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1958.

10 februari 1960

Appl. sci. Res. Section B, Vol. 8

THE SURFACE CHARGE OF A SEMI-INFINITE CYLINDER DUE TO AN AXIAL POINT CHARGE *)

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Summary
In this paper we determine the surface charge density of a semi-infinite conducting cylinder due to an axial point charge. The problem can be reduced to an integral equation of the Wiener-Hopf type. The relevant factorization problem is studied in detail.

§ 1. *Introduction.* The problem of the determination of the field of an axial point charge inside a hollow infinitely long conducting cylinder has been considered by various writers [1]).

If there is a point charge -1 at the origin and if in cartesian coordinates (x, y, z) the position of the cylinder is determined by $-\infty < x < \infty$, $r = \sqrt{y^2 + z^2} = 1$ the surface charge at the cylinder is

$$\sigma(x) = \frac{1}{2\pi^2} \int_0^\infty \frac{\cos tx}{\mathrm{I}_0(t)} \, \mathrm{d}t. \qquad (1.1)$$

In this paper we shall consider the equivalent problem for a semi-infinite cylinder $0 < x < \infty$, $r = 1$. This problem is much more complicated. The determination of the surface charge $\sigma(x)$, which also depends on the position of the point charge on the axis, involves the solving of a Wiener-Hopf equation of the first kind

$$\int_0^\infty h(x - t)\, \sigma(t)\, \mathrm{d}t = g(x),\ x > 0 \qquad (1.2)$$

with $h(x) \backsim -\ln x^2$ for $x \to 0$. If $\sigma(x)$ is known, the electrostatic

*) This paper is a revised version of a solution of a prize question put by the Wiskundig Genootschap in 1955.

field $V(r, x)$ may be easily determined (cf. (2.3)). We shall therefore confine our attention to the determination of $\sigma(x)$ from (1.2).

The Wiener-Hopf equation (1.2) cannot be solved by the familiar methods since in this case the strip of convergence is absent. There is only a line of convergence, the real axis, upon which the Fourier transform of the kernel function has a logarithmic singularity at the origin. However, the integral equation (1.2) may be solved by means of the methods developed by Muskhelishvili [2]; cf. also [3] and [4]. In particular the notion of sectionally holomorphic functions plays an important role in the solution of (1.2) [5]. In § 3 this will be discussed in detail. An explicit expression is obtained for the Fourier transform of the surface charge $\sigma(x)$ depending on the proper factorization of the Fourier transform $H(x)$ of $h(x)$.

The explicit factorization is carried out in § 4. Various integral representations and series expansions are derived. This may lead to an expansion of $\sigma(x)$ for small x in § 5, or for large x in § 7. If the point charge is at the open end (0, 0) of the cylinder $0 < x < \infty$, $r = 1$, we have e.g.

$$2\pi\sigma(x) = 0.337x^{-1/2} + 0.527x^{1/2} - 0.782^{3/2} - 0.961^{5/2} \ldots . \quad (1.3)$$

Special expressions are derived in § 6 for the case that the point charge is far inside or outside the cylinder.

§ 2. *Reduction of the problem to an integral equation.* The electrostatic field $V(r, x)$ of a point charge at $(0, a)$ of intensity -1 in the presence of a semi-infinite conducting cylinder $r = 1$, $0 < x < \infty$ is determined by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{\partial^2 V}{\partial x^2} = 0, \quad (2.1)$$

$$V = 0 \quad \text{for} \quad r = 1, x > 0,$$

$$V = -[r^2 + (x-a)^2]^{-\frac{1}{2}} + O(1) \quad \text{for} \quad r^2 + (x-a)^2 \to 0.$$

The induced charge density at the cylinder is

$$\sigma(x) = -\frac{1}{4\pi}\frac{\partial V}{\partial r}\bigg|_{1-0}^{1+0}. \quad (2.2)$$

Conversely, if $\sigma(x)$ is known, the potential $V(r, x)$ may be determined from

$$V(r, x) = -[r^2 + (x - a)^2]^{-\frac{1}{2}} + \\ + \int_0^\infty \sigma(t)\, dt \int_0^{2\pi} [r^2 + 1 + 2r\cos\theta + (x - t)^2]^{-\frac{1}{2}}\, d\theta$$

or

$$V(r, x) = -[r^2 + (x - a)^2]^{-\frac{1}{2}} + \int_0^\infty h(r, x - t)\, \sigma(t)\, dt, \tag{2.3}$$

where

$$h(r, x) = 4[(r + 1)^2 + x^2]^{-\frac{1}{2}}\, K\{2r^{\frac{1}{2}}[(r + 1)^2 + x^2]^{-\frac{1}{2}}\}. \tag{2.4}$$

The function $K(k)$ is the complete elliptic integral

$$K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^{-\frac{1}{2}}\, d\varphi = \tfrac{1}{2}\pi\, F(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2).$$

For $r = 1$, $x > 0$ from (2.3) the following integral equation for $\sigma(x)$ is obtained:

$$\int_0^\infty h(x - t)\, \sigma(t)\, dt = g(x), \tag{2.5}$$

where

$$h(x) = 4(x^2 + 4)^{-\frac{1}{2}}\, K[2(x^2 + 4)^{-\frac{1}{2}}], \tag{2.6}$$

and

$$g(x) = [1 + (x - a)^2]^{-\frac{1}{2}}. \tag{2.7}$$

Equation (2.5) is of the well-known Wiener-Hopf type. In view of

$$h(x) \backsim -\ln x^2, \quad x \to 0$$

the integral equation is singular.

§ 3. *Solution of the integral equation.* By $F^+(z)$ we shall understand a function of the complex variable $z = x + iy$ which is holomorphic in the upper half-plane $y > 0$ and which vanishes at infinity. In a similar way $F^-(z)$ is holomorphic in the lower half-plane $y < 0$ and vanishes at infinity. At the real axis we define

$$F^+(x) = \lim_{y \downarrow 0} F^+(x + iy) \quad F^-(x) = \lim_{y \uparrow 0} F^-(x + iy).$$

Consider now the Wiener-Hopf equation (2.5) in the form

$$\int_0^\infty h(x - t)\, \sigma(t)\, dt = \begin{cases} g(x) & x > 0, \\ \chi(x) & x < 0, \end{cases} \tag{3.1}$$

where $\sigma(x)$, $x > 0$ and $\chi(x)$, $x < 0$ are unknown functions. Let $H(x)$, $S^+(z)$, $G^+(z)$, $X^-(z)$ represent the following Fourier transforms:

$$H(x) = \int_{-\infty}^{\infty} e^{itx}\, h(t)\, dt,$$

$$S^+(z) = \int_0^{\infty} e^{itz}\, \sigma(t)\, dt, \quad G^+(z) = \int_0^{\infty} e^{itz}\, g(t)\, dt,$$

$$X^-(z) = \int_{-\infty}^{0} e^{itz}\, \chi(t)\, dt),$$

then Fourier transformation of (3.1) gives

$$H(x)S^+(x) = G^+(x) + X^-(x). \tag{3.2}$$

The kernel function $H(x)$ can be factorized as the product of limit functions of sectionally holomorphic functions $H^+(z)$ and $H^-(z)$:

$$H(x) = H^+(x)\, H^-(x). \tag{3.3}$$

Then (3.2) may be written as follows:

$$H^+(x)\, S^+(x) - \frac{X^-(x)}{H^-(x)} = \frac{G^+(x)}{H^-(x)}\,. \tag{3.4}$$

If the factorization (3.3) is carried out properly, the relation (3.4) is of the form

$$\phi^+(x) - \phi^-(x) = \varphi(x). \tag{3.5}$$

The solution of the problem (3.5) is unique if ϕ^+, ϕ^- and φ belong to any finite L^2-class, i.e. for $-A < x < A$ for any positive A. We have

$$\phi^\pm(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\varphi(t)}{t - z}\, dt, \tag{3.6}$$

and

$$\phi^+(x) = \tfrac{1}{2}\varphi(x) + \frac{1}{2\pi i} ⨍_{-\infty}^{\infty} \frac{\varphi(t)}{t - x}\, dt, \tag{3.7}$$

where

$$⨍_{-\infty}^{\infty} \frac{\varphi(t)}{t - x}\, dt \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\infty} \frac{\varphi(x + t) - \varphi(x - t)}{t}\, dt. \tag{3.8}$$

Thus the following explicit solution of (3.2) is obtained:

$$S^+(z) = \frac{1}{2\pi i H^+(z)} \int_{-\infty}^{\infty} \frac{G^+(t)}{H^-(t)} \frac{\mathrm{d}t}{t - z}. \tag{3.9}$$

From $S(z)$ the surface charge $\sigma(x)$ may be derived by inverse Fourier transformation.

In view of

$$\int_{-\infty}^{\infty} \phi^-(t) \frac{\mathrm{d}t}{t - z} = 0 \quad \text{for} \quad \operatorname{Im} z > 0$$

the right-hand side of (3.9) may be replaced by

$$S^+(z) = \frac{1}{2\pi i\, H^+(z)} \int_{-\infty}^{\infty} \frac{G(t)}{H^-(t)} \frac{\mathrm{d}t}{t - z}, \tag{3.10}$$

where

$$G(x) = \int_{-\infty}^{\infty} \mathrm{e}^{itx} g(t)\, \mathrm{d}t.$$

We have

$$G(x) = 2\mathrm{e}^{iax}\, \mathrm{K}_0\,(|x|) \tag{3.11}$$

and

$$H(x) = 4\pi \mathrm{I}_0(x) \mathrm{K}_0(|x|), \tag{3.12}$$

cf. Erdélyi, Tables of Integral Transforms (1.14.13).

§ 4. *Factorization of* $H(x)$. Introduce the function

$$L(x) \overset{\text{def}}{=} \frac{|x|}{2\pi} H(x) = 2|x|\, \mathrm{I}_0(x)\, \mathrm{K}_0(|x|). \tag{4.1}$$

For $x \to \infty$ we have

$$\ln L(x) = \frac{1}{8x^2} + \frac{13}{64x^4} + O(x^{-6}). \tag{4.2}$$

Then the factorization of $H(x)$ is obtained as follows

$$H^+(z) = \left(\frac{2\pi}{z}\right)^{\frac{1}{2}} \exp\left[\frac{\pi i}{4} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln L(t)}{t - z}\, \mathrm{d}t\right], \operatorname{Im} z \geqq 0, \tag{4.3}$$

$$H^-(z) = \left(\frac{2\pi}{z}\right)^{\frac{1}{2}} \exp\left[-\frac{\pi i}{4} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln L(t)}{t-z}\,\mathrm{d}t\right], \ \mathrm{Im}\, z \leqq 0. \quad (4.4)$$

From (4.3) we obtain for $z \to x$

$$H^+(x) = 2\,[\pi \mathrm{I}_0(x)\ \mathrm{K}_0(|x|)]^{\frac{1}{2}} \exp\left[\frac{\pi i}{4}\,\mathrm{sgn}\, x + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln L(t)}{t-x}\,\mathrm{d}t\right]; \quad (4.5)$$

also

$$H^-(x) = \overline{H^+(x)}, \quad (4.6)$$

$$\left.\begin{aligned} H^+(xe^{\pi i}) &= H^-(x), \\ H^-(xe^{-\pi i}) &= H^+(x), \end{aligned}\right\} \quad \text{for } x > 0. \quad (4.7)$$

The asymptotic behaviour of $H^+(z)$ may be determined as follows. We have

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln L(t)}{t-z}\,\mathrm{d}t = \sum_1 \frac{\alpha_k}{s^k}$$

where $s = -iz$, Re $s > 0$. The first few coefficients are

$$\alpha_1 = \frac{1}{\pi}\int_0^{\infty} \ln L(t)\,\mathrm{d}t = -0.0280, \quad \alpha_2 = -\frac{1}{16},$$

$$\alpha_3 = -\frac{1}{\pi}\int_0^{\infty} t^2\left[\ln L(t) - \frac{1}{8t^2}\right]\mathrm{d}t = -0.00298, \quad \alpha_4 = \frac{13}{128}.$$

Hence we have

$$z \to \infty \qquad H^+(z) = \left(\frac{2\pi}{s}\right)^{\frac{1}{2}} \exp \sum_1 \frac{\alpha_k}{s^k}, \quad s = -iz. \quad (4.8)$$

For $H^+(x)$ an alternative expression may be derived in the following way. We need the auxiliary function

$$\Omega(x) \stackrel{\mathrm{def}}{=} \arg\,[-\mathrm{Y}_0(x) + i\mathrm{J}_0(x)], \quad x > 0, \quad (4.9)$$

with $0 \leqq \Omega(x) \leqq \pi$. Without proof we mention the following simple properties of this function:

$$\Omega'(x) = \frac{2}{\pi x[\mathrm{J}_0^2(x) + \mathrm{Y}_0^2(x)]}. \quad (4.10)$$

For $x \to \infty$ we have

$$\Omega'(x) = 1 + \frac{1}{8x^2} - \frac{25}{128x^4} + O(x^{-6}). \tag{4.11}$$

For $x \to +0$ we have

$$\Omega(x) = \frac{\pi}{2 \ln (2/x)} + O\left(\ln^{-2} \frac{2}{x}\right). \tag{4.12}$$

At the *kth* zero β_k of $J_0(x)$, $k \geqq 1$, we have

$$\Omega(\beta_k - 0) = 0, \ \Omega(\beta_k + 0) = \pi. \tag{4.13}$$

We shall now prove the following formula:

$$\arg H^+(x) = \frac{x}{\pi} \int_0^\infty \frac{\Omega(t)}{t^2 + x^2} \, dt. \tag{4.14}$$

According to (4.5) we have

$$\arg H^+(x) = \frac{\pi}{4} \operatorname{sgn} x - \frac{x}{\pi} \int_0^\infty \frac{\ln L(t)}{t^2 - x^2} \, dt. \tag{4.15}$$

This odd function is the sine transform of the even function

$$\frac{1}{2y} + \frac{1}{\pi} \int_0^\infty \cos yt \ln L(t) \, dt, \quad y > 0.$$

This function may be reduced as follows

$$\frac{1}{2y} - \frac{1}{\pi y} \int_0^\infty \sin yt \, \frac{L'(t)}{L(t)} \, dt =$$

$$= \frac{1}{2y} - \frac{1}{\pi y} \operatorname{Im} \int_0^\infty e^{iyt} \left[\frac{1}{t} + \frac{I_0'(t)}{I_0(t)} + \frac{K_0'(t)}{K_0(t)} \right] dt =$$

$$= -\frac{1}{y} \sum_1^\infty e^{-\beta_k y} + \frac{1}{\pi y} \operatorname{Im} \int_0^\infty e^{-yt} \, \frac{J_0'(t) - iY_0'(t)}{J_0(t) - iY_0(t)} \, dt =$$

$$= -\frac{1}{y} \left[\sum_1^\infty e^{-\beta_k y} - \frac{1}{\pi} \int_0^\infty e^{-yt} \, \Omega'(t) \, dt \right] = \frac{1}{\pi} \int_0^\infty e^{-yt} \, \Omega(t) \, dt.$$

The expression last obtained is the sine transform of the righthand side of (4.14). Thus we have

$$H^+(x) = 2[\pi I_0(x) K_0(|x|)]^{\frac{1}{2}} \exp \frac{xi}{\pi} \int_0^\infty \frac{\Omega(t)}{t^2 + x^2} \, dt. \tag{4.16}$$

We note that for a purely imaginary z the function $H^+(z)$ is real. We have from (4.3) and (4.4)

$$H^+(si) = H^-(-si) = \left(\frac{2\pi}{s}\right)^{\frac{1}{2}} \exp \frac{s}{\pi} \int_0^\infty \frac{\ln L(t)}{t^2 + s^2} \, dt. \tag{4.17}$$

Finally we shall consider the behaviour of $H^+(z)$ near the origin. For $x > 0$ we have

$$\ln L(x) = \ln 2x + \ln \ln \frac{2}{x} + O\left(\ln^{-1} \frac{2}{x}\right).$$

Since

$$\frac{z}{\pi i} \int_0^\infty \frac{\ln 2t}{t^2 - z^2} \, dt = \tfrac{1}{2} \ln 2z - \frac{\pi i}{4}, \ \text{Im}\, z > 0,$$

and

$$\frac{z}{\pi i} \int_0^\infty \frac{\ln \ln 2/t}{t^2 - z^2} \, dt = \tfrac{1}{2} \ln \ln \frac{2}{z} + O\left(\ln^{-1} \frac{2}{z}\right), \ \text{Im}\, z > 0,$$

we obtain

$$z \to 0, \ H^+(z) = 2\left(\pi \ln \frac{2}{z}\right)^{\frac{1}{2}} \left[1 + O\left(\ln^{-1} \frac{2}{z}\right)\right]. \tag{4.18}$$

At the positive real axis we have

$$\begin{aligned} H^+(x) &= 2\left(\pi \ln \frac{2}{x}\right)^{\frac{1}{2}} \left[1 + \frac{\gamma - \frac{1}{2}\pi i}{2 \ln \frac{1}{2}x} + O\left(\ln^{-2} \frac{2}{x}\right)\right], \\ H^-(x) &= 2\left(\pi \ln \frac{2}{x}\right)^{\frac{1}{2}} \left[1 + \frac{\gamma + \frac{1}{2}\pi i}{2 \ln \frac{1}{2}x} + O\left(\ln^{-2} \frac{2}{x}\right)\right]. \end{aligned} \tag{4.19}$$

The argument of $H^+(x)$ as a function of x is given in table I.

TABLE I

x	$\arg H^+(x)$	x	$\arg H^+(x)$
0.00	0	1.4	0.739
0.05	0.278	1.6	0.749
0.1	0.349	1.8	0.757
0.15	0.401	2	0.762
0.2	0.441	3	0.772
0.3	0.506	4	0.778
0.4	0.554	5	0.779
0.5	0.593	6	0.780
0.6	0.624	7	0.781
0.8	0.670	8	0.781
1.0	0.701	9	0.782
1.2	0.723	10	0.782
		∞	$\pi/4 = 0.785$

§ 5. *Expansion of $\sigma(x)$ for small x.* The properties of the functions $H^{\pm}(z)$ derived in the previous section validate the derivation of the solution of the Wiener-Hopf equation (2.5). We have obtained according to (3.10) and (3.11).

$$S^+(z) = \frac{1}{\pi i H^+(z)} \int_{-\infty}^{\infty} e^{iat} \frac{K_0(|t|)}{H^-(t)} \frac{dt}{t-z}, \tag{5.1}$$

or

$$S^+(z) = \frac{1}{4\pi^2 i H^+(z)} \int_{-\infty}^{\infty} e^{iat} \frac{H^+(t)}{I_0(t)} \frac{dt}{t-z}. \tag{5.2}$$

The asymptotic expansion of $S^+(z)$ for $z \to \infty$ may be determined as follows.

According to (4.8) we have

$$\frac{1}{H^+(z)} = \left(\frac{s}{2\pi}\right)^{\frac{1}{2}} \exp - \sum_{1} \frac{\alpha_k}{s^k},$$

where $s = -iz$, Re $s > 0$. This may be brought into the form

$$\frac{1}{H^+(z)} = \left(\frac{s}{2\pi}\right)^{\frac{1}{2}} \sum_{0} \frac{a_k}{s^k}, \tag{5.3}$$

where

$$a_0 = 1,\ a_1 = -\alpha_1 = 0.0280,$$

$$a_2 = -\alpha_2 + \frac{\alpha_1^2}{2} = -0.0629,\ a_3 = -\alpha_3 + \alpha_1\alpha_2 - \frac{\alpha_1^3}{6} = 0.00473.$$

Furthermore we have the asymptotic expansion

$$\frac{1}{\pi i}\int_{-\infty}^{\infty} e^{iat}\,\frac{\mathrm{K}_0(|t|)}{H^-(t)}\,\frac{\mathrm{d}t}{t-z} = 2^{-\frac{1}{2}} \sum_0 \frac{b_k}{s^{k+1}}, \tag{5.4}$$

where

$$b_k = \frac{2^{-\frac{1}{2}}}{\pi}\,\mathrm{Re}\int_0^{\infty} \exp\left[i\left(\frac{k\pi}{2} - at\right)\right] t^k\,\frac{\mathrm{K}_0(t)}{H^+(t)}\,\mathrm{d}t$$

or, in view of (4.16),

$$b_k = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}\int_0^{\infty} t^k \left[\frac{K_0(t)}{\mathrm{I}_0(t)}\right]^{\frac{1}{2}} \cos\left[\frac{k\pi}{2} - A(t) - at\right]\mathrm{d}t, \tag{5.5}$$

where

$$A(x) \stackrel{\mathrm{def}}{=} \arg H^+(x) = \frac{x}{\pi}\int_0^{\infty} \frac{\Omega(t)}{t^2 + x^2}\,\mathrm{d}t. \tag{5.6}$$

By means of (5.3) and (5.4) we obtain from (5.1)

$$2\pi S^+(z) = \sqrt{\pi}\sum_0^{\infty} c_k s^{-k-\frac{1}{2}}, \quad z = si \tag{5.7}$$

where

$$c_k = \sum_0^{k} a_j b_{k-j}. \tag{5.8}$$

Accordingly we obtain for the inverse Fourier transform $\sigma(x)$ the following expansion, presumably convergent for small x:

$$2\pi\sigma(x) = \sum_0^{\infty} \frac{\Gamma(\frac{1}{2})}{\Gamma(k+\frac{1}{2})}\, c_k x^{k-\frac{1}{2}}. \tag{5.9}$$

In the case $a = 0$ the first few coefficients are

$$\begin{aligned} b_0 &= 0.337, & c_0 &= 0.337,\\ b_1 &= 0.254, & c_1 &= 0.263,\\ b_2 &= -0.614, & c_2 &= -0.586,\\ b_3 &= -1.803, & c_3 &= -1.803, \end{aligned}$$

so that

$$2\pi\sigma(x) = 0.337\,x^{-\frac{1}{2}}\,(1 + 1.56x - 2.32x^2 - 2.85x^3 \ldots). \tag{5.10}$$

From this we obtain $\sigma(x)$ for a few x values (table II).

TABLE II

x	$\sigma(x)$	x	$\sigma(x)$
0.001	10.69	0.02	2.46
0.002	7.57	0.05	1.62
0.003	6.19	0.1	1.21
0.004	5.37	0.2	0.92
0.005	4.81	0.5	0.57
0.01	3.43		

§ 6. *Expressions for large* $|a|$. We shall consider the case $a > 0$ and $a < 0$ separately. If $a > 0$, we have from (5.2)

$$S^{+}(z) = \frac{e^{iaz}}{2\pi I_0(z)} - \frac{1}{2\pi H^{+}(z)} \sum_{1}^{\infty} \frac{h_k e^{-a\beta_k}}{(iz + \beta_k)\, J_1(\beta_k)}, \tag{6.1}$$

where

$$h_k = H^{+}(i\beta_k).$$

The inverse transformation of (6.1) gives

$$\sigma(x) = \sigma_0(x - a) + \frac{1}{2\pi} \sum_{1}^{\infty} \frac{h_k e^{-a\beta_k}}{J_1(\beta_k)} B(\beta_k, x), \tag{6.2}$$

where $\sigma_0(x - a)$ represents the surface charge of a double-infinite cylinder due to a point charge -1 at $x = a$, $r = 1$, and where the function $B(\beta, x)$ is the inverse Fourier transform of

$$[H^{+}(z)(-iz - \beta)]^{-1}$$

or

$$B(\beta, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \frac{dt}{H^{+}(t)(-it - \beta)}. \tag{6.3}$$

If $a < 0$, we obtain from (5.1) by taking both sides of the negative imaginary axis as new path of integration

$$S^{+}(z) = \frac{1}{H^{+}(z)} \int_{0}^{\infty} e^{at} \frac{J_0(t)}{H^{+}(ti)} \frac{dt}{t - iz}, \tag{6.4}$$

from which

$$\sigma(x) = \int_{0}^{\infty} e^{at} \frac{J_0(t)}{H^{+}(ti)} B(-t, x)\, dt. \tag{6.5}$$

The expressions (6.2) and (6.5) are useful if $|a|$ is large. If, however, $|a|$ is small, we may proceed as follows. From (5.1) or (5.2) we obtain at once by inverse transformation

$$\sigma(x) = \sigma_0(x-a) + \frac{1}{2\pi^2 i}\int_L \frac{e^{-iux}}{H^+(u)}\,du \int_{-\infty}^{\infty} e^{iat}\,\frac{K_0(|t|)}{H^-(t)}\,\frac{dt}{t-u},$$

where L is a contour along both sides of the negative imaginary axis. This may be written as

$$\sigma(x) = \sigma_0(x-a) + \frac{1}{\pi}\int_{-\infty}^{\infty} e^{iat}\,\frac{K_0(|t|)}{H^-(t)}\,B(-ti, x)\,dt, \tag{6.6}$$

where

$$B(\beta, x) = \frac{1}{2\pi}\int_L e^{-itx}\,\frac{dt}{H^+(t)(-it-\beta)}, \quad \arg\beta = -\frac{\pi}{2}. \tag{6.7}$$

The functions $B(\beta, x)$ which are used in (6.2), (6.5) and (6.6) may be reduced as follows

a. β real and positive:

$$B(\beta, x) = \frac{1}{8\pi^2}\int_0^{\infty} e^{-itx}\,\frac{H^-(t)}{I_0(t)\,K_0(|t|)}\,\frac{dt}{-it-\beta} =$$

$$= \frac{1}{8\pi^2}\int_{-\infty}^{\infty} e^{-itx}\left[\frac{I_0'(t)}{I_0(t)} - \frac{t}{|t|}\,\frac{K_0'(|t|)}{K_0(|t|)}\right]\frac{tH^-(t)}{-it-\beta}\,dt =$$

$$= \frac{1}{4\pi}\sum_1^{\infty}\frac{\beta_j h_j}{\beta_j+\beta}\,e^{-\beta_j x} - \frac{1}{4\pi^2}\,\mathrm{Re}\int_0^{\infty} e^{-tx}\,\frac{K_0'(-ti)}{K_0(-ti)}\,\frac{tH^-(-ti)}{t+\beta}\,dt$$

and finally

$$B(\beta, x) = \frac{1}{4\pi}\sum_1^{\infty}\frac{\beta_j h_j}{\beta_j+\beta}\,e^{-\beta_j x} + \frac{1}{2\pi^3}\int_0^{\infty} e^{-tx}\,\frac{H^+(ti)}{J_0^2(t)+Y_0^2(t)}\,\frac{dt}{t+\beta}. \tag{6.8}$$

b. β real and negative:

In a similar way we obtain

$$B(\beta, x) = \frac{1}{4\pi}\sum_1^{\infty}\frac{\beta_j h_j}{\beta_j+\beta}\,e^{-\beta_j x} - \frac{\beta J_1(\beta)}{4\pi J_0(\beta)}\,H^+(-\beta i)\,e^{\beta x} +$$

$$+ \frac{1}{2\pi^3}\int_0^{\infty} e^{-tx}\,\frac{H^+(ti)}{J_0^2(t)+Y_0^2(t)}\,\frac{dt}{t+\beta} \tag{6.9}$$

c. β purely imaginary:

In this case the expression (6.8) may be used.

The expressions (6.8) and (6.9) are reasonable convergent if x is large. For small x we better take the expansion derived in the previous section.

§ 7. *Behaviour of* $\sigma(x)$ *for* $x \to \infty$. The behaviour of $\sigma(x)$ for $x \to \infty$ may be determined from the behaviour of $S^+(z)$ for $z \to 0$. From (5.2) we obtain for small z

$$S^+(z) = \frac{e^{iaz}}{2\pi I_0(z)} + \frac{1}{4\pi^2 i H^+(z)} \int_{L'} e^{iat} \frac{H^+(t)}{I_0(t)} \frac{dt}{t - z}, \tag{7.1}$$

where L' is the real axis with a semi-circular indentation at the origin which separates z from the poles $i\beta_k$, $k = 1,2 \ldots$ of $I_0^{-1}(t)$. Then for $z \to 0$ we obtain from (7.1) in view of (4.18)

$$2\pi S^+(z) = 1 - \frac{C}{2}\left(\pi \ln \frac{2}{z}\right)^{-\frac{1}{2}} + O\left[\left(\ln \frac{2}{z}\right)^{-3/2}\right], \tag{7.2}$$

where

$$C = -\frac{1}{2\pi i} \int_{L'} e^{iat} \frac{H^+(t)}{t I_0(t)} dt. \tag{7.3}$$

For $\sigma(x)$ we find accordingly

$$\sigma(x) = \frac{C}{8\pi^{\frac{1}{2}}} \frac{1}{x(\ln x)^{3/2}} + O\left[x^{-1} (\ln x)^{-5/2}\right]. \tag{7.4}$$

The coefficient C may be expressed as follows for $a > 0$:

$$C = \sum_{1}^{\infty} \frac{H^+(i\beta_k)\, e^{-a\beta_k}}{\beta_k J_1(\beta_k)}. \tag{7.5}$$

Received 10th February, 1960.

REFERENCES

1) Bouwkamp, C. J. and N. G. de Bruijn, J. Appl. Phys. **18** (1947) 562.
2) Muskhelishvili, N. I., Singular integral equations, Groningen, 1953.
3) Titchmarsh, E. C., Theory of Fourier integrals, Oxford 1937. See in particular Ch. 5 on Hilbert transforms.
4) Sparenberg, J. A., Proc. K. Ned. Akad. Wet. A **59** (1956) 29.
5) See ref.[2]), § 15 and § 26.