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THE INFLUENCE OF A DISTURBANCE UPON AN INFINITELY LARGE SHALLOW SEA OF CONSTANT DEPTH

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The influence of a disturbance upon an infinitely large shallow sea of constant depth

§ 1. Introduction

In a previous report [1] the problem of the influence of a non-stationary wind-field upon a sea bounded by a coast and an ocean has been considered from a general point of view. It has been shown that for some simple regions the Laplace transform $\bar{\zeta}$ of the elevation ζ of the level of the sea may be represented by an explicit formula. In the simplest case of a sea of infinite extensions, so that there is no boundary, we have for the solution vanishing at infinity

$$\bar{\zeta} = - \frac{1}{2\pi} \iint_{-\infty}^{\infty} K_0(k \sqrt{(x-\xi)^2+(y-\eta)^2})\, \bar{F}(\xi,\eta,p)\, d\xi\, d\eta , \qquad 1.1$$

where

$$k^2 = \frac{p\{(p+\lambda)^2+\Omega^2\}}{c^2(p+\lambda)} , \qquad 1.2$$

and where $\bar{F}$ is the Laplace transform of a function F which can be obtained from the wind-field.

In the above-mentioned report the determination of ζ from its Laplace transform had been left out of consideration in view of the complicated form of 1.2.

In this report where the case of an infinite sea is studied in more detail an explicit expression for the original $G(x,y,t)$ of the function $K_0(k\sqrt{x^2+y^2})$ occurring in 1.1 has been obtained. Thus we have from 1.1

$$\zeta(x,y,t) = - \frac{1}{2\pi} \iint_{-\infty}^{\infty} d\xi\, d\eta \int_{-\infty}^{\infty} F(\xi,\eta,t-\tau) G(x-\xi, y-\eta, \tau) d\tau . \qquad 1.3$$

In a number of non-trivial cases the function F reduces to a Dirac delta function which annihilates the integrations with respect to ξ and η. These cases are considered in § 3 and are listed below. In all those cases we have derived expressions for the initial response of the sea upon a unit step function in the time and for the asymptotic behaviour of the elevation as $t \to \infty$. If this step function is denoted by $\iota(t)$ i.e. $\iota(t)=0$ for $t<0$ and 1 for $t>0$, we consider the cases

<u>a</u> no wind, logarithmic barometric pressure b

$$b = - \iota(t) \ln \sqrt{x^2+y^2}.$$

<u>b</u> circular rotation-free wind-field

$$W_x = \frac{x}{x^2+y^2}\, \iota(t) \qquad W_y = \frac{y}{x^2+y^2}\, \iota(t).$$

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1) H.A. Lauwerier. The motion of a shallow sea under influence of a non-stationary wind-field. Report TW 31.

c circular divergence-free wind-field

$$W_x = \frac{-y}{x^2+y^2}\,\iota(t) \qquad W_y = \frac{x}{x^2+y^2}\,\iota(t).$$

d point-source wind

$$W_x = \delta(x)\,\delta(y)\,\iota(t) \qquad W_y = 0.$$

In § 4 an arbitrary circular wind-field is considered with

$$W_x = \varphi(\sqrt{x^2+y^2})\,f(t) \qquad W_y = 0.$$

If in particular

$$\varphi(\sqrt{x^2+y^2}) = -\ln\sqrt{x^2+y^2}$$

the following simple expression is obtained

$$\varrho c^2\,\zeta(r,\theta,p) = \frac{\bar{f}}{k^2}\left(\cos\theta - \frac{\Omega}{p+\lambda}\sin\theta\right)\frac{\partial}{\partial r}\left\{\ln r + K_0(kr)\right\} \tag{1.4}$$

where r, θ are polar coordinates, $x = r\cos\theta$ $y = r\sin\theta$.

The results obtained in §3 and §4 are mostly given without proof. Since they depend largely upon the theory of Laplace transformation a separate section, §5, is devoted to the study of the inverse Laplace transformation applied to $K_0(kr)$ and related functions.

§2. General theory

The linearised equations of motion are

$$\begin{cases} \left(\frac{\partial}{\partial t} + \lambda\right)w_x - \Omega w_y + c^2\left(\frac{\partial\zeta}{\partial x} + \frac{1}{\varrho}\frac{\partial b}{\partial x}\right) = \frac{1}{\varrho}W_x \\ \left(\frac{\partial}{\partial t} + \lambda\right)w_y + \Omega w_x + c^2\left(\frac{\partial\zeta}{\partial y} + \frac{1}{\varrho}\frac{\partial b}{\partial y}\right) = \frac{1}{\varrho}W_y \\ \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial\zeta}{\partial t} = 0, \end{cases} \tag{2.1}$$

where ζ is the elevation of the level of the sea, w_x and w_y the components of the stream, W_x and W_y the components of the wind-field, b the barometric pressure, λ the coefficient of friction, Ω the coefficient of Coriolis and c the velocity of propagation of a free wave.

If upon 2.1 Laplace transformation is applied, e.g.

$$\bar{\zeta}(x,y,p) = \int_{-\infty}^{\infty} e^{-pt}\,\zeta(x,y,t)\,dt, \tag{2.2}$$

for $\bar{\zeta}$ the following equation may be obtained

$$(\Delta - k^2)\bar{\zeta} = \bar{F} , \qquad 2.3$$

where
$$k^2 = \frac{p\{(p+\lambda)^2 + \Omega^2\}}{c^2(p+\lambda)} , \qquad 2.4$$

and
$$\rho c^2 \bar{F} = \left(\frac{\partial \bar{W}_x}{\partial x} + \frac{\partial \bar{W}_y}{\partial y}\right) - \frac{\Omega}{p+\lambda}\left(\frac{\partial \bar{W}_x}{\partial y} - \frac{\partial \bar{W}_y}{\partial x}\right) - \Delta \bar{b}. \qquad 2.5$$

If $\bar{F}$ represents a point-source disturbance at (ξ,η)

$$\bar{F} = 2\pi\,\delta(x-\xi)\,\delta(y-\eta), \qquad 2.6$$

where $\delta(z)$ is Dirac's delta function defined by

$$\int_{-\infty}^{\infty} f(u)\,\delta(u-z)du = f(z),$$

the solution of 2.3 which vanishes at infinity is

$$\bar{\zeta}_o(x,y,p) = -K_o(k\sqrt{(x-\xi)^2 + (y-\eta)^2}). \qquad 2.7$$

Therefore the solution of 2.3 for an arbitrary function $\bar{F}(x,y,p)$ at the right-hand side becomes

$$\bar{\zeta}(x,y,p) = -\frac{1}{2\pi}\iint_{-\infty}^{\infty} K_o(k\sqrt{(x-\xi)^2+(y-\eta)^2})\bar{F}(\xi,\eta,p)d\xi\,d\eta . \qquad 2.8$$

In the following section a few cases will be considered in which $\bar{F}$ reduces to a point source disturbance so that, apart from a constant factor, the solution is given by 2.7.

§3. Particular cases

a no wind, circular depression centred at the origin.

$$b = -f(t)\ln r, \qquad 3.1$$

where
$$r = \sqrt{x^2+y^2}.$$

Since $\Delta\bar{b} = -2\pi\,\delta(x-\xi)\,\delta(y-\eta)\,\bar{f}(p)$,

the solution is
$$\rho\bar{\zeta}(r,p) = -\bar{f}(p)\,K_o(kr). \qquad 3.2$$

If $G(r,t)$ represents the original of $K_o(kr)$ we have by means of the convolution theorem

$$\rho\,\zeta(r,t) = -\int_{-\infty}^{\infty} f(t-\tau)G(r,\tau)d\tau . \qquad 3.3$$

The function $G(r,t)$ has the following properties which will be proved in §5.

$$G(r,t) = 0 \; , \qquad t < r/c \qquad 3.4a$$

$$G(r,t) = \frac{1}{2t} - \frac{(\lambda^2+\Omega^2)r^2}{8\lambda c^2 t^2} + O(t^{-3}) \; , \qquad \lambda t \to \infty \; , \qquad 3.4b$$

$$G(r,t) = e^{-\frac{\lambda}{2}t}(t^2 - \frac{r^2}{c^2})^{-\frac{1}{2}} \left\{ 1 - \frac{4\Omega^2 - \lambda^2}{8}(t^2 - \frac{r^2}{c^2}) + \dots \right\} , \quad t - r/c \to +0. \qquad 3.4c$$

If f(t)=0 for t < 0, so that the disturbance 3.1 starts at t=0, the elevation $\zeta(r,t)$ at a distance r from the origin is zero for t < r/c. Thus the initial disturbance at t=0 travels radially away from the origin with constant velocity c.

In the special case where f(t) is the unit step-function

$$f(t) = \iota(t),$$

where

$$\iota(t) = 0 \qquad t < 0,$$
$$\iota(t) = 1 \qquad t > 0,$$

we have in particular

$$\rho\zeta(r,t) = -\int_{r/c}^{t} G(r,\tau)d\tau \; , \qquad 3.5$$

and, cf §5,

$$\rho\zeta(r,t) = -\tfrac{1}{2}\ln\lambda t + \tfrac{1}{2}\left\{\gamma + \ln\frac{(\lambda^2+\Omega^2)r^2}{4c^2}\right\} + O(t^{-1}), \quad \lambda t \to \infty \; , \qquad 3.6a$$

$$\rho\zeta(r,t) = (\frac{2c}{r})^{\frac{1}{2}} e^{-\frac{\lambda}{2}t}(t - \frac{r}{c})^{\frac{1}{2}}\left\{1 + O(t - \frac{r}{c})\right\} , \quad t - \frac{r}{c} \to +0. \qquad 3.6b$$

<u>b</u> circular rotation free wind-field, b=0.

$$W_x = \frac{x}{x^2+y^2} f(t) \; , \qquad W_y = \frac{y}{x^2+y^2} f(t). \qquad 3.7$$

The divergence is zero except at the origin where

$$\frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} = 2\pi\,\delta(x)\,\delta(y)\; f(t).$$

The elevation is accordingly determined by

$$\rho c^2 \bar{\zeta}(x,y,p) = -\bar{f}(p)\; K_0(kr), \qquad 3.8$$

which is the same result as in the previous case so that the same conclusions may be drawn.

<u>c</u> circular divergence free wind-field, b=0.

$$W_x = \frac{-y}{x^2+y^2} f(t), \qquad W_y = \frac{x}{x^2+y^2} f(t). \qquad 3.9$$

The rotation is zero except at the origin where

$$\frac{\partial W_x}{\partial y} - \frac{\partial W_y}{\partial x} = -2\pi\delta(x)\,\delta(y)\,f(t). \qquad 3.10$$

Thus we have for $\bar{\zeta}$

$$\rho c^2 \bar{\zeta}(x,y,p) = -\frac{\Omega}{p+\lambda}\;\bar{F}(p)\;K_o(kr), \qquad 3.11$$

from which $\zeta(x,y,t)$ may be obtained by means of the convolution theorem. Also in this case the disturbance initiated at t=0 travels radially away from the origin with velocity c.

If $f(t) = \iota(t)$ we have in particular

$$\rho c^2 \zeta(x,y,t) = -\frac{\Omega}{\lambda}\int_{r/c}^{t} \left(-e^{-\lambda(t-\tau)}\right) G(\tau)d\tau\,. \qquad 3.12$$

and, cf §5,

$$\rho c^2 \zeta(x,y,t) = -\frac{\Omega}{2\lambda}\ln\lambda t + \frac{\Omega}{2\lambda}\left\{\gamma + \ln\frac{(\lambda^2+\Omega^2)r^2}{4c^2}\right\} + O(t^{-1}),$$

$$\lambda t \to \infty, \qquad 3.13a$$

$$\rho c^2 \zeta(x,y,t) = -\frac{2\Omega}{3}\left(\frac{2c}{r}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{2}t}\left(t-\frac{r}{c}\right)^{3/2}\left\{1+O\left(t-\frac{r}{c}\right)\right\},\quad t-\frac{r}{c}\to +0.$$

$$3.13b$$

<u>d</u> point-source wind in a constant direction, b=0.

$$W_x = 2\pi\delta(x)\,\delta(y)\,f(t)\quad,\qquad W_y = 0. \qquad 3.14$$

We have

$$\rho c^2\,\bar{F} = \left(\frac{\partial}{\partial x} - \frac{\Omega}{p+\lambda}\frac{\partial}{\partial y}\right)\bar{W}_x. \qquad 3.15$$

If W_x is still an arbitrary function and if 3.15 is substituted into 2.8 we obtain after a partial integration

$$2\pi\,\rho\,c^2\,\bar{\zeta}(x,y,p) = \left(\frac{\partial}{\partial x} - \frac{\Omega}{p+\lambda}\frac{\partial}{\partial y}\right)\iint \bar{\zeta}_o\,\bar{W}_x\,d\xi\,d\eta\,, \qquad 3.16$$

and in particular for W_x given by 3.14

$$\rho c^2\,\bar{\zeta}(x,y,p) = -\left(\frac{\partial}{\partial x} - \frac{\Omega}{p+\lambda}\frac{\partial}{\partial y}\right)K_o(kr)\,\bar{F}(p). \qquad 3.17$$

Again the disturbance initiated at t=0 travels radially away from the origin with velocity c.

For $f(t) = \iota(t)$ we have in particular by means of 3.5 and 3.12

$$\rho c^2 \zeta(x,y,t) = \int_{r/c}^{t}\left\{-x + \frac{\Omega}{\lambda}y\left(1-e^{-\lambda(t-\tau)}\right)\right\}\frac{1}{r}\frac{\partial G(r,\tau)}{\partial r}\,d\tau\,,$$

$$3.18$$

and, cf §5,

$$\varrho c^2 \zeta(x,y,t) = (x - \frac{\Omega}{\lambda} y)\left\{ \frac{1}{r^2} - \frac{\lambda^2+\Omega^2}{4c^2 \lambda t} + O(t^{-2})\right\}, \quad \lambda t \to \infty, \qquad 3.19a$$

$$\varrho c^2 \zeta(x,y,t) = \frac{x}{r} (2cr)^{-\frac{1}{2}} e^{-\frac{\lambda}{2}t} (t - \frac{r}{c})^{-\frac{1}{2}} \left\{ 1 + O(t - \frac{r}{c})\right\}, \quad t - \frac{r}{c} \to +0, \qquad 3.19b$$

In this case the elevation at a distance r from the origin is zero until $t=r/c$ when it suddenly becomes plus infinity. However, this behaviour of ζ is not compatible with the linearisation of the hydrodynamical equations of motion. If instead of a step-function $\iota(t)$ for $f(t)$ a function is taken which is continuous at $t=0$, e.g. $f(t) = t\,\iota(t)$, ζ remains finite as $t - r/c \to +0$. In the latter case we have for $t - \frac{r}{c} \to +0$

$$\varrho c^2 \zeta(x,y,t) \backsim \frac{2x}{r} (2cr)^{-\frac{1}{2}} e^{-\frac{\lambda}{2}t} (t - \frac{r}{c})^{\frac{1}{2}}.$$

§4. An arbitrary circularly distributed wind-field of constant direction

We consider the case

$$W_x = \varphi(r)\, f(t), \qquad W_y = 0. \qquad 4.1$$

From 3.16 we obtain

$$2\pi \varrho c^2 \bar{\zeta}(x,y,p) = \bar{F}(p) \left(-\frac{\partial}{\partial x} + \frac{\Omega}{p+\lambda}\frac{\partial}{\partial y}\right) \iint \bar{\zeta}_0 \varphi \, d\xi \, d\eta .$$

If polar coordinates (r, θ) and (r_0, θ_0) are introduced we have

$$2\pi \varrho c^2 \bar{\zeta}(r,\theta,p) = \bar{F}(p)\left(-\cos\theta + \frac{\Omega}{p+\lambda}\sin\theta\right) \frac{\partial}{\partial r} \iint K_0(kR)\varphi(r_0)\, r_0 \, dr_0 \, d\theta_0 ,$$

where

$$R^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0).$$

The integration with respect to θ_0 may be carried out as follows [1)]

$$\int_{-\pi}^{\pi} K_0(kR)\, d\theta_0 = \int_{-\pi}^{\pi} K_0\left(k\sqrt{r^2+r_0^2-2rr_0\cos\psi}\right) d\psi =$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} d\psi \int_0^{\infty} \exp - \frac{u}{2}\left\{r^2+r_0^2-2rr_0\cos\psi + \frac{k^2}{u^2}\right\}\frac{du}{u} =$$

$$= \frac{1}{2}\int_0^{\infty} \exp - \frac{u}{2}\left\{r^2+r_0^2+\frac{k^2}{u^2}\right\}\frac{du}{u}\int_{-\pi}^{\pi} \exp rr_0 \cos\psi \, d\psi =$$

$$= \pi \int_0^{\infty} \exp - \frac{u}{2}\left\{r^2+r_0^2+\frac{k^2}{u^2}\right\}\frac{I_0(rr_0u)}{u}\, du =$$

$$= \begin{cases} 2\pi \, I_0(kr)\, K_0(kr_0) & \text{if} \quad r < r_0 \\ 2\pi \, I_0(kr_0) K_0(kr) & \text{if} \quad r > r_0 . \end{cases}$$

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1) cf Erdélyi c.s. Higher Transcendental Functions 7.12.23
Integral Transforms 4.17.4

Thus we have obtained

$$\rho c^2\bar{\zeta}(r,\theta,p) = \bar{f}(p)(-\cos\theta + \frac{\Omega}{p+\lambda}\sin\theta)\frac{\partial}{\partial r}\Big\{ I_o(kr)\int_r^\infty K_o(kr_o)\,\varphi(r_o)r_o dr_o + K_o(kr)\int_o^r I_o(kr_o)\varphi(r_o)r_o dr_o \Big\} . \qquad 4.2$$

This result will be applied on the following particular case

$$W_x = -\ln r\,\iota(t)\ , \qquad W_y = 0. \qquad 4.3$$

Since $\int I_o(kr)r\ln r\,dr = \frac{r\ln r\,I_o'(kr)}{k} - \frac{I_o(kr)}{k^2}$,

and similarly for $K_o(kr)$, we obtain without difficulty

$$\rho c^2\bar{\zeta} = \frac{1}{p}(\cos\theta - \frac{\Omega}{p+\lambda}\sin\theta)\frac{\partial}{\partial r}\Big\{ \frac{\ln r + K_o(kr)}{k^2} \Big\} . \qquad 4.4$$

In particular we have, cf §5,

$$\rho c^2\zeta(r,\theta,t) = (\cos\theta - \frac{\Omega}{\lambda}\sin\theta)\frac{r\ln\lambda t}{4} + O(1), \quad \lambda t \to \infty, \qquad 4.5$$

$$\rho c^2\zeta(r,\theta,t) = \frac{c^2t^2\cos\theta}{2r} + O(t^3)\ , \qquad t\to 0. \qquad 4.6$$

§5. <u>The inverse Laplace transform</u>

The original of $K_o(kr)$ where k is determined by 2.4 may be derived as follows.

<u>a</u> If $\lambda=0$ and $\Omega=0$ we have $k^2 = \frac{p^2}{c^2}$, so that

$$K_o(kr) \doteqdot \frac{\iota(t-r/c)}{\sqrt{t^2-\frac{r^2}{c^2}}} \qquad 5.1$$

<u>b</u> If $\lambda=0, \Omega\neq 0$ we have $k^2 = \frac{p^2+\Omega^2}{c^2}$

According to Erdélyi l.c. 5.16.45 we have

$$K_o(b\sqrt{p^2+a^2}) \doteqdot \frac{\cos a\sqrt{t^2-b^2}}{\sqrt{t^2-b^2}}$$

so that

$$K_o(kr) \doteqdot \frac{\cos\Omega\sqrt{t^2-\frac{r^2}{c^2}}}{\sqrt{t^2-\frac{r^2}{c^2}}} \qquad 5.2$$

<u>c</u> If $\lambda\neq 0, \Omega=0$ we have $k^2 = \frac{p(p+\lambda)}{c^2}$.

By means of the well-known formula

$$K_o(ab) = \tfrac{1}{2}\int_0^\infty \exp - \tfrac{1}{2}(a^2u + b^2u^{-1})\,\frac{du}{u}\,,$$

we have in this case

$$K_o(kr) = \tfrac{1}{2}\int_0^\infty \exp - p\left(\frac{u}{2} + \frac{r^2}{2c^2u}\right)\frac{e^{-\frac{\lambda u}{2}}}{u}\,du\,,$$

so that at first

$$p^{-1}\,K_o(kr) \doteqdot \tfrac{1}{2}\int_0^\infty \iota\left(-u^2+2tu - \frac{r^2}{c^2}\right)\frac{e^{-\frac{\lambda u}{2}}}{u}\,du$$

$$= \tfrac{1}{2}\int_{t-\sqrt{t^2-\frac{r^2}{c^2}}}^{t+\sqrt{t^2-\frac{r^2}{c^2}}} \frac{e^{-\frac{\lambda u}{2}}}{u}\,du\,,$$

and next by differentiation

$$\Omega = 0\,,\quad K_o(kr) \doteqdot \frac{e^{-\frac{\lambda t}{2}}\operatorname{ch}\frac{\lambda}{2}\sqrt{t^2-\frac{r^2}{c^2}}}{\sqrt{t^2-\frac{r^2}{c^2}}}\;\iota\left(t-\frac{r}{c}\right)\,. \qquad 5.3$$

d In the general case we have, putting $p+\lambda = s$

$$K_o(kr) = K_o\left\{\frac{r}{c}\left(\frac{s^2+\Omega^2}{s}\right)^{\frac{1}{2}}(s-\lambda)^{\frac{1}{2}}\right\}, \qquad 5.4$$

so that by means of the auxiliary formula of c

$$K_o(kr) = \tfrac{1}{2}\int_0^\infty \exp - \tfrac{1}{2}\left\{\left(s+\frac{\Omega^2}{s}\right)u + \frac{r^2(s-\lambda)}{c^2}\,u^{-1}\right\}\frac{du}{u}.$$

Since

$$1-\exp\left(-\frac{a}{p}\right) \doteqdot \sqrt{\frac{a}{t}}\;J_o(2\sqrt{at}),$$

we have

$$K_o(kr) - \left\{K_o(kr)\right\}_{\Omega=0} \doteqdot -\frac{\Omega}{2}\,e^{-\lambda t}\int_{u_1}^{u_2}\exp\frac{\lambda r^2}{2c^2u}\,\frac{J_1\left(\Omega\sqrt{-u^2+2tu-\frac{r^2}{c^2}}\right)}{\sqrt{-u^2+2tu-\frac{r^2}{c^2}}}\,du,$$

where u_1 and u_2 are the zeros of $u^2-2tu+\frac{r^2}{c^2}$ viz.

$$u_{1,2} = t \pm \sqrt{t^2-\frac{r^2}{c^2}}\,,\qquad u_1 \leq u_2.$$

On substituting $u=t+\sin\varphi\,\sqrt{t^2-r^2/c^2}$ we obtain in view of 5.3

$$G(r,t) = \left[\frac{e^{-\frac{\lambda t}{2}} \operatorname{ch} \frac{\lambda}{2}\sqrt{t^2-\frac{r^2}{c^2}}}{\sqrt{t^2-\frac{r^2}{c^2}}} - \right.$$

$$\left. - \frac{\Omega}{2} e^{-\lambda t} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_1(\Omega \cos\varphi \sqrt{t^2-\frac{r^2}{c^2}}) \exp \frac{\lambda r^2}{2c^2(t+\sin\varphi \sqrt{t^2-\frac{r^2}{c^2}})} d\varphi \right] \iota(t-\frac{r}{c}). \qquad 5.5$$

Thus for $t < r/c$ $G(r,t)=0$.

The asymptotic behaviour of $G(r,t)$ for large λt may be obtained either directly from 5.5 or from the behaviour of its Laplace transform near $p=0$.

For small p $K_o(kr)$ may be approximated by

$$K_o(kr) = -\ln\frac{kr}{2} - \gamma - \frac{k^2r^2}{4}\ln\frac{kr}{2} + (1-\gamma)\frac{k^2r^2}{4} \ldots \qquad 5.6$$

but near $p=0$ only the following terms do contribute

$$K_o(kr) = -\tfrac{1}{2}\ln p - \frac{(\lambda^2+\Omega^2)r^2}{8\lambda c^2}\, p\ln p + O(p^2 \ln p)$$

Thus we have at once

$$G(r,t) = \frac{1}{2t} - \frac{(\lambda^2+\Omega^2)r^2}{8\lambda c^2 t^2} + O(t^{-3}), \qquad \lambda t \to \infty . \qquad 5.7$$

The behaviour of $G(r,t)$ for $t-r/c \to +0$ may be obtained either from 5.5 or from the behaviour of its Laplace transform $K_o(kr)$ for large p. Writing

$$p + \frac{\lambda}{2} = s,$$

we have

$$k = \frac{s}{c}\left\{1 + \frac{4\Omega^2-\lambda^2}{8s^2} + \ldots\right\}. \qquad 5.8$$

From the well-known asymptotic expansion of the K_o-function we may derive in view of 5.8

$$K_o(kr) \backsim \left(\frac{\pi c}{2r}\right)^{\frac{1}{2}} \frac{e^{-\frac{r}{c}s}}{s^{\frac{1}{2}}}\left(1-\frac{a_1}{s}\ldots\right), \qquad 5.9$$

where
$$a_1 = \frac{r^2(4\Omega^2-\lambda^2)+c^2}{8rc}.$$

Thus we have for the original

$$G(r,t) = \left(\frac{c}{2r}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{2}t}\left(t-\frac{r}{c}\right)^{-\frac{1}{2}}\left\{1-2a_1\left(t-\frac{r}{c}\right) + \ldots\right\},$$

which conveniently may be written in the form

$$G(r,t) = \frac{e^{-\frac{\lambda}{2}t}}{\sqrt{t^2-\frac{r^2}{c^2}}}\left\{1 - \frac{4\Omega^2-\lambda^2}{8}\left(t^2-\frac{r^2}{c^2}\right)+\dots\right\}, \quad t-\frac{r}{c}\to +0. \qquad 5.10$$

Next we consider the original of $\frac{K_0(kr)}{p}$.
For small p we have

$$\frac{K_0(kr)}{p} = -\tfrac{1}{2}\,\frac{\ln p}{p} - \left\{\gamma + \tfrac{1}{2}\ln\frac{(\lambda^2+\Omega^2)r^2}{4\lambda c^2}\right\}\frac{1}{p} + O(\ln p),$$

so that for large t

$$\frac{K_0(kr)}{p} \doteqdot \tfrac{1}{2}\ln \lambda t - \tfrac{1}{2}\left\{\gamma + \ln\frac{(\lambda^2+\Omega^2)r^2}{4c^2}\right\} + O(t^{-1}), \quad \lambda t\to\infty. \quad 5.11$$

On the other hand we have for large p

$$\frac{K_0(kr)}{p} \backsim \left(\frac{\pi c}{2r}\right)^{\frac{1}{2}}\frac{e^{-\frac{r}{c}s}}{s^{3/2}}\left(1-\frac{a_1'}{s}\dots\right),$$

where $\qquad a_1' = a_1 - \frac{\lambda}{2}.$

For the original we have again

$$\frac{K_0(kr)}{p} \doteqdot \left(\frac{2c}{r}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{2}t}\left(t-\frac{r}{c}\right)^{\frac{1}{2}}\left\{1-\frac{2}{3}a_1'\left(t-\frac{r}{c}\right)\dots\right\}, \quad t-\frac{r}{c}\to +0.$$

For the original of $\frac{K_0(kr)}{p(p+\lambda)}$ we have in a similar way

$$\frac{K_0(kr)}{p(p+\lambda)} \doteqdot \frac{1}{2\lambda}\ln t - \frac{1}{2\lambda}\left\{\gamma + \ln\frac{(\lambda^2+\Omega^2)r}{2\lambda c}\right\} + O(t^{-1}), \quad \lambda t\to\infty$$

$$\doteqdot \frac{2}{3}\left(\frac{2c}{r}\right)^{\frac{1}{2}} e^{-\frac{\lambda}{2}t}\left(t-\frac{r}{c}\right)^{3/2}\left\{1-\frac{2}{5}a_1\left(t-\frac{r}{c}\right)\dots\right\}, \quad t-\frac{r}{c}\to +0.$$

The behaviour of the originals of $\frac{\partial}{\partial r}\frac{K_0(kr)}{p}$ and of $\frac{\partial}{\partial r}\frac{K_0(kr)}{p(p+\lambda)}$

may be obtained by means of the previous formulae, or in a more direct manner. By way of illustration we consider the original of the first function for large t.
For small p we have

$$\frac{\partial}{\partial r}\frac{K_0(kr)}{p} = -\frac{k}{p}K_1(kr)$$
$$= -\frac{1}{pr} - \frac{\lambda^2+\Omega^2}{4c^2\lambda}\ln p + O(1)$$
$$\doteqdot -\frac{1}{r} + \frac{\lambda^2+\Omega^2}{4c^2\lambda}\,\frac{1}{t} + O(t^{-2}).$$

The function $G(r,t)$ has been computed for a few t and r values both for $\Omega=0$ and $\Omega=0.44$.
If further $c=1$ and $\lambda=0.08$ we have the following table (see figure 1)

		$\Omega=0$	$\Omega=0.44$
t=2	r= 0	0.463	0.308
	0.75	0.499	0.351
	1.50	0.698	0.586
	1.90	1.479	1.423
	2	∞	∞
t=6	r= 0	0.135	-0.059
	2.8	0.152	-0.065
	4.5	0.201	-0.016
	5.6	0.367	0.218
	5.95	1.018	0.959
	6	∞	∞
t=24	r= 0	0.024	0.015
	12	0.025	-0.007
	16	0.027	0.001
	20	0.033	0.019
	22	0.043	-0.017
	22.5	0.048	-0.036
	23	0.058	-0.051
	23.5	0.080	-0.039
	23.98	0.391	0.355
	24	∞	∞

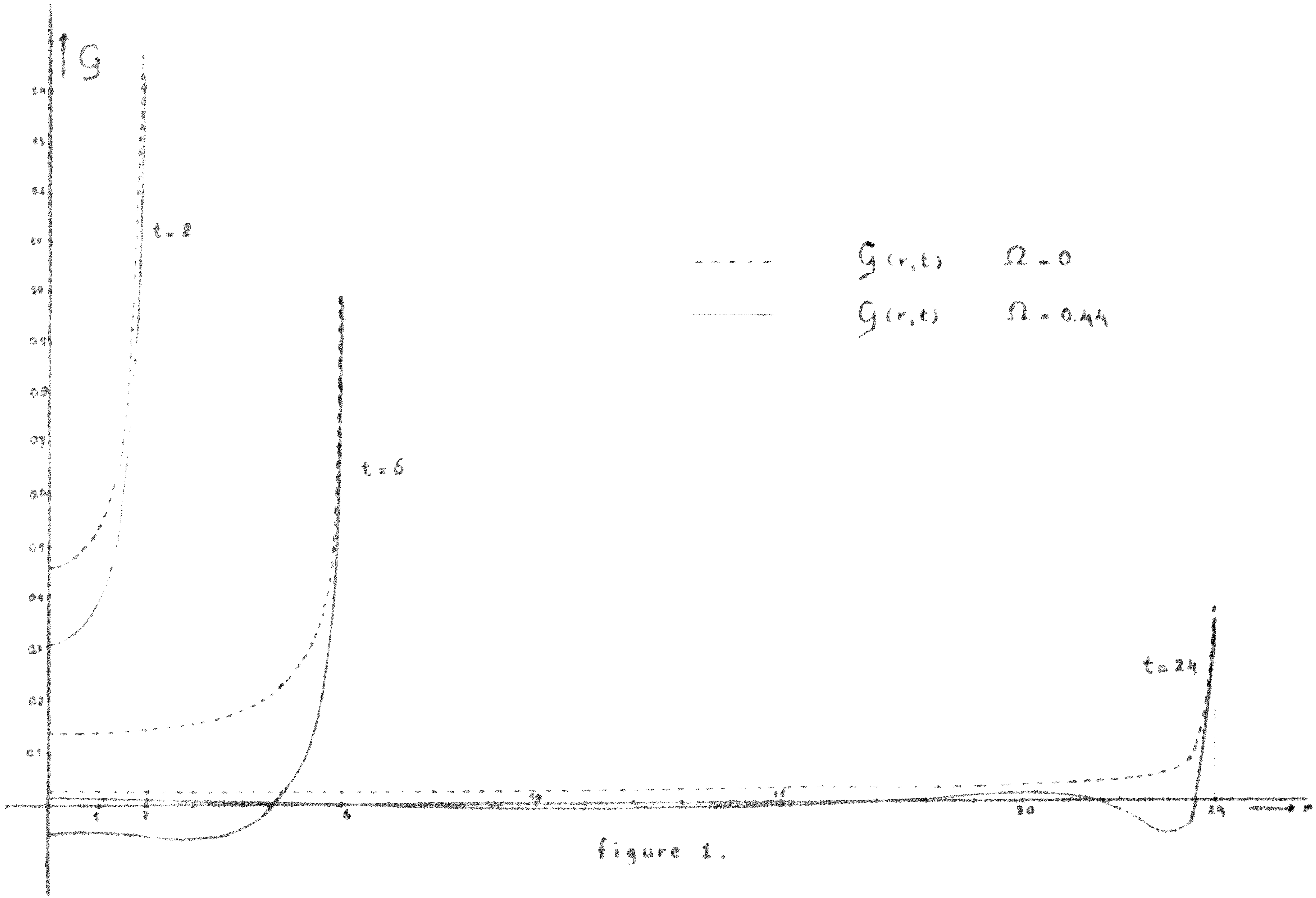

figure 1.