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THE EXPANSION OF A FUNCTION INTO A FOURIER SERIES WITH PRESCRIBED PHASES, VALID IN THE HALF-PERIOD INTERVAL

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CONTENTS

§1. Introduction

A real function of a real variable x may be expanded in the interval $(0,\pi)$ under fairly general conditions either into a cosine series or in a sine series

$$f(x) = \sum_0^\infty \alpha_n \cos nx \quad , \quad f(x) = \sum_1^\infty \beta_n \sin nx. \qquad 1\text{-}1$$

In the first case a constant term α_0 is needed, in the second case this is not necessary, and we may expand unity into a sine series

$$1 = \frac{4}{\pi}\left(\frac{\sin x}{1} - \frac{\sin 3x}{3} + \frac{\sin 5x}{5} - \dots \right) , \quad 0<x<\pi \qquad 1\text{-}2$$

We shall now investigate the possibility of an a trigonometric expansion of $f(x)$ of the following type

$$f(x) = \sum_1^\infty a_n(\sin nx + \gamma_n \cos nx) , \quad 0<x<\pi, \qquad 1\text{-}3$$

where γ_n may be complex.

This may be considered as a Fourier expansion with prescribed phases valid only in the half-period interval $(0,\pi)$. Sometimes we shall write for 1-3

$$f(x) = \sum_1^\infty b_n \sin (nx + \mu_n\pi) , \quad 0<x<\pi, \qquad 1\text{-}4$$

where

$$\operatorname{tg} \mu_n\pi = \gamma_n \quad , \quad |\operatorname{Re} \mu_n| \leq \tfrac{1}{2} \quad \text{and} \quad b_n \cos \mu_n\pi = a_n . \qquad 1\text{-}5$$

In the following sections the case of a positive and real $\gamma_n = \gamma$ which is independent of n is considered first. It appears that generally the expansion 1-3 is always possible and that the coefficients a_n are of order $n^{-1+2\mu}$, where $\mu_n = \mu$ for all n, so that the series 1-3 converges conditionally. In the case $f(x)\equiv 1$ we find in particular

$$1 = \sum_1^\infty (-1)^{n-1} e_n(2\mu) \left(\cos nx + \frac{1}{\gamma} \sin nx\right) , \qquad 1\text{-}6$$

where

$$\sum_1^\infty e_n(2\mu)\lambda^n = \left(\frac{1+\lambda}{1-\lambda}\right)^{2\mu}-1. \qquad 1\text{-}7$$

It appears further that a function exists which is orthogonal to all functions $\sin(nx+\mu\pi)$, and we have found

$$\int_0^\pi (\operatorname{tg} \tfrac{1}{2}x)^{1-2\mu} \sin(nx+\mu\pi)dx=0, \qquad n\geq 1 . \qquad 1\text{-}8$$

If $f(x)$ satisfies the condition

$$\int_0^{\pi} (\mathrm{tg}\ \tfrac{1}{2}x)^{1-2\mu}\ f(x)dx = 0 \qquad 1\text{-}9$$

the series 1-3 converges uniformly and $\sum |a_n|$ and $\sum n|a_n|^2$ are finite. Also the inverse is true. If e.g. $\sum n|a_n|^2 < \infty$ the condition 1-9 is satisfied. If this is the case the coefficients a_n appear to be of the order $n^{-1-2\mu}$.

If $\gamma_n = \gamma$ is a complex constant with Re $\gamma > 0$ similar conclusions may be drawn since also $0 <$ Re $\mu < \frac{1}{2}$. If Re $\gamma < 0$ we may perform the transformation $x \rightarrow \pi - x$, $\gamma \rightarrow -\gamma$ by means of which this case is reduced to the previous one.

If γ is purely imaginary with $|\gamma| > 1$ we have Re $\mu = \frac{1}{2}$ and the coefficients b_n of 1-4 in general do not tend to zero, so that in this case a convergent expansion of type 1-4, i.e. without constant term, does not exist. If, however, the condition 1-9 is satisfied the series 1-4 converges and $b_n = \mathcal{O}(\frac{1}{n^2})$.

If γ is purely imaginary with $|\gamma| < 1$ we have Re $\mu = 0$ and the expansion 1-4 is possible with $b_n = \mathcal{O}(\frac{1}{n})$ without subsidiary conditions 1-9. However, in this case the orthogonality relation 1-8 breaks down and should be replaced by the <u>two</u> relations 1-10 the first of which may be considered as the analytic continuation of 1-8.

$$\begin{cases} \displaystyle\int_0^{\pi} (\mathrm{tg}\ \tfrac{1}{2}x)^{1-2\mu} \left\{ \sin(nx+\mu\pi) - \sin(n\pi+\mu\pi) \right\} dx = (-1)^{n-1}\pi, \\ \displaystyle\int_0^{\pi} (\mathrm{tg}\ \tfrac{1}{2}x)^{-1-2\mu} \left\{ \sin(nx+\mu\pi) - \sin\mu\pi \right\} dx = \pi. \end{cases} \qquad 1\text{-}10$$

Thus, there is no function orthogonal to all $\sin(nx+\mu\pi)$, $n \geq 1$, in the ordinary sense.

If again we require that $\sum n|a_n|^2 < \infty$ <u>two</u> conditions are imposed upon $f(x)$

$$\begin{cases} \displaystyle\frac{1}{\pi}\int_0^{\pi} (\mathrm{tg}\ \tfrac{1}{2}x)^{1-2\mu} \left\{ f(\pi) - f(x) \right\} dx = \frac{f(\pi)}{\sin\mu\pi}, \\ \displaystyle\frac{1}{\pi}\int_0^{\pi} (\mathrm{tg}\ \tfrac{1}{2}x)^{1+2\mu} \left\{ f(x) - f(0) \right\} dx = \frac{f(0)}{\sin\mu\pi}. \end{cases} \qquad 1\text{-}11$$

Next we shall consider the case of variable γ_n. For simplicity we shall assume that γ_n is real and

$$\gamma_n = \gamma + r_n$$

where $0 < \gamma < \infty$ and $r_n = \mathcal{O}(\frac{1}{n^2})$.

We introduce the auxiliary function

$$s(x) = \sum_{1}^{\infty} a_n \sin nx$$

from which the coefficients a_n can be determined easily. It appears that the problem of expanding $f(x)$ into a trigonometrical series 1-3 is equivalent to the problem of solving a Cauchy singular integral equation for $s(x)$. This singular integration may be reduced to an ordinary Fredholm equation with bounded kernel

$$F(x) = G(x) - \frac{\cos \mu\pi}{\pi} \int_0^{\pi} K(t,x)\, F(t)\, dt , \qquad 1\text{-}12$$

where
$$F(x) = (1 + \cos x)^{\mu}\, s(x) .$$

$G(x)$ is a known function dependent on $f(x)$ only, and $K(t,x)$ is uniformly bounded in the square $0 \leq t,x \leq \pi$.
If γ_n is a constant, $K(t,x)$ vanishes identically and 1-12 reduces to an explicit expression for $s(x)$ and hence for the coefficients a_n.

If $f(x) \equiv 1$ is developed into an adjoint series

$$1 = \sum_{1}^{\infty} h_n(\sin nx + \gamma_n^{-1} \cos nx) ,$$

the function $h(x)$ defined by

$$h(x) = \sum_{1} h_n \sin nx$$

is orthogonal to all functions $\sin nx + \gamma_n \cos nx$.
The orthogonal function may be determined from the following ordinary Fredholm equation

$$H(x) = H_o(x) + \frac{\sin \mu\pi}{\pi} \int_0^{\pi} L\,(t,x)\, H(t)\, dt , \qquad 1\text{-}13$$

where

$$H(x) = (1 + \cos x)^{\frac{1}{2}-\mu}\, h(x) ,$$

$$H_o(x) = (1-\cos x)^{\frac{1}{2}-\mu}\; \frac{\sin \mu\pi}{\pi} \int_0^{\pi} h(t)\, dt .$$

and $L(t,x)$ is a uniformly bounded kernel.
If $\gamma_n = \gamma$ constant, $L(t,x)$ vanishes identically and 1-13 gives merely $h(x) = C(\operatorname{tg} \frac{1}{2}x)^{1-2\mu}$, where C is a constant, in accordance with 1-9.

If γ_n is complex similar conclusions may be drawn since the principle of analytic continuation may be applied. If in particular γ_n is purely imaginary, the integral equations 1-12 and 1-13 remain solvable.

Finally, a twodimensional potential problem is considered the solution of which may depend upon the expansion of a given function into a trigonometrical series in the half-period interval. By means of Green's theorem the potential problem may be reduced to a singular integral equation, which is another way of showing that the expansion problem 1-3 is equivalent to a singular integral equation.

This report will be followed by another report in which special attention is given to the integral equations 1-12 and 1-13, and to the numerical determination of the orthogonal function h(x).

§2 The Cauchy integral [1)]

We need a few properties of Cauchy integrals

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(t)dt}{t-z} \qquad 2\text{-}1$$

where L=L(a,b) is an interval on the real axis with two endpoints a and b, t any point of L, z any point of the complex z(x,y) plane.

If $\varphi(t)$ is R-integrable on L, bounded on L with the possible exception of the endpoints and if $\int_L |\varphi(t)|dt < \infty$ the Cauchy integral $\Phi(z)$ is a holomorphic function in the entire region excluding L, and for large $|z|$

$$\Phi(z) = \mathcal{O}\left(\frac{1}{|z|}\right).$$

On the interval L we have the two limits

$$\begin{cases} \Phi^+(x) = \lim \Phi(x+yi) & y \downarrow 0, \\ \Phi^-(x) = \lim \Phi(x+yi) & y \uparrow 0. \end{cases}$$

The integral

$$\int_a^b \frac{\varphi(t)dt}{t-x},$$

where $a < x < b$ does not exist in the ordinary sense, but if $\varphi(t)$ satisfies a Hölder condition on L

$$|\varphi(t_2) - \varphi(t_1)| \leqq A\,|t_2-t_1|^\alpha, \qquad \alpha > 0$$

the following limit exists

$$\lim_{\varepsilon\to 0} \left\{ \int_a^{x-\varepsilon} \frac{\varphi(t)dt}{t-x} + \int_{x+\varepsilon}^b \frac{\varphi(t)dt}{t-x} \right\},$$

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1) Cf Muskhelishvili. Singular Integral Equations Ch.2.

and will be written as

$$\int_a^b \frac{\varphi(t)dt}{t-x} .$$

Plemelj [1] gave the following formulae which are fundamental

$$\Phi^+(x) - \Phi^-(x) = \varphi(x) , \qquad 2\text{-}2$$

$$\Phi^+(x) + \Phi^-(x) = \frac{1}{\pi i} \int_L \frac{\varphi(t)dt}{t-x} . \qquad 2\text{-}3$$

We shall now calculate some Cauchy integrals which will be needed later on.

Some elementary integrals are given first

$$\int_{-1}^{1} (1-t)^{\alpha} (1+t)^{\beta} dt = \frac{2^{\alpha+\beta+1}\, \Gamma(1+\alpha)\Gamma(1+\beta)}{(1+\alpha+\beta)} , \qquad 2\text{-}4$$

$$\int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\alpha} \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{\cos \alpha\pi} , \qquad |\mathrm{Re}\ \alpha| < \tfrac{1}{2}, \qquad 2\text{-}5$$

$$\int_{0}^{\pi} (\mathrm{tg}\ \tfrac{1}{2}t)^{2\alpha} dt = \frac{\pi}{\cos \alpha\pi} , \qquad |\mathrm{Re}\ \alpha| < \tfrac{1}{2}. \qquad 2\text{-}6$$

It can be shown easily that

$$\int_{-1}^{1} (1-t)^{-\frac{1}{2}+\alpha} (1+t)^{-\frac{1}{2}-\alpha} \frac{dt}{t-z} = - \frac{\pi}{\cos \alpha\pi} \left(\frac{z-1}{z+1}\right)^{\alpha} \frac{1}{\sqrt{z^2-1}}, \quad |\mathrm{Re}\,\alpha| < 1. \qquad 2\text{-}7$$

Proof.

Consider the integral $F(w) = \frac{1}{2\pi i} \int_C (w-1)^{-\frac{1}{2}+\alpha} (w+1)^{-\frac{1}{2}-\alpha} \frac{dw}{w-z}$ where C is a closed contour round the section $(-1,1)$, taken in positive sense, and such that z lies outside C. From the calculus of residues

$$F(w) = -(z-1)^{-\frac{1}{2}+\alpha} (z+1)^{-\frac{1}{2}-\alpha} ,$$

but on the other hand

$$F(w) = \frac{1}{2\pi} (e^{\alpha\pi i} + e^{-\alpha\pi i}) \int_{-1}^{1} (1-t)^{-\frac{1}{2}+\alpha} (1+t)^{-\frac{1}{2}-\alpha} \frac{dt}{t-z} .$$

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1) l.c. § 17.

If Plemelj's formula 2-3 is applied to 2-7 we obtain

$$\fint_{-1}^{1} (1-t)^{-\frac{1}{2}+\alpha}(1+t)^{-\frac{1}{2}-\alpha}\frac{dt}{t-x} = -\pi\,\mathrm{tg}\,\alpha\pi\left(\frac{1-x}{1+x}\right)^{\alpha}\frac{1}{\sqrt{1-x^2}}\,, \quad |\mathrm{Re}\,\alpha| < \tfrac{1}{2} \qquad 2\text{-}8$$

If in 2-7 and 2-8 we put $t \to -\cos t$ we obtain

$$\int_{0}^{\pi}(\mathrm{tg}\,\tfrac{1}{2}t)^{-2\alpha}\frac{dt}{\cos t-z} = -\frac{\pi}{\cos\alpha\pi}\left(\frac{z+1}{z-1}\right)^{\alpha}\frac{1}{\sqrt{z^2-1}}\,, \quad |\mathrm{Re}\,\alpha| < \tfrac{1}{2}, \qquad 2\text{-}9$$

$$\sin x \fint_{0}^{\pi}(\mathrm{tg}\,\tfrac{1}{2}t)^{-2\alpha}\frac{dt}{\cos t-\cos x} = \pi\,\mathrm{tg}\,\alpha\pi\,(\mathrm{tg}\,\tfrac{1}{2}x)^{-2\alpha}\,, \quad |\mathrm{Re}\,\alpha| < \tfrac{1}{2}. \qquad 2\text{-}10$$

If in 2-7 we put

$$\frac{1}{t-z} = \frac{1}{1-z} + \frac{1-t}{(1-z)(t-z)}$$

we obtain, replacing α by $\alpha-\frac{1}{2}$

$$\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{-\alpha}\frac{dt}{t-z} = \frac{\pi}{\sin\alpha\pi}\left\{\left(\frac{z-1}{z+1}\right)^{\alpha}-1\right\}, \quad |\mathrm{Re}\,\alpha| < 1 \qquad 2\text{-}11$$

and similarly

$$\fint_{-1}^{1}(1-t)^{\alpha}(1+t)^{-\alpha}\frac{dt}{t-x} = \frac{\pi}{\sin\alpha\pi}\left\{\cos\alpha\pi\left(\frac{1-x}{1+x}\right)^{\alpha}-1\right\}. \quad |\mathrm{Re}\,\alpha| < 1 \qquad 2\text{-}12$$

From the Cauchy integrals obtained above the following pair easily follows

$$\fint_{-1}^{1}\frac{\sqrt{1-t^2}}{t-x}\,dt = -\pi x\,, \qquad -1 < x < 1\,, \qquad 2\text{-}13$$

$$\fint_{-1}^{1}\frac{1}{t-x}\frac{dt}{\sqrt{1-t^2}} = 0\,, \qquad -1 < x < 1\,. \qquad 2\text{-}14$$

and also

$$\int_{-1}^{1}\frac{\sqrt{1-t^2}}{t-z}\,dt = -\pi\left(z-\sqrt{z^2-1}\right), \qquad 2\text{-}15$$

$$\int_{-1}^{1}\frac{1}{t-z}\frac{dt}{\sqrt{1-t^2}} = -\frac{\pi}{\sqrt{z^2-1}}\,. \qquad 2\text{-}16$$

A trigonometrical substitution gives

$$\fint_0^{\pi} \frac{\sin^2 t\, dt}{\cos t - \cos x} = -\pi \cos x , \qquad 0 < x < \pi , \qquad 2\text{-}17$$

$$\fint_0^{\pi} \frac{dt}{\cos t - \cos x} = 0 , \qquad 0 < x < \pi . \qquad 2\text{-}18$$

A repeated Cauchy integral may be inverted by means of the formula of Poincaré-Bertrand for the proof of which we refer to Muskhelishvili's book [1].

$$\fint_{-1}^{1} \frac{dt}{t-x} \fint_{-1}^{1} \frac{\varphi(t,u)du}{t-u} = \pi^2 \varphi(x,x) + \fint_{-1}^{1} du \fint_{-1}^{1} \frac{\varphi(t,u)dt}{(t-x)(t-u)}. \qquad 2\text{-}19$$

A trigonometrical substitution gives

$$\fint_0^{\pi} \frac{\sin^2 t\, dt}{\cos t - \cos x} \fint_0^{\pi} \frac{\psi(t,u)du}{\cos t - \cos u} = \pi^2 \psi(x,x) + \fint_0^{\pi} du \fint_0^{\pi} \frac{\sin^2 t\; \psi(t,u)dt}{(\cos t - \cos x)(\cos t - \cos u)}$$

2-20

If in 2-20 $\psi(t,u)$ does not depend on t the right-hand side may be simplified by means of 2-17. We obtain

$$\fint_0^{\pi} \frac{\sin^2 t\, dt}{\cos t - \cos x} \fint_0^{\pi} \frac{\psi(u)du}{\cos t - \cos u} = \pi^2 \psi(u) - \pi \int_0^{\pi} \psi(u)du. \qquad 2\text{-}21$$

§3 γ a positive real constant

We shall consider the convergent trigonometrical series

$$f(x) = \sum_1^{\infty} a_n(\sin nx + \gamma \cos nx), \qquad 0 < x < \pi , \qquad 3\text{-}1$$

or in equivalent notation

$$f(x) = \sum_1^{\infty} b_n' \sin(nx + \mu\pi) , \qquad 0 < x < \pi , \qquad 3\text{-}2$$

where

$$a_n = b_n \cos \mu\pi \quad \text{and} \quad \gamma = \operatorname{tg} \mu\pi \quad \text{with} \quad 0 < \mu < \tfrac{1}{2}.$$

We introduce the analytic function

$$\Phi_0(z) = \sum_1^{\infty} b_n e^{nzi} , \qquad z = x+yi , \qquad y > 0 . \qquad 3\text{-}3$$

This function satisfies the following conditions

$$x = 0,\pi \qquad \operatorname{Im} \Phi_0 = 0, \qquad 3\text{-}4$$

$$y \to \infty \qquad \Phi_0 \to 0. \qquad 3\text{-}5$$

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1) l.c. § 23.

The expansion 3-2 of f(x) gives

$$y=0 \qquad \operatorname{Im} e^{\mu\pi i}\Phi_0 = f(x)\ , \quad 0 < x < \pi \ . \tag{3-6}$$

In order that 3-2 converges for all x in $(0,\pi)$ we should also have

$$y=0 \qquad \left| \int_0^{\pi} \Phi_0(z)dz \right| < \infty . \tag{3-7}$$

We shall see that, given a function f(x) which satisfies a Hölder condition on (0,x), the function $\Phi(z)$ is uniquely determined by the conditions 3-4 3-5 3-6 and 3-7. Thus any function of the Hölder type may be represented by a trigonometric series 3-1 or 3-2 and the coefficients a_n or b_n may be found by expanding $\Phi(z)$ in powers of e^{zi}.

The halfstrip $0 < x < \pi$ $y > 0$ may be mapped upon the upper half-plane by means of the mapping function

$$w = -\cos z . \tag{3-8}$$

We note the following relations which easily follow from 3-8

$$e^{iz} = -w + \sqrt{w^2-1} ,$$

$$w-1 = -\tfrac{1}{2}\, e^{-iz}\, (1+e^{iz})^2 , \qquad w+1 = -\tfrac{1}{2}\, e^{-iz}\, (1-e^{iz})^2 . \tag{3-9}$$

If we write $w = u+vi$ the conditions 3-4 3-5 3-6 become

$$|u| > 1 \quad v=0 \qquad \operatorname{Im} \Phi = 0 \ , \tag{3-10}$$

$$w \to \infty \qquad \Phi \to 0 \ , \tag{3-11}$$

$$|u| < 1 \quad v=0 \qquad \operatorname{Im} e^{\mu\pi i}\Phi = \varphi(u) \ , \tag{3-12}$$

where $$\varphi(-\cos x) = f(x), \tag{3-13}$$

and $$\Phi(-\cos z) = \Phi_0(z) .$$

If $\varphi(u) = 0$ the solution of 3-10 and 3-12 is a linear combination of functions

$$X(w) = \left(\frac{w+1}{w-1}\right)^{\mu} (w+1)^{-\nu_1} (w-1)^{-\nu_2} , \tag{3-14}$$

where ν_1 and ν_2 are integers. The condition 3-11 gives $\nu_1 + \nu_2 \geq 1$. In the z-plane we have near z=0 and $z=\pi$

$$X(-\cos z) \sim C_1(1-e^{iz})^{2\mu-2\nu_1} \ , \quad z \sim 0 ,$$

$$X(-\cos z) \sim C_2(1+e^{iz})^{-2\mu-2\nu_2} \ , \quad z \sim \pi ,$$

where C_1 and C_2 are constant factors.
The condition of integrability 3-7 requires

$$\nu_1 < \mu + \tfrac{1}{2} \quad \text{and} \quad \nu_2 < -\mu + \tfrac{1}{2} ,$$

so that $\nu_1 + \nu_2 < 1$ in contradiction to the result found above.
This shows that, if $f(x) \equiv 0$, the problem considered above has only the zero solution $\Phi = 0$.
This also implies that it is impossible to represent zero by a convergent trigonometric series 3-1. Of course zero may be represented by a trigonometric series 3-1 which is divergent but Abel-summable. Every function $X(w)$ from 3-14 with $\nu_1 + \nu_2 \geq 1$ defines an A-summable trigonometric series.

A particular solution of 3-10, 3-11, 3-12 may be obtained by a Cauchy integral

$$\Phi(w) = \left(\frac{w+1}{w-1}\right)^{\mu} \frac{1}{\pi} \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)}{t-w}\, dt \,, \qquad 3\text{-}15$$

for we have

$$e^{\mu\pi i}\, \Phi = 2i \left(\frac{1+u}{1-u}\right)^{\mu} \left\{ \frac{1}{2\pi i} \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)}{t-w}\, dt \right\}_{w=u+oi} ,$$

and by Plemelj's formulae 2-2 and 2-3

$$\mathrm{Re} \left\{ \frac{1}{2\pi i} \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)}{t-w}\, dt \right\}_{w=u+oi} = \frac{1}{2} \left(\frac{1-u}{1+u}\right)^{\mu} \varphi(u).$$

Since the problem with $\varphi(u) \equiv 0$ has the zero solution only, the solution 3-15 is unique. We shall show that near $w = \pm 1$

$$\Phi(w) \backsim \text{constant}, \quad w \backsim -1 \,,$$

$$\Phi(w) \backsim (w-1)^{-\mu}, \quad w \backsim 1 \,,$$

so that the condition 3-7 also is satisfied.
The behaviour of $\Phi(w)$ near $w=1$ may be determined as follows.
The identity

$$\frac{1-t}{t-w}\, \varphi(t) = \frac{1-w}{t-w}\, \varphi(w) - \varphi(t) + \frac{1-w}{t-w} \left\{ \varphi(t) - \varphi(w) \right\} , \qquad 3\text{-}16$$

is substituted into 3-15. We obtain

$$\Phi(w) = - \left(\frac{w+1}{w-1}\right)^{\mu} \frac{1}{\pi} \int_{-1}^{1} (1-t)^{-1+\mu} (1+t)^{-\mu} \varphi(t)\, dt \; +$$

$$+ (w-1)^{1-\mu} (w+1)^{\mu} \varphi(w) \frac{1}{\pi} \int_{-1}^{1} (1-t)^{-1+\mu} (1+t)^{-\mu} \frac{dt}{t-w} \; +$$

$$- (w-1)^{1-\mu} (w+1)^{\mu} \frac{1}{\pi} \int_{-1}^{1} (1-t)^{-1+\mu} (1+t)^{-\mu} \frac{\varphi(t) - \varphi(w)}{t-w}\, dt \,.$$

The second term on the right-hand side may be evaluated by means of 2-7. If $f(x)$ at $x=0$ and $x=\pi$ satisfies a Lipschitz condition of order ε ($\varepsilon>0$) $\varphi(t)$ satisfies a Lipschitz condition of order $\frac{1}{2}\varepsilon$ at the endpoints ± 1. Thus the third term is $\mathcal{O}(|w-1|^{\alpha})$ where $\alpha = \mathrm{Min}\,(1-\mu, \frac{1}{2}\varepsilon)$, [1] and we may write

$$w \backsim 1, \quad \Phi(w) = -A\left(\frac{w+1}{w-1}\right)^{\mu} + \frac{\varphi(w)}{\sin\mu\pi} + \mathcal{O}(|w-1|^{\alpha}), \qquad 3\text{-}17$$

where

$$A = \frac{1}{\pi}\int_{-1}^{1}(1-t)^{-1+\mu}\,(1+t)^{-\mu}\varphi(t)\,dt\,. \qquad 3\text{-}18$$

A similar result may be obtained at $w=-1$. Starting from the identity

$$\frac{\varphi(t)}{t-w} = \frac{\varphi(-1)}{t-w} + \frac{\varphi(t)-\varphi(-1)}{t+1} + \frac{w+1}{t-w}\;\frac{\varphi(t)-\varphi(-1)}{t+1} \qquad 3\text{-}19$$

we obtain in view of 2-11

$$w \backsim -1, \quad \Phi(w) = \frac{\varphi(-1)}{\sin\mu\pi} + B\left(\frac{w+1}{w-1}\right)^{\mu} + \mathcal{O}(|w+1|^{\beta}), \qquad 3\text{-}20$$

where

$$B = \frac{1}{\pi}\int_{-1}^{1}\left(\frac{1-t}{1+t}\right)^{\mu}\frac{\varphi(t)-\varphi(-1)}{t+1}\,dt - \frac{\varphi(-1)}{\sin\mu\pi} \qquad 3\text{-}21$$

and

$$\beta = \mathrm{Min}\,(1+\mu, \tfrac{1}{2}\varepsilon).$$

If $\varphi(t)$ is an analytic function or if, at least, $\varphi(t)$ is differentiable a sufficient number of times, the formulae 3-17 and 3-20 may be continued into an expansion of $\Phi(w)$ according to ascending powers of $w \mp 1$ near ± 1.

We need the following lemma.

<u>Lemma 3</u> If for every integer $k \geqslant 0$ $\psi_k(t)$ is defined by

$$t\,\psi_{k+1} \stackrel{\mathrm{def}}{=} \psi_k - \frac{\psi^{(k)}(0)}{k!}, \qquad k \geqslant 1, \qquad 3\text{-}22$$

$$\psi_0 \stackrel{\mathrm{def}}{=} \psi(t),$$

then we have the following identity

$$\frac{\psi(t)-\psi(w)}{t-w} = \sum_{0}^{N-1} w^k \psi_{k+1}(t) + \frac{w^N}{t-w}\left\{\psi_N(t) - \psi_N(w)\right\}. \qquad 3\text{-}23$$

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1) Cf Muskhelishvili l.c. § 29.

Proof

Either by means of induction from N to N+1 or as follows.
From 3-22 we get

$$t^k \psi_k = \psi - \sum_0^{k-1} c_j t^j , \qquad k \geqslant 1 ,$$

where $c_j = \frac{\psi^{(j)}(0)}{j!}$. Then we have the following reduction.

$$\sum_0^{N-1} w^k \psi_{k+1}(t) = \psi(t) \sum_0^{N-1} \frac{w^k}{t^{k+1}} - \sum_0^{N-1} \frac{w^k}{t^{k+1}} \sum_0^k c_j t^j =$$

$$= (1 - \frac{w^N}{t^N}) \frac{\psi(t)}{t-w} - \sum_0^{N-1} c_j t^j \sum_j^{N-1} \frac{w^k}{t^{k+1}} =$$

$$= (1 - \frac{w^N}{t^N}) \frac{\psi(t)}{t-w} - \frac{1}{t-w} \sum_0^{N-1} c_j w^j + \frac{1}{t-w} \frac{w^N}{t^N} \sum_0^{N-1} c_j t^j =$$

$$= \frac{\psi(t) - \psi(w)}{t-w} - w^N \frac{\psi_N(t) - \psi_N(w)}{t-w} .$$

The identity 3-23 may lead to the formal expansion

$$\frac{\psi(t) - \psi(w)}{t-w} = \sum_0^{\infty} w^k \psi_{k+1}(t). \qquad 3\text{-}24$$

The functions $t^k \psi_k(t)$ are the remainders of the Taylor expansion of $\psi(t)$ at $t=0$. A similar expansion may be found for the remainders of the Taylor expansion of $\psi(t)$ at an arbitrary point t_o if upon 3-24 or 3-23 a linar transformation $t \rightarrow t-t_o$ is applied.
If $t^k \varphi_k(t)$ and $t^k \varphi_k^*(t)$ are the remainders of the Taylor expansion of $\varphi(t)$ at $t=1$ and $t=-1$ respectively we have from 3-24

$$\frac{\varphi(t) - \varphi(w)}{t-w} = \sum_0^{\infty} (w-1)^k \varphi_{k+1}(t) = \sum_0^{\infty} (w+1)^k \varphi_{k+1}^*(t) . \qquad 3\text{-}25$$

The behaviour of $\Phi(w)$ at $w=1$ can be determined as follows.
From 3-25 we may derive

$$\frac{1-t}{t-w} \varphi(t) = - \frac{1-w}{t-w} \varphi(w) - \sum_0^{\infty} (w-1)^k \varphi_k(t) ,$$

so that

$$\Phi(w) = \frac{\varphi(w)}{\sin \mu\pi} - (\frac{w+1}{w-1})^{\mu} \sum_0^{\infty} A_k (w-1)^k , \qquad w \rightarrow 1 , \qquad 3\text{-}26$$

where

$$A_k = \frac{1}{\pi} \int_{-1}^{1} (1-t)^{-1+\mu} (1+t)^{-\mu} \varphi_k(t) \, dt . \qquad 3\text{-}27$$

Similarly the expansion

$$\frac{\varphi(t)}{t-w} = \frac{\varphi(w)}{t-w} + \sum_{0}^{\infty} (w+1)^k \varphi^*_{k+1}(t) ,$$

from 3-25 gives

$$\Phi(w) = \frac{\varphi(w)}{\sin \mu\pi} \left\{ 1 - \left(\frac{w+1}{w-1}\right)^{\mu} \right\} + \left(\frac{w+1}{w-1}\right)^{\mu} \sum_{0}^{\infty} B_k (w+1)^k, \quad w \to -1 \qquad 3\text{-}28$$

where

$$B_k = \frac{1}{\pi} \int_{-1}^{1} (1-t)^{\mu} (1+t)^{-\mu} \varphi^*_{k+1}(t) \, dt. \qquad 3\text{-}29$$

§4 Special cases

The coefficient of s^n in the expansion of $\left(\frac{1+s}{1-s}\right)^{\alpha}$ will be denoted by $e_n(\alpha)$.

$$\left(\frac{1+s}{1-s}\right)^{\alpha} = \sum_{0}^{\infty} e_n(\alpha) \, s^n , \qquad |s| < 1 . \qquad 4\text{-}1$$

We note the following elementary properties

$$e_n(-\alpha) = (-1)^n e_n(\alpha) , \qquad 4\text{-}2$$

$$n \, e_n(\alpha) = 2\alpha \, e_{n-1}(\alpha) + (n-2) \, e_{n-2}(\alpha) , \qquad 4\text{-}3$$

$$e_0 = 1 , \quad e_1 = 2 , \quad e_2 = 2\alpha^2 , \quad e_3 = \frac{4\alpha^3 + 2\alpha}{3} \ \ldots ,$$

$$e_n(\alpha+1) = e_n(\alpha) + 2 \sum_{0}^{n-1} e_k(\alpha). \qquad 4\text{-}4$$

If in 4-1 $s = e^{ix}$ is substituted we obtain by separating real and imaginary part

$$\cos \frac{\alpha\pi}{2} \ (\mathrm{tg}\ \tfrac{1}{2}x)^{-\alpha} = \sum_{0}^{\infty} e_n(\alpha) \cos nx , \qquad |\mathrm{Re}\ \alpha| < 1. \qquad 4\text{-}5$$

and

$$\sin \frac{\alpha\pi}{2} \ (\mathrm{tg}\ \tfrac{1}{2}x)^{-\alpha} = \sum_{1}^{\infty} e_n(\alpha) \sin nx , \qquad |\mathrm{Re}\ \alpha| < 1. \qquad 4\text{-}6$$

From 4-5 and 4-6 we obtain for $|\mathrm{Re}\ \alpha| < 1$

$$\int_{0}^{\pi} (\mathrm{tg}\ \tfrac{1}{2}x)^{-\alpha} \sin nx \, dx = \frac{\pi}{2} \frac{e_n(\alpha)}{\sin \frac{\alpha\pi}{2}} , \qquad n > 0 , \qquad 4\text{-}7$$

$$\int_0^{\pi} (\mathrm{tg}\ \tfrac{1}{2}x)^{-\alpha} \cos nx\, dx = \frac{\pi}{2} \frac{e_n(\alpha)}{\cos \frac{\alpha\pi}{2}}\ , \qquad n > 0\ , \tag{4-8}$$

Also we have from 4-5 and 4-6

$$1 = - \sum_1^{\infty} e_n(\alpha)\ (\cos nx - \mathrm{cotg}\ \tfrac{\alpha\pi}{2} \sin nx)\ , \tag{4-9}$$

and from 4-7 and 4-8

$$\int_0^{\pi} (\mathrm{tg}\ \tfrac{1}{2}x)^{-\alpha} (\sin nx - \mathrm{cotg}\ \tfrac{\alpha\pi}{2} \cos nx)\, dx = 0 \tag{4-10}$$

for all $n \geq 1$.

If α is real and positive the coefficients $e_n(\alpha)$ are positive. We shall prove that

$$e_n(\alpha) \leq \frac{2\alpha}{n^{1-\alpha}}\ . \qquad n \geq 1\ , \quad 0 < \alpha \leq 1. \tag{4-11}$$

The assertion is clearly true for n=1,2. In order to apply the principle of induction we have to prove the inequality

$$\frac{2\alpha}{(n-1)^{1-\alpha}} + \frac{1}{(n-2)^{-\alpha}} \leq n^{\alpha} \qquad\qquad n \geq 3$$

Consider the function $V(x) = (1-2x)^{\alpha} + 2\alpha x\ (1-x)^{\alpha-1}$ for $0 < x < 1/3$. Its Taylor expansion is

$$V(x) = 1 - \binom{\alpha}{3}\ (2^3-6)\ x^3 + \binom{\alpha}{4}\ (2^4-8)\ x^4 .....$$

i.e. 1 followed by negative terms. From this we get $V(x) \leq 1$ or $V(\frac{1}{n}) \leq 1$ so that the principle of induction holds.

Next we shall derive an asymptotic estimation for $e_n(\alpha)$ in the case $0 \leq \mathrm{Re}\ \alpha < 1$.

Starting from

$$e_n(\alpha) = \frac{1}{2\pi i} \oint \left(\frac{1+s}{1-s}\right)^{\alpha} \frac{ds}{s^{n+1}}, \tag{4-12}$$

we have by a suitable deformation of the contour

$$\frac{\pi}{\sin \alpha\pi}\ e_n(\alpha) = \int_1^{\infty} \left(\frac{t+1}{t-1}\right)^{\alpha} \frac{dt}{t^{n+1}} - (-1)^n \int_1^{\infty} \left(\frac{t-1}{t+1}\right)^{\alpha} \frac{dt}{t^{n+1}}\ . \tag{4-13}$$

In the first integral of the right-hand side we substitute $t=e^u$

$$\int_1^{\infty} \left(\frac{t+1}{t-1}\right)^{\alpha} \frac{dt}{t^{n+1}} = \int_0^{\infty} e^{-nu} \left(\frac{e^u+1}{e^u-1}\right)^{\alpha} du\ .$$

Since $$\left(\frac{e^u+1}{e^u-1}\right)^{\alpha} = \left(\frac{2}{u}\right)^{\alpha} \left\{1 + \mathcal{O}(u^2)\right\}$$

the last integral equals $\frac{2^{\alpha}\Gamma(1-\alpha)}{n^{1-\alpha}} + \mathcal{O}(n^{-3+\alpha})$.

In a similar way the second integral of 4-13 is $\frac{2^{-\alpha}\Gamma(1+\alpha)}{n^{1+\alpha}} + \mathcal{O}(n^{-3-\alpha})$.

Thus we have the following asymptotic expression

$$e_n(\alpha) = \frac{2^{\alpha}}{\Gamma(\alpha)\, n^{1-\alpha}} + \frac{(-1)^n\, 2^{-\alpha}}{\Gamma(-\alpha)\, n^{1+\alpha}} + \mathcal{O}(n^{-3+\alpha}) , \quad 0 \leq \mathrm{Re}\,\alpha < 1.$$

We note the following expression for $e_n(\alpha)$

$$e_n(\alpha) = (-1)^{n-1}\, 2\alpha\ F(1-n,\ 1+\alpha,\ 2,2) ,\ n \geq 1, \quad -1 < \mathrm{Re}\,\alpha < 1. \tag{4-14}$$

which may be obtained from 4-12 as follows

$$\frac{1}{2\pi i}\oint \left(\frac{1+s}{1-s}\right)^{\alpha}\frac{ds}{s^{n+1}} = \frac{\sin\alpha\pi}{\pi}\int_{-1}^{1}(1+t)^{\alpha}(1-t)^{-\alpha}t^{n-1}\,dt =$$

$$= \frac{(-1)^{n-1}\ 2\sin\alpha\pi}{\pi}\int_0^1 t^{\alpha}(1-t)^{-\alpha}(1-2t)^{n-1}\,dt.$$

We return now to the original expansion

$$f(x) = \sum_1^{\infty} a_n(\sin nx + \gamma\cos nx) , \tag{4-15}$$

or

$$f(x) = \sum_1 b_n \sin(nx + \mu\pi) .$$

If we take $f(x) \equiv 1$, whence by 3-13 $\varphi(u)=1$, 3-15 gives by 2-11

$$\sin\mu\pi\ \Phi(w) = 1 - \left(\frac{w+1}{w-1}\right)^{\mu} ,$$

or (cf 3-13)

$$\sin\mu\pi\ \Phi_o(z) = 1 - \left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{2\mu},$$

from which the coefficients a_n or b_n follow easily. We obtain

$$1 = \sum_1^{\infty}(-1)^{n-1}e_n(2\mu)\left(\cos nx + \frac{1}{\gamma}\sin nx\right) , \tag{4-16}$$

which is identical with 4-9 if $\alpha = -2\mu$.

If we take $f(x) \equiv (\mathrm{tg}\,\frac{1}{2}x)^{2\sigma}$, 3-15 gives

$$\sin(\mu-\sigma)\pi\ \Phi(w) = \left(\frac{w+1}{w-1}\right)^{\sigma} - \left(\frac{w+1}{w-1}\right)^{\mu}$$

$$= \left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{2\sigma} - \left(\frac{1-e^{iz}}{1+e^{iz}}\right)^{2\mu},$$

again with $w = -\cos z$, so that the coefficients of

$$(\mathrm{tg}\ \tfrac{1}{2}x)^{2\sigma} = \sum_{1}^{\infty} b_n(\sigma) \sin(nx + \mu\pi) \qquad 4\text{-}17$$

become

$$b_n(\sigma) = \frac{e_n(-2\sigma) - e_n(-2\mu)}{\sin(\mu - \sigma)\pi} . \qquad 4\text{-}18$$

The particular case $\sigma = \mu$ gives

$$b_n(\mu) = \frac{-1}{\mu \cos \mu\pi} \frac{d}{d\mu} e_n(-2\mu) , \qquad 4\text{-}19$$

with the asymptotic estimate

$$b_n(\mu) \backsim \frac{(-1)^{n-1}\, 2^{2\mu}}{\pi\, \Gamma(2\mu - 1)} \frac{\ln n}{n^{1-2\mu}} . \qquad 4\text{-}20$$

§5 <u>The expansion of f(x)</u>

The coefficients of the expansion of a function f(x) which satisfies a Hölder condition on $(0,\pi)$ are determined by the expansion of $\varphi_0(z)$ from 3-3 in powers of e^{zi}.

If again

$$f(x) = \sum_{1}^{\infty} b_n \sin(nx + \mu\pi), \qquad 5\text{-}1$$

we have

$$b_n = \frac{1}{2\pi i} \oint \frac{\Phi_1(s)}{s^{n+1}}\, ds , \qquad 5\text{-}2$$

where

$$\Phi_1(e^{zi}) = \varphi_0(z).$$

We know that $\Phi(w)$ from 3-15 is holomorphic in the entire w-region except at $w = \pm 1$. The asymptotic behaviour of b_n for large n is determined by the singularities of $\Phi_1(s)$ at $s = \pm 1$ which correspond to those of $\Phi(w)$ at $w = \pm 1$ by means of the conformal transformation

$$w = -\tfrac{1}{2}\left(s + \frac{1}{s}\right) , \quad s = -w + \sqrt{w^2 - 1}. \qquad 5\text{-}3$$

The transformation 5-3 maps the w-plane upon the interior of the unit circle in the s-plane. Circles $|s| =$ constant correspond to ellipses with foci ± 1 in the w-plane. The section $(-1,1)$ in the w-plane corresponds to the unit-circle in the s-plane.

The behaviour at the singularities $s = \pm 1$ follows from 3-17 and 3-20. If f(x) at x=0 and $x = \pi$ satisfies a Hölder condition of order ε $(\varepsilon > 0)$ we have

$$s \to 1 \qquad \Phi_1(s) = \frac{f(0)}{\sin \mu\pi} + B \left(\frac{1-s}{1+s}\right)^{2\mu} + \mathcal{O}(|1-s|^{2\beta}) , \qquad 5\text{-}4$$

$$s \to -1 \qquad \Phi_2(s) = -A\left(\frac{1-s}{1+s}\right)^{2\mu} + \frac{f(\pi)}{\sin\mu\pi} + \mathcal{O}\left(|1+s|^{2\alpha}\right), \qquad 5\text{-}5$$

where A and B are given by 3-18 and 3-21, or

$$A = \frac{1}{\pi}\int_0^\pi (\mathrm{tg}\,\tfrac{1}{2}x)^{1-2\mu}\; f(x)\,dx, \qquad 5\text{-}6$$

$$B = \frac{1}{\pi}\int_0^\pi (\mathrm{tg}\,\tfrac{1}{2}x)^{-1-2\mu}\;\left\{f(x) - f(0)\right\} dx - \frac{f(0)}{\sin\mu\pi}, \qquad 5\text{-}7$$

and

$$\alpha = \mathrm{Min}\,(1-\mu,\ \tfrac{1}{2}\varepsilon),$$

$$\beta = \mathrm{Min}\,(1+\mu,\ \tfrac{1}{2}\varepsilon).$$

By a similar procedure as in the previous section we obtain for b_n the following asymptotic estimate

$$b_n = \frac{(-1)^{n+1}\,2^{2\mu}}{\Gamma(2\mu)}\,\frac{A}{n^{1-2\mu}} + \frac{2^{-2\mu}}{\Gamma(-2\mu)}\,\frac{B}{n^{1+2\mu}} + \mathcal{O}\left(n^{-1-2\alpha}\right). \qquad 5\text{-}8$$

The second term in the right-hand side of 5-8 is significant only if $f(x)$ is hölderian at $x=\pi$ of an order which exceeds 2μ. This is true e.g. if $f(x)$ is differentiable at $x=\pi$.

§6 γ is a complex constant

We shall now consider the expansion

$$f(x) = \sum_1^\infty a_n\,(\sin nx + \gamma\cos nx), \qquad 0 \leq x \leq \pi, \qquad 6\text{-}1$$

where γ is complex.
We put

$$\gamma = \mathrm{tg}\,\mu\pi, \qquad \mu = \frac{1}{2\pi i}\ln\frac{1+\gamma i}{1-\gamma i}, \qquad 6\text{-}2$$

and we shall consider the strip $-\frac{1}{2} \leq \mathrm{Re}\,\mu \leq \frac{1}{2}$.
The right-hand half-plane of γ corresponds to $0 < \mathrm{Re}\,\mu < \frac{1}{2}$.
The unit-circle $|\gamma|=1$ with the exception of $\gamma = \pm i$ corresponds to $\mathrm{Re}\,\mu = \pm\frac{1}{4}$.
The imaginary axis of γ with $|\mathrm{Im}\,\gamma| > 1$ corresponds to $\mathrm{Re}\,\mu = \pm\frac{1}{2}$.
The imaginary axis of γ with $|\mathrm{Im}\,\gamma| < 1$ corresponds to $\mathrm{Re}\,\mu = 0$.

The results obtained in the preceding sections for γ real and positive may be extended easily to the case of complex γ. Since $\Phi(w)$ from 3-15 is an analytic function of μ the principle of analytic continuation may be applied throughout.

We shall now consider the main result, the asymptotic estimation of a_n (or b_n) from formula 5-8. The coefficients A and B are given by 3-18 and 3-21. The expression for B is valid for the entire region $|\mathrm{Re}\,\mu| \leq \frac{1}{2}$ whereas the expression for A is only valid for $0 < \mathrm{Re}\,\mu \leq \frac{1}{2}$. However, an analytic continuation may easily be obtained.

We have for $|\mathrm{Re}\,\mu| \leq \frac{1}{2}$

$$A = -\frac{1}{\pi} \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)-\varphi(1)}{t-1}\, dt + \frac{\varphi(1)}{\sin \mu\pi} ,$$

$$B = \frac{1}{\pi} \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\varphi(t)-\varphi(-1)}{t+1}\, dt - \frac{\varphi(-1)}{\sin \mu\pi} ,$$

or

$$A = \frac{1}{\pi} \int_{0}^{\pi} (\mathrm{tg}\, \tfrac{1}{2}x)^{1-2\mu} \left\{ f(x)-f(\pi) \right\} dx + \frac{f(\pi)}{\sin \mu\pi} , \qquad 6\text{-}3$$

$$B = \frac{1}{\pi} \int_{0}^{\pi} (\mathrm{tg}\, \tfrac{1}{2}x)^{-1-2\mu} \left\{ f(x)-f(0) \right\} dx - \frac{f(0)}{\sin \mu\pi} . \qquad 6\text{-}4$$

For simplicity the following conclusions will be founded upon the assumption that $f(x)$ is differentiable in $0 \leq x \leq \pi$.

<u>a</u> $0 < \mathrm{Re}\,\mu < \frac{1}{2}$.

The leading term of the asymptotic expansion of a_n is proportional to $A\, n^{-1+2\mu}$ so that the following conclusion may be drawn:

The trigonometrical expansion 6-1 is possible and

$$a_n = \mathcal{O}\,(n^{-1+2\mathrm{Re}\,\mu}) .$$

The condition $\sum |a_n| < \infty$ or $\sum n\, |a_n|^2 < \infty$ requires $A = 0$ or

$$\int_{0}^{\pi} (\mathrm{tg}\, \tfrac{1}{2}x)^{1-2\mu} f(x)\, dx = 0, \qquad 6\text{-}5$$

and if $A=0$ we have

$$a_n = \mathcal{O}\,(n^{-1-2\mathrm{Re}\,\mu}) .$$

<u>b</u> $-\frac{1}{2} < \mathrm{Re}\,\mu < 0$.

This case may be reduced to the previous one by means of the substitution $x \rightarrow \pi - x$, $\mu \rightarrow -\mu$. The condition 6-5 becomes

$$\int_{0}^{\pi} (\mathrm{tg}\, \tfrac{1}{2}x)^{-1-2\mu} f(x)\, dx = 0. \qquad 6\text{-}6$$

c $\mathrm{Re}\,\mu=0$.

In this case corresponding to a pure imaginary γ with $|\gamma|<1$ the leading term of the asymptotic expansion contains both An^{-1} and Bn^{-1}.

The expansion 6-1 is possible and

$$a_n = \mathcal{O}(n^{-1}) .$$

The condition $\sum |a_n| < \infty$ or $\sum n|a_n|^2 < \infty$ requires both A=0 and B=0, thus from 6-3 and 6-4

$$\begin{cases} \frac{1}{\pi}\int_0^{\pi} (\mathrm{tg}\,\tfrac{1}{2}x)^{1-2\mu} \{f(x)-f(\pi)\} + \frac{f(\pi)}{\sin\mu\pi} = 0, \\ \frac{1}{\pi}\int_0^{\pi} (\mathrm{tg}\,\tfrac{1}{2}x)^{-1-2\mu} \{f(x)-f(0)\} - \frac{f(0)}{\sin\mu\pi} = 0 . \end{cases} \qquad 6\text{-}7$$

Thus, if and only if the two conditions 6-7 are satisfied we have

$$a_n = \mathcal{O}(n^{-2}) .$$

If in 6-1 we take $f(x) = \sin nx + \gamma \cos nx$ one of the conditions 6-5, 6-6, 6-7 should be satisfied and thus another proof is obtained for the following relation of orthogonality.

$$\int_0^{\pi} (\mathrm{tg}\,\tfrac{1}{2}x)^{1-2\mu} (\sin nx + \gamma\cos nx)\, dx=0,\ n\geq 1,\quad \mathrm{Re}\,\mu > 0, \qquad 6\text{-}8$$

$$\int_0^{\pi} (\mathrm{tg}\,\tfrac{1}{2}x)^{-1-2\mu} (\sin nx + \gamma\cos nx)\, dx=0,\ n\geq 1,\quad \mathrm{Re}\,\mu < 0; \qquad 6\text{-}9$$

$$\begin{cases} \int_0^{\pi} (\mathrm{tg}\,\tfrac{1}{2}x)^{1-2\mu} \{\sin(nx+\mu\pi)-\sin(n\pi+\mu\pi)\}\, dx= (-1)^{n-1}\pi , \\ \qquad n\geq 1,\ \mathrm{Re}\,\mu=0 \\ \int_0^{\pi} (\mathrm{tg}\,\tfrac{1}{2}x)^{-1-2\mu} \{\sin(nx+\mu\pi)- \sin\mu\pi\}\, dx = \pi . \\ \qquad n\geq 1,\ \mathrm{Re}\,\mu=0 \end{cases} \qquad 6\text{-}10$$

d $\mathrm{Re}\,\mu=\frac{1}{2}$.

In this case, and only here, a convergent expansion 6-1 is not possible unless A=0.

If $A\neq 0$ we have $a_n=\mathcal{O}(1)$ and $\lim_{n\to\infty} a_n$ does not exist.

If A=0 we have $a_n=\mathcal{O}(n^{-2})$.

However, in this case the relation of orthogonality 6-8 remains true.

Taking up the general case $0<\mathrm{Re}\,\mu<\frac{1}{2}$ we consider an expansion which also includes a constant term

$$f(x) = a_o' + \sum_1^\infty a_n' (\sin nx + \gamma_n \cos nx) \qquad 6\text{-}11$$

If a convergent expansion 6-1 without constant term exists the convergency may be improved by adding an appropriate constant. If a_o' is chosen in such a way that for the function $f(x)-a_o'$ the condition 6-5 is fulfilled we have

$$a_o' = A \sin \mu\pi \qquad a_n' = \mathcal{O}(n^{-1-2\mathrm{Re}\mu})$$

and
$$a_n' = a_n + (-1)^n \cos \mu\pi \, e_n(2\mu) A$$

If $\mathrm{Re}\,\mu = \frac{1}{2}$ the expansion 6-1 is divergent but 6-11 converges. If $\mathrm{Re}\,\mu = 0$ nothing is gained since both a_n and a_n' are $\mathcal{O}(n^{-1})$.

§7 γ_n is variable

We consider the general problem of the following expansion

$$f(x) = \sum_1^\infty a_n(\sin nx + \gamma_n \cos nx), \quad 0 < x < \pi, \qquad 7\text{-}1$$

where $f(x)$ satisfies a Hölder condition in the interval $0 \leq x \leq \pi$, and where γ_n is a function of n of the following type

$$\gamma_n = \gamma + r_n, \quad r_n = \mathcal{O}(n^{-2}), \quad \mathrm{Re}\,\gamma > 0. \qquad 7\text{-}2$$

If we introduce the auxiliary functions

$$s(x) = \sum_1^\infty a_n \sin nx, \qquad 7\text{-}3$$

$$c(x) = \sum_1^\infty a_n \gamma_n \cos nx, \qquad 7\text{-}4$$

we have

$$f(x) = s(x) + c(x), \qquad 7\text{-}5$$

and for a_n we have

$$\frac{\pi}{2} a_n = \int_0^\pi s(x) \sin nx \, dx = \frac{1}{\gamma_n} \int_0^\pi c(x) \cos nx \, dx. \qquad 7\text{-}6$$

Thus, for all $n \geq 1$ we have the relations

$$\gamma_n \int_0^\pi s(x) \sin nx \, dx - \int_0^\pi c(x) \cos nx \, dx = 0. \qquad 7\text{-}7$$

The infinite number of relations 7-7 can be shown to be equivalent to a single integral equation of the Cauchy type. We need a few lemmas.

lemma 7-1 $$\frac{1}{2} + \sum_{1}^{N} \cos nt = \frac{\sin (N+\frac{1}{2})t}{2 \sin \frac{1}{2}t} .$$

$$\sum_{1}^{N} \sin nt = \frac{\cos \frac{1}{2}t - \cos(N+\frac{1}{2})t}{2 \sin \frac{1}{2} t} .$$

Proof Take real and imaginary part of $\sum_{1}^{N} e^{int}$.

lemma 7-2 $$\lim_{N\to\infty} \int_0^{\pi} \varphi(t) \sum_{1}^{N} \cos n(t-x)dt = \begin{cases} \pi\varphi(x)-\frac{1}{2}\int_0^{\pi}\varphi(t)dt , & \text{if } 0<x<\pi , \\ -\frac{1}{2}\int_0^{\pi}\varphi(t)\, dt , & \text{if } -\pi<x<0. \end{cases}$$

Proof from Riemann's theorem for a Fourier series.

lemma 7-3 $$\lim_{N\to\infty} \int_0^{\pi} \varphi(t) \sum_{1}^{N} \sin n(t-x)\, dt = \frac{1}{2} ⨍_0^{\pi} \operatorname{cotg} \tfrac{1}{2}(t-x)\varphi(t)\, dt,$$

where $⨍$ represents a Cauchy integral if $0<x<\pi$.

Proof. If $-\pi<x<0$ the relation follows from Riemann's theorem. The Cauchy integral reduces in this case to a Riemann integral.

If $0<x<\pi$ consider $\lim_{N\to\infty} \int_0^{x-\varepsilon} + \int_{x+\varepsilon}^{\pi}$.

If the relations 7-7 are multiplied either by sin nx or cos nx then summation yields an integral equation.

Taking the sine multiplication first we obtain from 7-7

$$\int_0^{\pi} s(t) \sum_{1}^{N} \gamma_n \{ \cos n(t-x) - \cos n(t+x) \} dt +$$

$$+ \int_0^{\pi} c(t) \sum_{1}^{N} \{ \sin n(t-x) - \sin n(t+x) \} dt = 0.$$

For $N\to\infty$ we find by means of lemma 7-2 and 7-3

$$\gamma s(x) - \frac{1}{\pi} ⨍_0^{\pi} \frac{\sin x\, c(t)dt}{\cos t-\cos x} + \frac{1}{\pi} \int_0^{\pi} K_1(t,x)\, s(t)\, dt = 0 , \qquad 7\text{-}8$$

where

$$K_1(t,x) = 2 \sum_{1}^{\infty} r_n \sin nt \sin nx . \qquad 7\text{-}9$$

If the cosine multiplication is taken we obtain in a similar way

$$c(x) + \frac{\gamma}{\pi} ⨍_0^{\pi} \frac{\sin t\, s(t)\, dt}{\cos t-\cos x} - \frac{1}{\pi} \int_0^{\pi} K_2(t,x)\, s(t)\, dt = 0, \qquad 7\text{-}10$$

where
$$K_2\,(t,x) = 2\sum_1^{\infty} r_n \sin nt \cos nx. \qquad 7\text{-}11$$

Since K_1 and K_2 are of the form

$$K_1 = \sum_1^{\infty} \alpha_n \sin nx\ , \qquad K_2 = \sum_1^{\infty} \alpha_n \cos nx\ ,$$

$K_2(x)$ is the series conjugate to $K_1(x)$, and hence

$$K_2(x) = \frac{1}{2\pi}\int_0^{\pi} \operatorname{cotg}\frac{u}{2}\left\{ K_1(x+u) - K_1(x-u)\right\} du.$$

Since $K_1(x)$ is odd this expression may be written in the form

$$K_2(x) = -\frac{1}{\pi}\int_0^{\pi} \frac{\sin u\, K_1(u)\, du}{\cos u - \cos x}\ . \qquad 7\text{-}12$$

The integral equations 7-8 and 7-10 are not entirely equivalent since the condition

$$\int_0^{\pi} c(x)\, dx = 0\ , \qquad 7\text{-}13$$

expressing the absence of a constant term in the expansion 7-1 is used in the derivation of 7-10 but not in that of 7-8.
Conversely 7-13 follows from 7-10, for, if we integrate from 0 to π the second term vanishes in view of 2-18 and the third term vanishes in view of $\int_0^{\pi} \cos nx\, dx = 0$ for each $n \geq 1$.

By means of the formula of Poincaré-Bertrand of § 2 the integral equation 7-10 can be shown to be equivalent to the integral equation 7-8 plus the condition 7-13. If e.g. upon 7-8 the following singular operator Ω is applied

$$\Omega\, \varphi(x) = \frac{1}{\pi}\int_0^{\pi} \frac{\varphi(t)\sin t\, dt}{\cos t - \cos x}\ ,$$

a repeated Cauchy integral is obtained upon which formula 2-21 may be applied which gives 7-10.

The singular integral equation 7-10 will now be studied more closely according to the methods given by Muskhelishvili [1]. The equation may be written as

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1) l.c. part V ch. 14.

$$s(x) - \frac{\gamma}{\pi} \int_0^{\pi} \frac{\sin t \; s(t)\, dt}{\cos t - \cos x} + \frac{1}{\pi} \int_0^{\pi} K_2(t,x)\, s(t)\, dt = f(x) \qquad 7\text{-}14$$

The dominant equation of 7-14 is

$$s_d(x) - \frac{\gamma}{\pi} \int_0^{\pi} \frac{\sin t \; s_d(t)\, dt}{\cos t - \cos x} = f(x). \qquad 7\text{-}15$$

If we put

$$u = -\cos x\,,$$
$$s_d(x) = \varphi(u)\,,$$
$$f(x) = \psi(u)\,,$$

7-15 passes into

$$\varphi(u) + \frac{\gamma}{\pi} \int_{-1}^{1} \frac{\varphi(t)\, dt}{t-u} = \psi(u). \qquad 7\text{-}16$$

If $\Phi(w)$ represents the Cauchy integral of $\varphi(u)$

$$\Phi(w) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{\varphi(t)}{t-w}\, dt$$

7-16 may be brought in the form

$$(1 + \gamma i)\, \Phi^+ - (1 - \gamma i)\, \Phi^- = \psi\,.$$

This problem is very similar to that considered in section 2 and the solution of 7-16 may be written down immediately (cf formula 3-15)

$$\Phi(w) = \cos \mu\pi \left(\frac{w+1}{w-1}\right)^{\mu} \frac{1}{2\pi i} \int_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)}{t-w}\, dt. \qquad 7\text{-}17$$

In fact, we have from Plemelj's formulae 2-2, 2-3

$$\Phi^+(u) = e^{-\mu\pi i} \cos \mu\pi \left(\frac{1+u}{1-u}\right)^{\mu} \left\{ \tfrac{1}{2} \left(\frac{1-u}{1+u}\right)^{\mu} \psi(u) + \frac{1}{2\pi i} \fint_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)}{t-u}\, dt \right\},$$

$$\Phi^-(u) = e^{\mu\pi i} \cos \mu\pi \left(\frac{1+u}{1-u}\right)^{\mu} \left\{ -\tfrac{1}{2} \left(\frac{1-u}{1+u}\right)^{\mu} \psi(u) + \frac{1}{2\pi i} \fint_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)}{t-u}\, dt \right\},$$

so that, again from 2-2, $\varphi(u)$ becomes

$$\varphi(u) = \cos^2 \mu\pi\; \psi(u) - \frac{\sin 2\mu\pi}{2\pi} \left(\frac{1+u}{1-u}\right)^{\mu} \fint_{-1}^{1} \left(\frac{1-t}{1+t}\right)^{\mu} \frac{\psi(t)}{t-u}\, dt\,, \qquad 7\text{-}18$$

or for s(x)

$$s(x) = \cos^2\mu\pi\, f(x) + \frac{\sin 2\mu\pi}{2\pi}(\mathrm{tg}\,\tfrac{1}{2}x)^{2\mu}\fint_0^{\pi}(\mathrm{tg}\,\tfrac{1}{2}t)^{-2\mu}\frac{\sin t\, f(t)\, dt}{\cos t - \cos x}. \qquad 7\text{-}19$$

If γ_n is constant the function $K_2(t,x)$ vanishes and 7-19 represents the unique solution of the integral equation 7-10 [1]. If γ_n is variable f(x) in 7-19 may be replaced by

$$f(x) - \frac{1}{\pi}\int_0^{\pi} K_2(t,x)\, s(t)\, dt,$$

and the following integral equation of the Fredholm type is obtained

$$F(x) = G(x) - \frac{\cos\mu\pi}{\pi}\int_0^{\pi} K(t,x)\, F(t)\, dt, \qquad 7\text{-}20$$

where

$$F(x) = (1+\cos x)^{\mu}\, s(x), \qquad 7\text{-}21$$

$$G(x) = \cos^2\mu\pi\,(1+\cos x)^{\mu} f(x) + \frac{\sin 2\mu\pi}{2\pi}(1-\cos x)^{\mu}\fint_0^{\pi}\left(\frac{1+\cos t}{1-\cos t}\right)^{\mu}\frac{\sin t\, f(t)\, dt}{\cos t - \cos x}, \qquad 7\text{-}22$$

$$K(t,x) = (1+\cos t)^{-\mu}(1-\cos x)^{\mu}\sum_1^{\infty} r_n \sin nt\, k_n(x), \qquad 7\text{-}23$$

$$k_n(x) = 2\cos\mu\pi\left(\frac{1+\cos x}{1-\cos x}\right)^{\mu}\cos nx + \frac{2\sin\mu\pi}{\pi}\fint_0^{\pi}\left(\frac{1+\cos u}{1-\cos u}\right)^{\mu}\frac{\sin u\cos nu\, du}{\cos u - \cos x}. \qquad 7\text{-}24$$

We shall use the following lemmas.

<u>lemma 7-4</u>

$$1 + 2\sum_1^{\infty}\lambda^n\cos nx = \frac{1-\lambda^2}{1-2\lambda\cos x + \lambda^2}, \qquad 7\text{-}25$$

$$\sum_1^{\infty}\lambda^n\sin nx = \frac{\lambda\sin x}{1-2\lambda\cos x+\lambda^2}. \qquad 7\text{-}26$$

<u>Proof</u>

Take real and imaginary part of $\sum_0^{\infty}(\lambda e^{ix})^n = (1-\lambda e^{ix})^{-1}$.

<u>lemma 7-5</u>

If $|\theta_n| < 1$, $0 < \alpha < 1$, we have

----

1) cf § 3, 3-15.

$$\left| \sum_1^{\infty} \theta_n \frac{\sin nx}{n^{1+\alpha}} \right| \leq \frac{5 \sin^{\alpha} x}{4\alpha(1-\alpha)} , \qquad 0 < x < \pi \qquad 7\text{-}27$$

<u>Proof</u> If $N = \left[\frac{1}{\sin x} \right]$ we have $\left| \frac{\theta_{N+1} \sin(N+1)x}{(N+1)^{1+\alpha}} \right| \leq (N+1)^{-1-\alpha} \leq \frac{\sin^{\alpha} x}{4\alpha(1-\alpha)}$

$$\left| \sum_1^{N} \frac{\theta_n \sin nx}{n^{1+\alpha}} \right| \leq \sin x \sum_1^{N} \frac{1}{n^{\alpha}} \leq \sin x \int_0^N \frac{dt}{t} = \frac{N^{1-\alpha} \sin x}{1-\alpha} \leq \frac{\sin^{\alpha} x}{1-\alpha} ,$$

$$\left| \sum_{N+2}^{\infty} \frac{\theta_n \sin nx}{n^{1+\alpha}} \right| \leq \sum_{N+2}^{\infty} \frac{1}{n^{1+\alpha}} \leq \int_{N+1}^{\infty} \frac{dt}{t^{1+\alpha}} = \frac{(N+1)^{-\alpha}}{\alpha} \leq \frac{\sin^{\alpha} x}{\alpha} .$$

The expression 7-24 for $k_n(x)$ can be simplified considerably. If the following generating function is introduced

$$k(x, \lambda) = \frac{k_o}{2} + \sum_1^{\infty} k_n \lambda^n , \qquad 1, \qquad 7\text{-}28$$

we obtain by means of 7-25

$$(1-\lambda^2)^{-1} k(x,\lambda) = \frac{\cos \mu\pi \, (\operatorname{tg} \tfrac{1}{2}x)^{-2\mu}}{1-2\lambda \cos x + \lambda^2} +$$

$$+ \frac{\sin \mu\pi}{\pi} \int_0^{\pi} \left(\frac{1+\cos u}{1-\cos u}\right)^{\mu} \frac{\sin u \, du}{(1-2\lambda \cos u + \lambda^2)(\cos u - \cos x} .$$

The second term in the right-hand side may be written as

$$\frac{\sin \mu\pi}{\pi} \frac{1}{1-2\lambda \cos x + \lambda^2} \left\{ \int_{-1}^{1} \left(\frac{1+t}{1-t}\right)^{\mu} \frac{dt}{t-\cos x} + \right.$$

$$\left. + 2\lambda \int_{-1}^{1} \left(\frac{1+t}{1-t}\right)^{\mu} \frac{dt}{1-2\lambda t + \lambda^2} \right\} .$$

If the formulae 2-11 and 2-12 are used we find without difficulty

$$k(x,\lambda) = \left(\frac{1+\lambda}{1-\lambda}\right)^{2\mu} \frac{1-\lambda^2}{1-2\lambda \cos x + \lambda^2} , \qquad 7\text{-}29$$

from which

$$k_n(x) = e_n(2\mu) + 2 \sum_1^{n} e_k(2\mu) \cos(n-k)x . \qquad 7\text{-}30$$

We may also write

$$k(x,\lambda) = \left(\frac{1+\lambda}{1-\lambda}\right)^{2\mu-1} \left\{ 1 + \operatorname{cotg} \tfrac{1}{2}x \, \frac{2\lambda \sin x}{1-2\lambda \cos x + \lambda^2} \right\}, \qquad 7\text{-}31$$

so that by means of lemma 7-4

$$k_n(x) = c_n(2\mu-1) + 2 \operatorname{cotg} \tfrac{1}{2}x \sum_{1}^{n} c_k(2\mu-1) \sin(n-k)\, x. \qquad 7\text{-}32$$

We shall next derive an estimate for $k_n(x)$. We shall prove the following property. There is a constant C not dependent on x such that

$$|k_n(x)| \leq \frac{C}{(\sin \frac{1}{2}x)^{2\mu} \, (\cos \frac{1}{2}x)^{1-2\mu}}, \qquad 0 < x < \pi \qquad 7\text{-}33$$

<u>Proof</u> For $k_n(x)$ we may write

$$k_n(x) = \frac{1}{2\pi i} \oint \left(\frac{1+w}{1-w}\right)^{2\mu} \frac{1-w^2}{1-2w \cos x + w^2} \, \frac{dw}{w^{n+1}},$$

where the path of integration is a small circle round w=0. The contour may be deformed as in section 4 and we get contributions from the residues at $w=e^{\pm ix}$ and the cuts from 1 to $+\infty$ and -1 to $-\infty$.
The residues give

$$\left(\frac{1+e^{ix}}{1-e^{ix}}\right)^{2\mu} e^{-nix} + \left(\frac{1+e^{-ix}}{1-e^{-ix}}\right)^{-2\mu} e^{nix} = \mathcal{O}\left(\frac{1}{(\sin \frac{1}{2}x)^{2\mu}}\right).$$

The integral

$$\int_1^\infty \left(\frac{u+1}{u-1}\right)^{2\mu} \frac{u^2-1}{u^2-2u \cos x + 1} \quad \frac{du}{u^{n+1}}$$

is less than

$$2^{1+2\mu} \int_0^\infty \frac{v^{1-2\mu}\, dv}{v^2+2(1-\cos x)(1+v)},$$

which in turn is less than

$$\frac{2}{(\sin \frac{1}{2}x)^{2\mu}} \int_0^\infty \frac{v^{1-2\mu}\, dv}{v^2+1},$$

thus the contribution from the positive real axis is also $\mathcal{O}\left(\frac{1}{(\sin \frac{1}{2}x)^{2\mu}}\right)$.

In a similar way the contribution from the negative real axis is shown to be $\mathcal{O}\left(\frac{1}{(\cos \frac{1}{2}x)^{1-2\mu}}\right)$.

If $\frac{\pi}{2} \leq x \leq \pi$ we may use formula 7-32 to get a better estimate near $x=\pi$. In section 4 we have derived the estimate $e_n(2\mu-1)=\mathcal{O}(n^{-2\mu})$. Thus we have also

$$|k_n(x)| \leq \frac{C'}{\sin\frac{1}{2}x}\, n^{1-2\mu}, \qquad 0<x\leq\pi \tag{7-34}$$

We shall, however, be satisfied with the estimate

$$|k_n(x)| \leq \frac{C''}{(\sin\frac{1}{2}x)^{2\mu}}\, n^{1-2\mu}, \qquad 0<x\leq\pi \tag{7-35}$$

for, using this, we find from lemma 7-5

$$\left|\sum_1^\infty r_n \sin nt\, k_n(x)\right| \leq C''' \frac{(\sin t)^{2\mu}}{(\sin\frac{1}{2}x)^{2\mu}},$$

and finally form $K(t,x)$

$$|K(t,x)| \leq C''' \sin^{2\mu}\tfrac{1}{2}t. \tag{7-36}$$

Thus we have obtained the important result that the kernel $K(t,x)$ is uniformly bounded in the square $0\leq t,x\leq\pi$.

§8 The orthogonal function

We shall again consider the expansion

$$f(x) = \sum_1^\infty a_n(\sin nx + \gamma_n \cos nx), \tag{8-1}$$

where

$$\gamma_n = \gamma + r_n, \quad \mathrm{Re}\,\gamma>0, \quad r_n = \mathcal{O}(\tfrac{1}{n^2}).$$

The expansion

$$f(x) = \sum_1^\infty a_n^* \left(\sin nx + \frac{1}{\gamma_n}\cos nx\right), \tag{8-2}$$

will be called the adjoint expansion. In particular we shall consider the adjoint expansion of unity

$$1 = \sum_1 h_n\left(\sin nx + \frac{1}{\gamma_n}\cos nx\right). \tag{8-3}$$

The problem of determining the coefficients h_n of 8-3 has been considered in the previous section. If now we introduce the function

$$h(x) = \sum_1^\infty h_n \sin nx = 1 - \sum_1^\infty \frac{h_n}{\gamma_n}\cos nx, \qquad 0<x<\pi \tag{8-4}$$

it is clear that

$$\int_0^{\pi} h(x)\sin nx\,dx = \frac{\pi}{2}h_n\,, \qquad n \geq 1 \qquad 8\text{-}5$$

$$\int_0^{\pi} h(x)\cos nx\,dx = -\frac{\pi}{2}\frac{h_n}{\gamma_n}\,, \qquad n \geq 1 \qquad 8\text{-}6$$

From this we obtain

$$\int_0^{\pi} h(x)(\sin nx + \gamma_n \cos nx)\,dx = 0 \qquad n \geq 1 \qquad 8\text{-}7$$

which shows that the function $h(x)$ constructed in this way is orthogonal to all $\sin nx + \gamma_n \cos nx$. If γ_n is constant the formulae 8-3, 8-5, 8-6, 8-7 reduce to the formulae 4-9, 4-7, 4-8, 4-10 considered in section 4. If 8-1 converges uniformly we may apply 8-7 term by term and the following condition will be obtained

$$\int_0^{\pi} h(x) f(x)\,dx = 0. \qquad 8\text{-}8$$

In a similar way as in the previous section $h(x)$ may be shown to satisfy a singular integral equation. We may either start from 8-3, apply the theory of the previous section or we may choose an independent argument. We prefer the latter and we start from

$$\lim_{N\to\infty} \int_0^{\pi} h(t) \sum_1^N \left(\cos nt + \frac{1}{\gamma_n}\sin nt\right)\cos nx\,dt = 0. \qquad 8\text{-}9$$

Without difficulty we obtain

$$h(x) - \frac{1}{\gamma\pi} \fint_0^{\pi} \frac{\sin t\, h(t)\,dt}{\cos t - \cos x} = g(x)\,, \qquad 8\text{-}10$$

with

$$g(x) = \frac{1}{\pi}\int_0^{\pi} K_3(t,x)\,h(t)\,dt + \frac{1}{\pi}\int_0^{\pi} h(t)\,dt\,, \qquad 8\text{-}11$$

where

$$K_3(t,x) = 2\sum_1^{\infty}\left(\frac{1}{\gamma} - \frac{1}{\gamma_n}\right)\sin nt \cos nx. \qquad 8\text{-}12$$

Equation 8-10 is of the type 7-14 and will be solved in the same way. The solution of 8-10 becomes for known $g(x)$

$$h(x) = \sin^2\mu\pi\, g(x) + \frac{\sin 2\mu\pi}{2\pi}\left(\operatorname{tg}\tfrac{1}{2}x\right)^{1-2\mu} \fint_0^{\pi} \left(\operatorname{tg}\tfrac{1}{2}u\right)^{-1+2\mu} \frac{\sin u\, g(u)\,du}{\cos u - \cos x}. \qquad 8\text{-}13$$

If g(x) is a constant, say g(x)=1, 8-13 gives

$$h(x) = \sin\mu\pi \, (\mathrm{tg}\, \tfrac{1}{2}x)^{1-2\mu},$$

which is the orthogonal function for constant γ_n.
If for g(x) we substitute 8-11 formula 8-10 reduces to

$$h(x) = \frac{\sin\mu\pi}{\pi} (\mathrm{tg}\, \tfrac{1}{2}x)^{1-2\mu} \int_0^\pi h(t)dt + \frac{\sin\mu\pi}{\pi} \int_0^\pi K_4(t,x)\, h(t)\, dt, \qquad 8\text{-}14$$

where

$$K_4(t,x) = \sin\mu\pi \; K_3(t,x) + \frac{\cos\mu\pi}{\pi} (\mathrm{tg}\, \tfrac{1}{2}x)^{1-2\mu} .$$

$$\cdot \fint_0 \left(\frac{1+\cos u}{1-\cos u}\right)^{\frac{1}{2}-\mu} \frac{\sin u\, K_3(t,u) du}{\cos u - \cos x} . \qquad 8\text{-}15$$

Finally we have as in the previous section the following result:
If we put

$$H(x) = (1+\cos x)^{\frac{1}{2}-\mu} \; h(x) , \qquad 8\text{-}16$$

$$H_o(x) = (1-\cos x)^{\frac{1}{2}-\mu} \; \frac{\sin\mu\pi}{\pi} \int_0^\pi h(t)\, dt , \qquad 8\text{-}17$$

$$L(t,x) = (1-\cos x)^{\frac{1}{2}-\mu} \; (1+\cos t)^{-\frac{1}{2}+\mu} \sum_1^\infty \left(\frac{1}{\gamma} - \frac{1}{\gamma_n}\right) \sin nt \;\; k_n(x,\tfrac{1}{2}-\mu), \qquad 8\text{-}18$$

where $\sum_0^\infty k_n(x,\frac{1}{2}-\mu)\, \lambda^n = \left(\frac{1+\lambda}{1-\lambda}\right)^{1-2\mu} \frac{1-\lambda^2}{1-2\lambda \cos x + \lambda^2}$,

H(x) satisfies the ordinary Fredholm equation

$$H(x) = H_o(x) + \frac{\sin\mu\pi}{\pi} \int_0^\pi L(t,x)\, H(t)\, dt \qquad 8\text{-}19$$

with a uniformly bounded kernel in the square $0 \leq t,x \leq \pi$.

§9 A potential problem

In this section we shall consider a potential problem which gives rise to the expansion of a given function into a trigonometrical series

$$f(x) = \sum_1^\infty a_n(\sin nx + \gamma_n \cos nx) . \qquad 9\text{-}1$$

Let the function $F(x,y)$ in the region $0 < x < \pi$, $y > 0$ satisfy the potential equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0, \qquad 9\text{-}2$$

and the following boundary conditions

$$x = 0, \pi \qquad F = 0 , \qquad 9\text{-}3$$
$$y \to \infty \qquad F \to 0 , \qquad 9\text{-}4$$
$$y = 0 \qquad \gamma \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} + \frac{1}{\pi} \int_{-\pi}^{\pi} R(x-\xi) \left(\frac{\partial F}{\partial x}\right)_{\xi} d\xi = f(x), \qquad 9\text{-}5$$

where $\gamma > 0, f(x)$ and $R(x)$ are given functions, and where in particular

$$R(x) = \sum_1^{\infty} r_n \cos nx . \qquad 9\text{-}6$$

If we represent $F(x,y)$ by

$$F(x,y) = \sum_1^{\infty} c_n \sin nx\, e^{-ny} , \qquad 9\text{-}7$$

only the condition 9-5 has to be considered.
Since

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos m(x-\xi) \cos n\xi \, d\xi = \delta_{mn} \cos nx$$

we get

$$f(x) = \sum_1^{\infty} nc_n (\sin nx + \gamma_n \cos nx) , \qquad 9\text{-}8$$

where

$$\gamma_n = \gamma + r_n .$$

If $a_n = nc_n$ this is exactly the representation 9-1.

Thus the solution of the potential problem considered above may be reduced to the solution of the representation problem 9-1.
In the preceding sections it has been shown that the representation problem 8-1 may be reduced to a singular integral equation. We shall now show that this singular integral equation also can be derived from the potential problem by means of Green's theorem.

If $R(x) \equiv 0$ the coefficients c_n may be determined by explicit analytic expressions. We have seen that for $n \to \infty$

$$c_n = \mathcal{O}(n^{-2+2\mu}) , \qquad \mu\pi = \operatorname{arc\,tg} \gamma .$$

This means that both $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ become infinite at $x=\pi$. For $0 \leq x < \pi$, however, they are bounded. In particular at $x=\pi$ F_x, $F_y = \mathcal{O}\{(\pi-x)^{-2\mu}\}$. If f(x) satisfies the condition

$$\int_0^\pi (\mathrm{tg}\,\tfrac{1}{2}x)^{1-2\mu}\ f(x)\ dx = 0$$

F_x and F_y are uniformly bounded in the closed interval $(0,\pi)$. If $R(x) \not\equiv 0$ and if $r_n = \mathcal{O}(\frac{1}{n^2})$ similar conclusions may be drawn.

Let $G(x,y,\xi,\eta)$ be a Green function in the region $0 < x < \pi$, $y > 0$ and let

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0$$

$$\begin{array}{ll} x = 0,\pi & G = 0 \\ y \rightarrow \infty & G \rightarrow 0 \\ (x-\xi)^2+(y-\eta)^2 \rightarrow 0 & G \sim \frac{1}{2\pi} \ln \sqrt{(x-\xi)^2 + (y-\eta)^2} \end{array}$$

then applying Green's theorem on the region considered above one easily obtains

$$F(\xi,\eta) = \int_0^\pi G(x,0,\xi,\eta)\ \frac{\partial}{\partial y} F(x,0)\ dx - \int_0^\pi \left\{\frac{\partial}{\partial y} G(x,0,\xi,\eta)\right\} F(x,0)\,dx \qquad 9\text{-}9$$

Taking $\eta=0$ an integral equation is obtained between the partial derivatives of F at zero for which also the relation 9-5 exists.

Let $G_o(x,y,\xi,\eta)$ represent the standard Green function of the full strip $0 < x < \pi$. Obviously we have

$$G_o(x,y,\xi,\eta) = \frac{1}{2\pi} \mathrm{Re} \left\{ \ln \sin \frac{(x-\xi)+i(y-\eta)}{2} - \ln \sin \frac{(x+\xi)+i(y-\eta)}{2} \right. \qquad 9\text{-}10$$

Simplification may be expected if in 9-9 the following Green functions are chosen

$$G_1 = G_o(x,y,\xi,\eta) - G_o(x,y,\xi,-\eta) \qquad 9\text{-}11$$

$$G_2 = G_o(x,y,\xi,\eta) + G_o(x,y,\xi,-\eta) \qquad 9\text{-}12$$

so that at y=0 $G_1=0$ and $\frac{\partial G_2}{\partial y} = 0$.

We note the following expressions

$$G_o(x,y,\xi,\eta) = \frac{1}{4\pi} \ln \frac{\sin^2 \frac{x-\xi}{2} + \mathrm{sh}^2 \frac{y-\eta}{2}}{\sin^2 \frac{x+\xi}{2} + \mathrm{sh}^2 \frac{y+\eta}{2}}$$

$$2\pi \frac{\partial}{\partial x} G_o(x,0,\xi,0) = \frac{\sin \xi}{\cos \xi - \cos x}$$

$$2\pi \frac{\partial}{\partial \xi} G_0(x,0,\xi,0) = \frac{\sin x}{\cos x - \cos \xi}$$

$$\frac{\partial G_1}{\partial \eta} + \frac{\partial G_2}{\partial y} = 0.$$

If in 9-9 we substitute G_1 the first integral on the right-hand side vanishes. After a differentiation with respect to η we get

$$\frac{\partial F(\xi,\eta)}{\partial \eta} = -\fint_0^\pi \frac{\partial^2 G_1}{\partial y \partial \eta} F(x,0)\, dx$$

but $-\frac{\partial^2 G_1}{\partial y \partial \eta} = \frac{\partial^2 G_2}{\partial y^2} = -\frac{\partial^2 G_2}{\partial x^2}$, so that after a partial integration and change of notation

$$y=0 \quad \left(\frac{\partial F}{\partial y}\right)_x + \frac{1}{\pi}\fint_0^\pi \left(\frac{\partial F}{\partial x}\right)_t \frac{\sin x\, dt}{\cos t - \cos x} = 0 \qquad 9\text{-}13$$

From 9-5 and 9-13 the following singular integral equation for $\frac{\partial F}{\partial x}$ is obtained

$$y=0 \quad \gamma \frac{\partial F}{\partial x} + \frac{1}{\pi}\fint_0^\pi \left(\frac{\partial F}{\partial x}\right)_t \frac{\sin x\, dt}{\cos t - \cos x} + \frac{1}{\pi}\int_{-\pi}^{\pi} R(x-t)\left(\frac{\partial F}{\partial x}\right)_t dt = f(x) \qquad 9\text{-}14$$

This integral equation is clearly equivalent with that obtained in section 7 formula 7-8.
If in 9-9 G_2 is substituted we obtain at once

$$\frac{\partial F(\xi,\eta)}{\partial \xi} = \fint_0^\pi \frac{\partial G_2}{\partial \xi} \left(\frac{\partial F}{\partial y}\right)_x dx$$

or at y=0

$$y=0 \quad \left(\frac{\partial F}{\partial x}\right)_x - \frac{1}{\pi}\fint_0^\pi \left(\frac{\partial F}{\partial y}\right)_t \frac{\sin t\, dt}{\cos t - \cos x} = 0\ . \qquad 9\text{-}15$$

If again in 9-15 $\frac{\partial F}{\partial y}$ is expressed by means of 9-5 another singular integral equation is obtained which turns out to be equivalent to that of formula 7-10.

§10 Conclusions

The problem of expanding a function f(x), which is hölderian in the interval $0 \leq x \leq \pi$, into a trigonometrical series of the following kind

$$f(x) = \sum_{1}^{\infty} a_n(\sin nx + \gamma_n \cos nx) \qquad 10\text{-}1$$

has been solved completely for the case of $\gamma_n = \gamma$ which is not dependent on n.

In the case of γ_n variable, however, we have only shown that, if

$$\gamma_n = \gamma + \mathcal{O}(n^{-2}) \qquad 10\text{-}2$$

the problem considered above is equivalent to an ordinary Fredholm integral equation with bounded kernel. Thus in a particular case of a given function f(x) and numerically given phase factors γ_n the coefficients a_n may be determined by any numerical process by which the solution of a Fredholm equation is obtained. The discussion of the Fredholm equation will be given in another report, but we remark already here that there might be exceptional cases of sets $\{\gamma_n\}$ satisfying 10-2 for which the problem 10-1 does not admit a solution or admits an infinity of solutions. Those cases correspond to the proper values of the Fredholm equation.
A similar remark may be made in connection with the function h(x) which is orthogonal to the functions $\sin nx + \gamma_n \cos nx$, $n \geqslant 1$. Also the discussion of the numerical determination of h(x), if existing, will be given later on.

If the assumption 10-2 is dropped the problem 10-1 becomes much more difficult, and apart from the following case $f(x) \in L_2$, $|\gamma_n| < \gamma < 1$ considered by Paley and Wiener almost nothing seems to be known about the possibility of the expansion 10-1.