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Report TW 39

A note on the formulae of Plemelj

by

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July 1957

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§ 1. Introduction

If f(x) is a function defined for all real x and satisfying certain conditions as regards integrability (in the Lebesgue sense) and behaviour at infinity we may consider the following pair of analytic functions

$$\begin{cases} \phi^{+}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt & \text{Im } z > 0 , \\ \phi^{-}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt & \text{Im } z < 0 , \end{cases} \qquad 1.1$$

which are regular respectively in the upper and lower half plane. For real z the following limiting values may exist

$$\lim_{y \downarrow 0} \phi^{+}(x+iy) = \phi^{+}(x) , \qquad \lim_{y \uparrow 0} \phi^{-}(x+iy) = \phi^{-}(x), \qquad 1.2$$

and we have formally

$$\phi^{+}(x) - \phi^{-}(x) = f(x) \qquad 1.3$$

$$\phi^{+}(x) + \phi^{-}(x) = \frac{1}{\pi i} ⨍_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt , \qquad 1.4$$

where $⨍$ denotes a Cauchy integral i.e.

$$g(x) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{f(t)}{t-x}\, dt = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon}^{\infty} \frac{f(x+t)-f(x-t)}{t}\, dt. \qquad 1.5$$

The formulae 1.3 and 1.4 are called the Plemelj formulae after T. Plemelj who introduced them in 1908 [1] . It is not difficult to prove that the Cauchy integral 1.5 exists and the P emelj formulae hold if f(x) belongs to a class of hölderian functions. In Muskhelishvili's book a generalisation is made in so far that the real axis is replaced by an arbitrary smooth line.

This report is the result of an attempt to generalise the class of functions f(x) for which 1.5 exists and the Plemelj formulae 1.3 and 1.4 hold or at least hold almost everywhere. It appears that the relevant theorems are more or less explicitly contained in the chapter on Hilbert transforms in Titchmarsh [2] . They will be given here in a slightly adapted and simplified version with special reference to the concept of the Cauchy integral and the validity of the Plemelj formulae. In section 2 a simple theory

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Research carried out under the direction of Prof.Dr D. van Dantzig

will be given for functions of the $L^2(-\infty,\infty)$ class. It will be shown that if the limiting values $\phi^+(x)$, $\phi^-(x)$, $g(x)$ are interpreted as limits in the mean the Plemelj formulae hold almost everywhere. In section 3 it will be shown that $\phi^+(x)$, $\phi^-(x)$, $g(x)$ exist not only as limits in the mean but also for almost all x as ordinary limits.

Finally in section 4 it will be shown that also for the class $L(-\infty,\infty)$ the formulae of Plemelj hold for almost all x. We note that convergence in the mean does not imply convergence at any point. Wiener [3] gives an example where the limit in the mean is zero whereas the limit in the ordinary sense does not exist in any point. On the other hand convergence in the ordinary sense does not imply convergence in the mean, even if the ordinary limit exists everywhere. However, if both the limit in the mean and the ordinary limit exist, at least almost everywhere, both limits are equal almost everywhere.

In these sections we shall consider Fourier transforms of functions belonging to either the L^1 or the L^2 class. If $f(x)$ belongs to $L^2(-\infty,\infty)$ the Fourier transform of $f(x)$ is defined by

$$F(x) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-itx} f(t)dt \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} f(t)dt.$$

By the norm of $f(x)$ we shall understand

$$\|f(x)\| \overset{\text{def}}{=} \left\{ \int_{-\infty}^{\infty} |f(t)|^2 \, dt \right\}^{\frac{1}{2}} .$$

[1] Muskhelishvili. Singular integral equations. Groningen 1953. 17. Cf.also § 3 and § 43.

[2] Titchmarsh. Introduction to the theory of Fourier integrals. Oxford 1937. Ch.V.

[3] Wiener. The Fourier integral. New York 1933. p.29.

§ 2. Pure L^2 theory

Let $f(x)$ belong to $L^2(-\infty,\infty)$. Then its Fourier transform

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} f(t)dt \qquad 2.1$$

also belongs to $L^2(-\infty,\infty)$.

Since for Im $z>0$ $\frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{itu} \frac{1}{t-z} dt = \begin{cases} e^{iuz} & u>0 \\ 0 & u<0 , \end{cases}$

we obtain by means of Parseval's theorem for the function $\phi^+(z)$ from 1.1 and similarly for $\phi^-(z)$

$$\begin{cases} \phi^+(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{itz} F(t)dt & \text{Im } z>0 , \\ \phi^-(z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{itz} F(t)dt & \text{Im } z<0 . \end{cases} \qquad 2.2$$

For real x we define (also in the l.i.m. sense)

$$\begin{cases} \phi^+(x) \overset{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{itx} F(t)dt, \\ \phi^-(x) \overset{\text{def}}{=} -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{itx} F(t)dt . \end{cases} \qquad 2.3$$

<u>Theorem 1</u>

$$\begin{aligned} &\underset{y \downarrow 0}{\text{l.i.m.}}\ \phi^+(x+iy) = \phi^+(x) , \\ &\underset{y \uparrow 0}{\text{l.i.m.}}\ \phi^-(x+iy) = \phi^-(x) . \end{aligned} \qquad 2.4$$

<u>Proof</u> If $\phi_A^+(z) = \frac{1}{\sqrt{2\pi}} \int_0^{A} e^{itz} F(t)dt$ and similarly $\phi_A^+(x)$, then

$$\|\phi^+(z) - \phi^+(x)\| \le \|\phi^+(z) - \phi_A^+(z)\| + \|\phi_A^+(z) - \phi_A^+(x)\| + \|\phi_A^+(x) - \phi^+(x)\|$$

For $A \to \infty$ $\phi_A^+(z)$ converges in the mean to $\phi^+(z)$ uniformly in y since

$$\|\phi^+(z) - \phi_A^+(z)\|^2 \le \int_A^{\infty} |F(t)|^2 dt \to 0 .$$

Thus A may be determined such that the first and third term on the right-hand side of the inequality given above are less than $\varepsilon/3$, say. For a fixed A we have $\underset{y \downarrow 0}{\text{l.i.m.}}\ \phi_A^+(x+iy) = \phi_A^+(x)$ since

$$\| \phi_A^+(x+iy) - \phi_A^+(x) \|^2 = \| \frac{1}{\sqrt{2\pi}} \int_0^A e^{itx} (1-e^{-ty}) F(t)dt \|^2 =$$

$$= \int_0^A (1-e^{-ty})^2 \, | F(t) |^2 \, dt \longrightarrow 0 \text{ for } y \downarrow 0.$$

Thus also the second term of the inequality is less than $\varepsilon/3$ for y sufficiently small and positive, and the norm of $\phi^+(z) - \phi^+(x)$ is less than ε for y sufficiently small and positive. This proves the first part of the theorem. Similarly for $\phi^-(z)$.

Theorem 2

$$\phi^+(x) - \phi^-(x) = f(x) \tag{2.5}$$

$$\phi^+(x) + \phi^-(x) = \underset{\varepsilon \to 0}{\text{l.i.m}} \frac{1}{\pi i} \int_\varepsilon^\infty \frac{f(x+t)-f(x-t)}{t} dt. \tag{2.6}$$

almost everywhere.

Proof The first part of this theorem is an obvious consequence of 2.1 and 2.3. For the second part we have

$$\phi^+(x) + \phi^-(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{itx} F(t) \operatorname{sgn} t \, dt.$$

On the other hand

$$\frac{1}{\pi i} \int_\varepsilon^\infty \frac{f(x+t)-f(x-t)}{t} dt = \frac{1}{\pi i} \int_{-\infty}^\infty \frac{\varepsilon(t)}{t} f(x-t)dt ,$$

where $\varepsilon(t)$ is zero for $|t| < \varepsilon$ and -1 for $|t| > \varepsilon$.

Since $\displaystyle\int_{-\infty}^\infty e^{-itx} \frac{\varepsilon(t)}{t} dt = 2i \int_\varepsilon^\infty \frac{\sin tx}{t} dt,$

we obtain by means of the convolution theorem

$$\frac{1}{\pi i} \int_{-\infty}^\infty \frac{\varepsilon(t)}{t} f(x-t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{itx} F(t) \left\{ \frac{2}{\pi} \int_\varepsilon^\infty \frac{\sin tu}{u} du \right\} dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{itx} F(t) \operatorname{sgn} t \, dt - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{itx} F(t) . \left\{ \frac{2}{\pi} \int_0^\varepsilon \frac{\sin tu}{u} du \right\} dt.$$

The square norm of the last term equals

$$\int_{-\infty}^\infty |F(t)|^2 \left\{ \frac{2}{\pi} \int_0^\varepsilon \frac{\sin tu}{u} du \right\}^2 dt$$

and obviously tends to zero as $\varepsilon \to 0$. Thus

$$\underset{\varepsilon \to 0}{\text{l.i.m.}} \frac{1}{\pi i} \int_\varepsilon^\infty \frac{f(x+t)-f(x-t)}{t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{itx} F(t) \operatorname{sgn} t \, dt$$ and the

second part of the theorem has been proved.

§ 3 Mixed L^2 theory

Let again $f(x)$ belong to $L^2(-\infty,\infty)$. Then also

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} F(t) \operatorname{sgn} t \, dt \qquad 3.1$$

belongs to $L^2(-\infty,\infty)$.

We know already that

$$\underset{\varepsilon \to 0}{\text{l.i.m.}} \frac{1}{\pi} \int_{\varepsilon}^{\infty} \frac{f(x+t)-f(x-t)}{t} dt = g(x) .$$

We shall now prove

Theorem 3 x)

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon}^{\infty} \frac{f(x+t)-f(x-t)}{t} dt = g(x) \qquad 3.2$$

for almost all x.

We need the following two lemmas

Lemma 1 xx) If $\frac{\varphi(x)}{1+x^2}$ belongs to $L(-\infty,\infty)$ then

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon}{t^2+\varepsilon^2} \varphi(x-t)dt = \varphi(x) \qquad 3.3$$

for almost all x.

Proof

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon}{t^2+\varepsilon^2} \varphi(x-t)dt = \varphi(x) + \frac{\varepsilon}{\pi} \int_{0}^{\infty} \frac{\varphi(x+t)+\varphi(x-t)-2\varphi(x)}{t^2+\varepsilon^2} dt.$$

Since φ is an integrable function, we have for almost all x

$\lim_{u \to 0} \omega(u)=0$ xxx)

where

$$\omega(u) = \frac{1}{u} \int_{0}^{u} \left|\varphi(x+t) + \varphi(x-t)-2\varphi(x)\right| dt.$$

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x) Cf. Titchmarsh, Theory of Fourier integrals theorem 91.

xx) A generalisation of this lemma is given in Titchmarsh l.c. theorem 13.

xxx) The point set where this is true is a Lebesgue set. Cf. Titchmarsh. Theory of functions 11.6.

For such an x

$$\int_0^\infty \frac{|\varepsilon|}{t^2+\varepsilon^2}\,|\varphi(x+t)+\varphi(x-t)-2\varphi(x)|\,dt = \int_0^u + \int_u^\infty ;$$

$$\int_0^u = \frac{u|\varepsilon|}{u^2+\varepsilon^2}\,\omega(u) + \int_0^u \frac{2t^2|\varepsilon|}{(t^2+\varepsilon^2)^2}\,\omega(t)dt \leq \tfrac{1}{2}\omega(u) + \tfrac{\pi}{2}\omega(u') ,$$

where $u' < u$;

$$\int_u^\infty \leq \frac{|\varepsilon|(u^2+1)}{u^2+\varepsilon^2}\int_u^\infty \frac{|\varphi(x+t)+\varphi(x-t)-2\varphi(x)|}{t^2+1}\,dt , \text{ provided } |\varepsilon|<1.$$

Since $\int_0^u$ can be made arbitrarily small for suitable u and since for any fixed u $\int_u^\infty \to 0$ as $\varepsilon \to 0$, we have for almost all x

$$\lim_{\varepsilon\to 0}\int_0^\infty \frac{\varepsilon}{t^2+\varepsilon^2}\left\{\varphi(x+t)+\varphi(x-t)-2\varphi(x)\right\}dt = 0$$ which proves the lemma.

<u>Lemma 2</u> x) If $\frac{\varphi(x)}{1+|x|}$ belongs to $L(-\infty,\infty)$

$$\lim_{\varepsilon\to 0}\left\{\int_\varepsilon^\infty \frac{\varphi(x+t)-\varphi(x-t)}{t}\,dt - \int_0^\infty \frac{t}{t^2+\varepsilon^2}\left\{\varphi(x+t)-\varphi(x-t)\right\}dt\right\} = 0 \qquad 3.4$$

for almost all x.

<u>Proof</u>

For almost all x we have $\lim_{u\to 0}\frac{\theta(u)}{u} = 0$ where

$$\theta(u) = \int_0^u |\varphi(x+t)-\varphi(x-t)|\,dt .$$

Let x be such a point. Then the expression between brackets in 3.4 may be written as

$$\int_\varepsilon^\infty \frac{\varepsilon^2}{t(t^2+\varepsilon^2)}\left\{\varphi(x+t)-\varphi(x-t)\right\}dt - \int_0^\varepsilon \frac{t}{t^2+\varepsilon^2}\left\{\varphi(x+t)-\varphi(x-t)\right\}dt,$$

which is absolutely less than

$$\int_\varepsilon^1 \frac{\varepsilon^2}{t(t^2+\varepsilon^2)}\,d\theta(t) + \varepsilon^2\int_1^\infty \frac{|\varphi(x+t)-\varphi(x-t)|}{t}\,dt + $$
$$+ \frac{1}{2\varepsilon}\int_0^\varepsilon |\varphi(x+t)-\varphi(x-t)|\,dt.$$

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x) Cf. Titchmarsh. F.I. theorem 92.

The second and third term tend to zero as $\varepsilon \to 0$. For the first term we have

$$\int_{\varepsilon}^{1} \frac{\varepsilon^2}{t(t^2+\varepsilon^2)}\, d\,\theta(t) \le \varepsilon^2\,\theta(1) + \tfrac{1}{2}\,\frac{\theta(\varepsilon)}{\varepsilon} + \int_{\varepsilon}^{1} \frac{\varepsilon^2(3t^2+\varepsilon^2)\,\theta(t)}{t^2(t^2+\varepsilon^2)^2}\,dt$$
$$= o(1) + \int_{1}^{1/\varepsilon} \frac{3t^2+1}{t(t^2+1)^2}\,\frac{\theta(\varepsilon t)}{\varepsilon t}\,dt = o(1).$$

We now proceed to the proof of the theorem.
Since

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}\,\frac{\varepsilon}{t^2+\varepsilon^2}\,dt = \sqrt{\frac{\pi}{2}}\,e^{-\varepsilon|x|},$$

and

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}\,\frac{t}{t^2+\varepsilon^2}\,dt = i\sqrt{\frac{\pi}{2}}\,e^{-\varepsilon|x|}\,\mathrm{sgn}\,x,$$

we have according to Parseval's formula

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{t^2+\varepsilon^2}\,f(x-t)dt = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varepsilon}{t^2+\varepsilon^2}\,g(x-t)dt. \qquad 3.5$$

According to lemma 1 the right-hand side of this equality tends to $g(x)$ as $\varepsilon \to 0$ for almost all x. According to lemma 2 we have for almost all x

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon}^{\infty}\frac{f(x+t)-f(x-t)}{t}\,dt = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{0}^{\infty}\frac{t}{t^2+\varepsilon^2}\left\{f(x+t)-f(x-t)\right\}dt =$$
$$= -\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{t^2+\varepsilon^2}\,f(x-t)dt = g(x).$$

<u>Theorem 4</u>[x)]

$$\begin{cases} \lim\limits_{y\downarrow 0}\ \phi^+(x+iy) = \phi^+(x) = f(x) - ig(x) \\ \lim\limits_{y\uparrow 0}\ \phi^-(x+iy) = \phi^-(x) = -f(x) - ig(x) \end{cases} \qquad 3.6$$

for almost all x.

<u>Proof</u>

<u>a</u> $\phi^+(x+i\varepsilon) - \phi^-(x-i\varepsilon) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varepsilon}{(t-x)^2+\varepsilon^2}\,f(t)dt \longrightarrow f(x)$

for almost all x according to lemma 1.

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x) Cf. Titchmarsh. F.I. theorem 93.

b $\phi^{+}(x+i\varepsilon) + \phi^{-}(x-i\varepsilon) = -\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{t}{t^2+\varepsilon^2} f(x-t)dt =$

$$= \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\varepsilon}{t^2+\varepsilon^2} g(x-t)dt \longrightarrow -ig(x)$$

for almost all x according to 3.5 and lemma 1.

Combining both theorems we have for $\phi^{+}(x)$ and $\phi^{-}(x)$ as defined by $\lim_{y\to 0} \phi^{\pm}(x+iy)$

Theorem 5

$$\begin{cases} \phi^{+}(x) - \phi^{-}(x) = f(x) , & 3.7 \\ \phi^{+}(x) + \phi^{-}(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi i}\int_{\varepsilon}^{\infty}\frac{f(x+t)-f(x-t)}{t}\,dt. & 3.8 \end{cases}$$

§ 4 L^1 theory

If f(x) belongs to $L(-\infty,\infty)$ we have

$$\lim_{\varepsilon \to 0} \left\{ \phi^+(x+i\varepsilon) - \phi^-(x-i\varepsilon) \right\} = f(x)$$

as the proof of the first relation of theorem 4 still holds in this case.

It is, however, questionable whether the second relation of theorem 5 remains valid. The Fourier transform of f(x) exists, but does not necessarily belong to $L(-\infty,\infty)$ so that it is uncertain whether g(x) exists as the inverse transform of i F(x)sgnx. Still the results of the preceding section remain true and we have the remarkable theorem

Theorem 6 ᵡ)

If f(x) belongs to $L(-\infty,\infty)$ then

$$\phi^+(x) = \lim_{y \downarrow 0} \phi^+(x+iy)$$

and

$$\phi^-(x) = \lim_{y \uparrow 0} \phi^-(x+iy)$$

exist for almost all x and almost everywhere

$$\phi^+(x) + \phi^-(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{\varepsilon}^{\infty} \frac{f(x+t)-f(x-t)}{t}\, dt \qquad 4.1$$

We need the following lemma

Lemma 3

If $\psi(z)$ is regular for Im $z > 0$ and if

$$|\psi(z)| < C|z|^{\gamma} \qquad \text{Im } z > 0$$

for some $C > 0$ and $\gamma \geq 0$ then

$$\lim_{y \downarrow 0} \psi(x+iy)$$

exists almost everywhere.

Proof Consider the function $\psi_0(z) = \dfrac{\psi(z)}{(z+i)^{\gamma+2}}$.

The integral

$$\int e^{-izt}\, \psi_0(z)dz$$

taken along the line Im $z = y > 0$ clearly does not depend on y so

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ᵡ) Cf. Titchmarsh. F.I. theorem 105.

that there is a function $\varphi(t)$ with

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt}\, \psi_0(x+iy)dx = e^{-yt}\, \varphi(t) \qquad 4.2$$

By Parseval's theorem

$$\int_{-\infty}^{\infty} |\psi_0(x+iy)|^2\, dx = \int_{-\infty}^{\infty} e^{-2yt}\, |\varphi(t)|^2 dt\ .$$

Since the left-hand side is bounded as $y \to \infty$ we must have a.e. $\varphi(t)=0$ for $t<0$. Since it is also bounded as $y \to 0$ $\varphi(t)$ belongs to $L^2(0,\infty)$ and we have from 4.2

$$\psi_0(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{itz}\, \varphi(t)dt \qquad \text{Im } z > 0.$$

From the preceding section, theorem 4, we know that $\lim_{y \downarrow 0} \psi_0(x+iy)$ exists almost everywhere. The same is true for $\psi(x+iy)$.

We now proceed to the proof of the theorem. We have

$$\phi^+(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt$$

$$= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2+y^2}\, f(x-t)dt + \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2+y^2}\, f(x-t)dt.$$

Without loss of generality $f(x)$ may be supposed to be real and non-negative. Then

$$\phi^+(x+iy) = U + iV$$

where $U \geq 0$. The function $\exp -\phi^+(z)$ satisfies the requirements of the preceding lemma since it is uniformly bounded for $\text{Im } z > 0$. Hence $\lim_{y \downarrow 0} \exp -\phi^+(z)$ exists almost everywhere. We know already that U tends to the limit $f(x)$ for almost all x so that $\phi^+(z)$ has a finite limit for almost all x.

In particular we have proved that

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2+\varepsilon^2}\, f(x-t)dt$$

exists almost everywhere. The rest of the proof follows easily from lemma 2 of the preceding section.