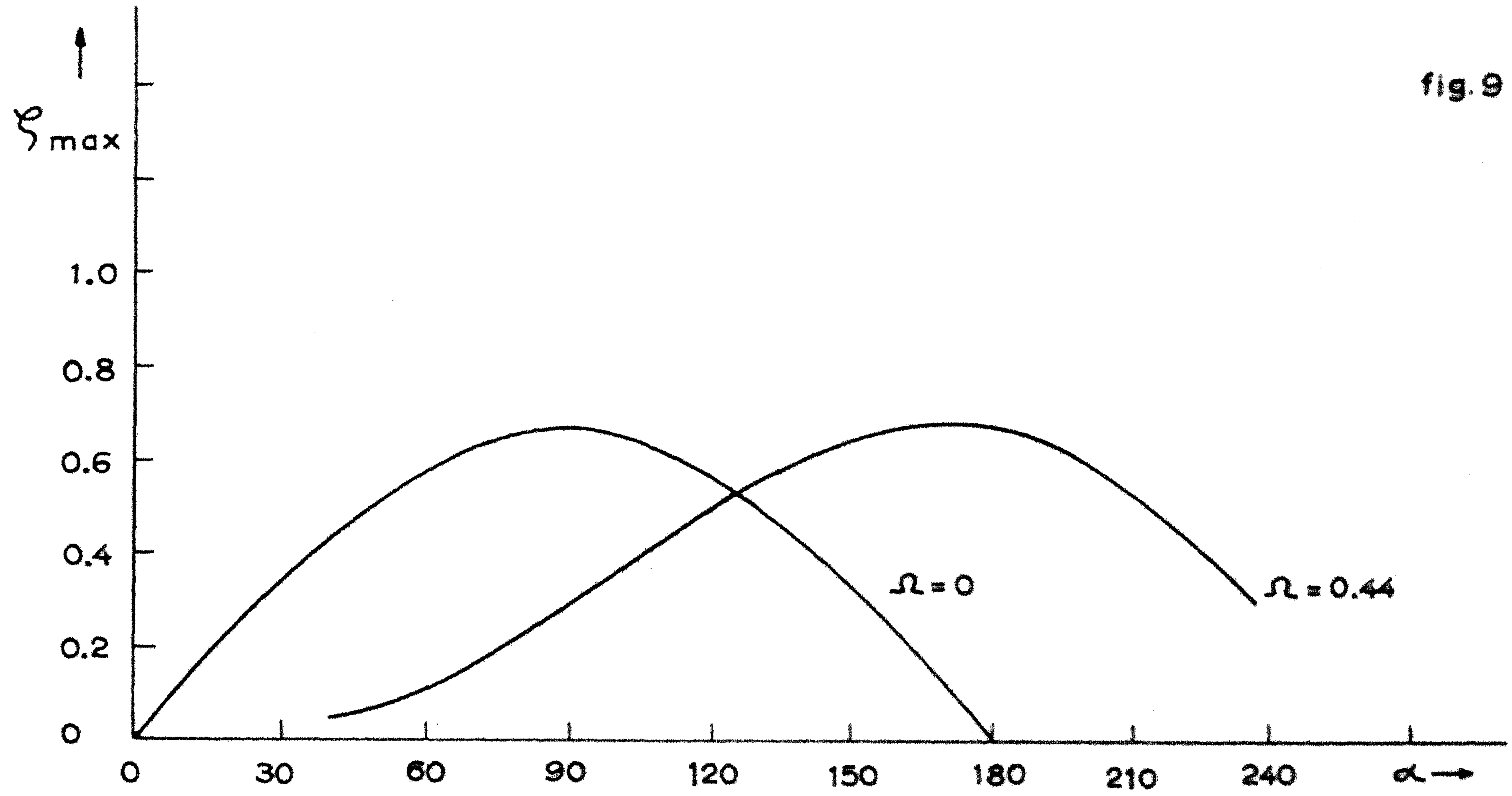


fig. 9
$\zeta_{max}$
1.0
0.8
0.6
0.4
0.2
0
0
30
60
90
120
150
180
210
240
$\alpha$
$\Omega = 0$
$\Omega = 0.44$

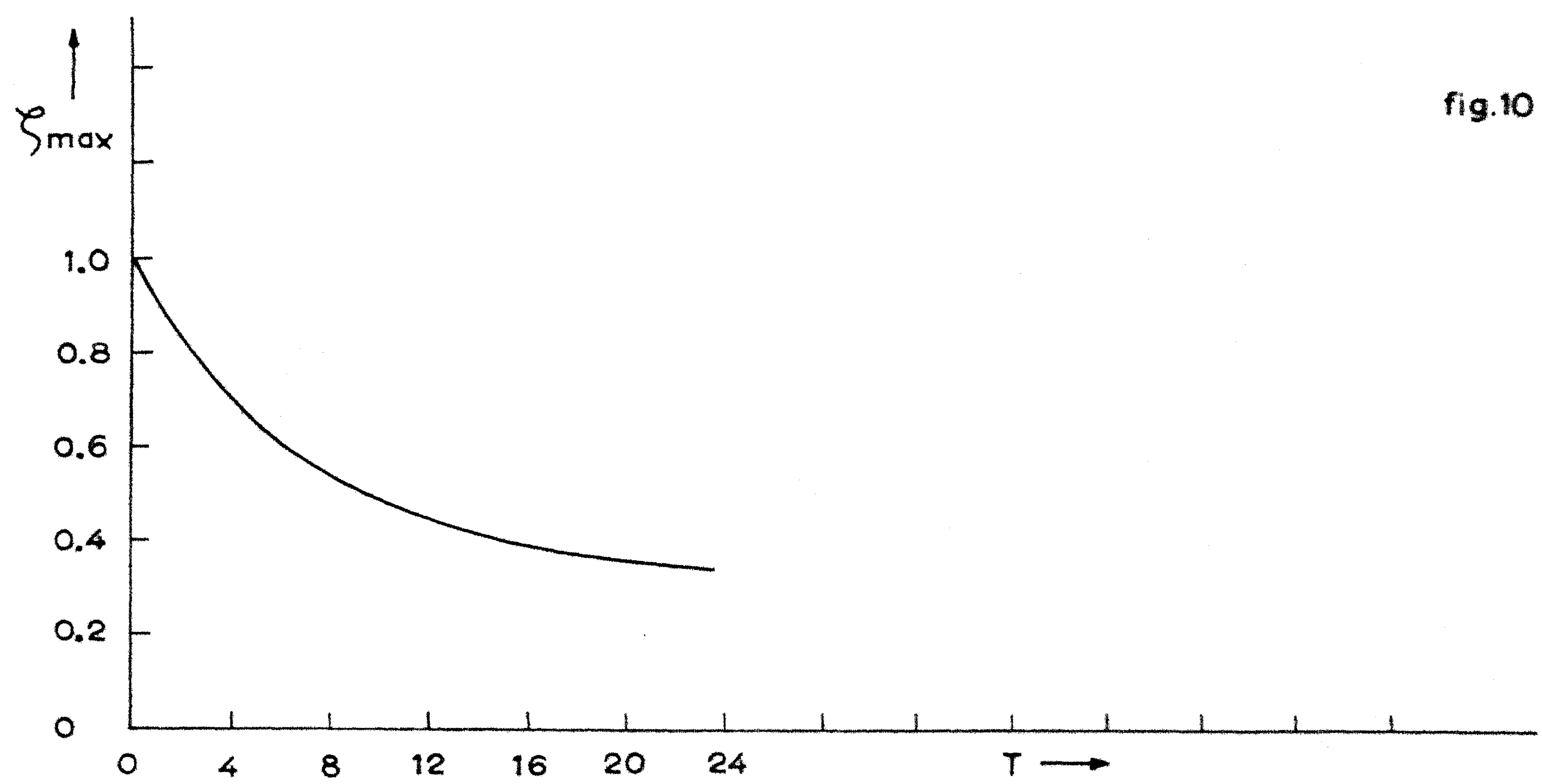



fig.10
$\zeta_{max}$
1.0
0.8
0.6
0.4
0.2
0
0
4
8
12
16
20
24
T

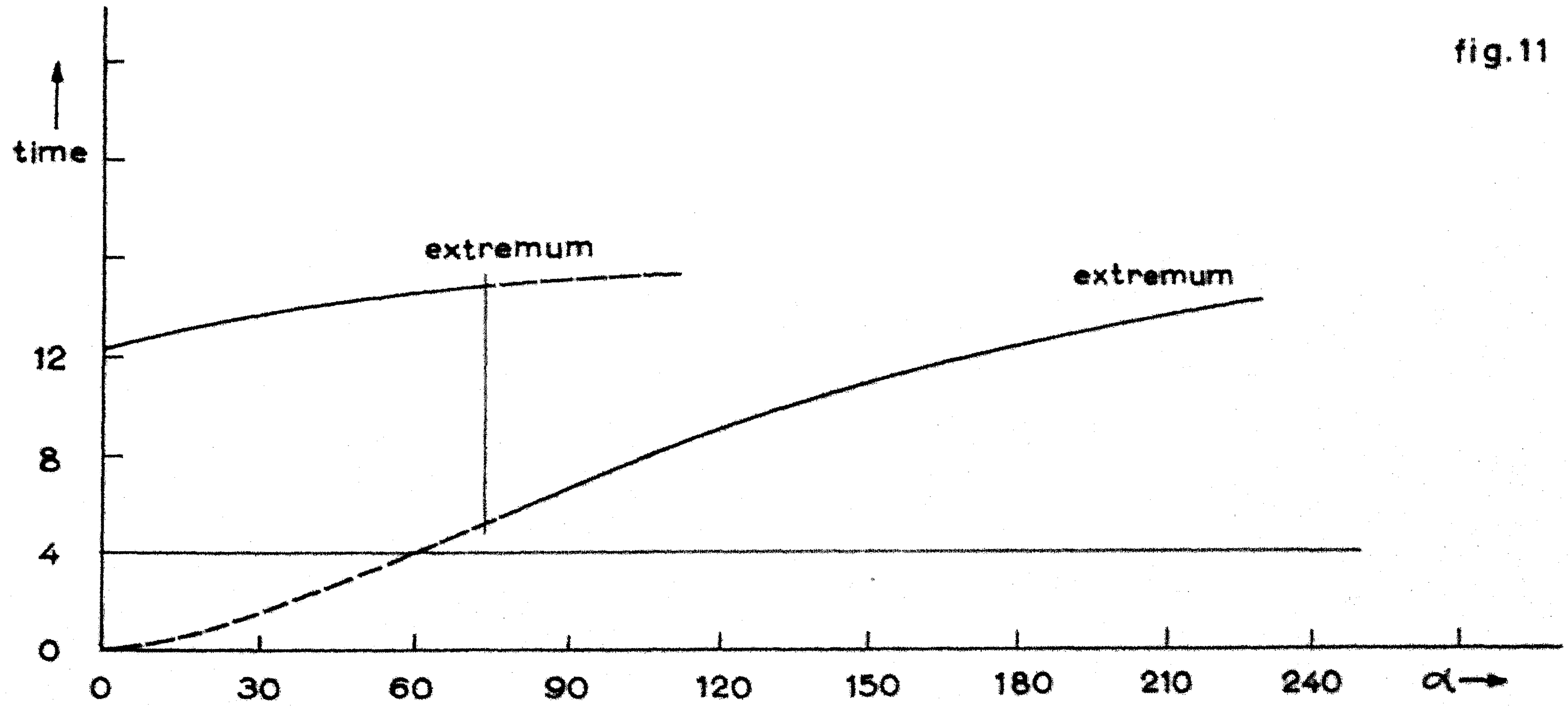



fig.11
time
extremum
extremum
12
8
4
0
0
30
60
90
120
150
180
210
240
$\alpha$