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2e BOERHAAVESTRAAT 49
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Free motions in a rotating sea which has the form of a semi-infinite strip

by

H.A. Lauwerier

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Contents

§ 1. Introduction

In this report some aspects of the work carried out by the Applied Mathematics Division of the Amsterdam Mathematical Centre in connection with the problem of the motion of the North Sea are considered.

We shall study the free motions of a rotating shallow sea of constant depth which has the form of a semi-infinite strip $0 < x < a$, $0 < y < \infty$.

The differential equations and the boundary equations are

$$\begin{cases} (\frac{\partial}{\partial t} + \lambda)u - \Omega v + gd \frac{\partial \zeta}{\partial x} = 0 \\ (\frac{\partial}{\partial t} + \lambda)v + \Omega u + gd \frac{\partial \zeta}{\partial y} = 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0 \end{cases} \qquad 1.1$$

$$\begin{cases} x = 0 \text{ and } x = a & u = 0, \\ y = 0 & v = 0. \end{cases} \qquad 1.2$$

Here g is the acceleration of gravity, λ a friction coefficient, Ω the coefficient of Coriolis, d the depth, u and v integrals over a vertical of the horizontal components of the velocity, ζ the elevation of the sealevel above the undisturbed level.

The North Sea is usually represented by a rectangle $0 < x < a$, $0 < y < b$ where $x=0$, $x=a$ and $y=0$ are coasts and $y=b$ the open end at the ocean. In this report the influence of the ocean is not taken into account. The numerical values of the constants are those for the North Sea.

$$\lambda = 0.09 \text{ h}^{-1}, \quad \Omega = 0.48 \text{ h}^{-1}, \quad d = 65 \text{ m}, \quad a = 425 \text{ km}$$

If the following units are introduced

t	$a/\pi\sqrt{gd}$	h	x, y	a/π	km
u, v	$d\sqrt{gd}$	km/h	ζ	d	m

1.1 and 1.2 become

$$\begin{cases} \left(\frac{\partial}{\partial t}+\lambda\right)u - \Omega v + \frac{\partial \zeta}{\partial x} = 0 \\ \left(\frac{\partial}{\partial t}+\lambda\right)v + \Omega u + \frac{\partial \zeta}{\partial y} = 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0 \,. \end{cases} \qquad 1.3$$

$u=0$ for $x=0$, $y>0$ $\quad$ $v=0$ for $y=0$, $0<x<\pi$ $\quad$ 1.4
$u=0$ for $x=\pi$, $y>0$.

In the numerical case the units are as follows

t	1.5 h	u,v	91 km
x,y	135 km	ζ	65 m ,

and

$\lambda = 0.14$, $\Omega = 0.71$; (we shall take $\lambda^2=0.02$, $\Omega^2=0.5$).

We shall consider free motions which are proportional to e^{pt}. If in u,v,ζ in 1.3 the time factor is omitted we have

$$\begin{cases} (p+\lambda)u - \Omega v + \frac{\partial \zeta}{\partial x} = 0 \\ (p+\lambda)v + \Omega u + \frac{\partial \zeta}{\partial y} = 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + p\zeta = 0 \,, \end{cases} \qquad 1.5$$

with $u=0$ for $x=0$ and $x=\pi$, $\quad v=0$ for $y=0$. $\quad$ 1.6

Only solutions of 1.5, 1.6 will be considered which are bounded as $y \to \infty$.

From the equations 1.5 we obtain easily

$$(p+\lambda)(uu^* + vv^*) + p^*\zeta\zeta^* = \Omega(u^*v - uv^*) - \frac{\partial}{\partial x}(u^*\zeta) - \frac{\partial}{\partial y}(v^*\zeta) \,, \qquad 1.7$$

where u^*, v^*, p^* are the complex conjugates of u,v,p.

If

$$E_u \overset{\text{def}}{=} \lim_{b\to\infty} \frac{1}{\pi b} \int_0^{\pi} dx \int_0^{b} uu^* \, dy \quad \text{etc.}$$

then we obtain from 1.7 in view of 1.6

$$(p+\lambda)(E_u+E_v) + p^* E\zeta = F,$$

where F is purely imaginary. Therefore we have

$$(\mathrm{Re}\, p+\lambda)(E_u+E_v) + \mathrm{Re}\, p\; E\zeta = 0. \qquad 1.8$$

Thus we have for a free motion

$$-\lambda < \mathrm{Re}\ p < 0 \ . \qquad 1.9$$

This means that any free motion is damped out as $t \to \infty$.
If $\lambda=0$ the free motions are purely oscillatory.

§2. The case $\lambda=0$

The equations 1.5 become

$$\begin{cases} pu - \Omega v + \frac{\partial \zeta}{\partial x} = 0 \\ pv + \Omega u + \frac{\partial \zeta}{\partial y} = 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + p\zeta = 0 \ , \end{cases} \qquad 2.1$$

with

$$u=0 \text{ at } x=0 \text{ and } x=\pi \ , \quad v=0 \text{ at } y=0 \ .$$

We may take

$$p=\omega i \ , \qquad \omega > 0 \ .$$

The solutions of 2.1 may be normalised in such a way that

$$u(x,y,-\omega) = u^*(x,y,\omega) \quad \text{etc.}$$

If $\Omega=0$ the free oscillations may be written down at once

$$\begin{cases} u = k \sin kx \cos y\sqrt{\omega^2-k^2} \\ v = \sqrt{\omega^2-k^2} \cos kx \sin y \sqrt{\omega^2-k^2} , \end{cases} \qquad 2.2$$

where $\qquad k = 0,1 \ldots [\omega]$.

The motion 2.2 is in general ellipsoidal. For k=0 we obtain a motion in the y-direction

$$u = 0 \ , \quad v = \omega \sin \omega y.$$

For $\omega=k$ we obtain a motion in the x-direction

$$v = 0 \ , \quad u = k \sin kx.$$

Similar phenomena may be expected for $\Omega \neq 0$.
The general solution of 2.1 may be written as follows

$$u = \sum_{1}^{\infty} \frac{\Omega^2+k^2}{k\nu_k} C_k \sin kx\, e^{-\nu_k y} - \sum_{1}^{N} \frac{\Omega^2+k^2}{k\nu_k} C_k' \sin kx\, e^{\nu_k y}. \quad 2.3$$

$$v = A \exp\left\{\Omega(x-\tfrac{\pi}{2}) - py\right\} - B \exp\left\{-\Omega(x-\tfrac{\pi}{2}) + py\right\} + \\ + \sum_{1}^{\infty} C_k(\cos kx + \frac{p\Omega}{k\nu_k}\sin kx)e^{-\nu_k y} + \sum_{1}^{N} C_k'(\cos kx - \frac{p\Omega}{k\nu_k}\sin kx)e^{\nu_k y}. \quad 2.4$$

$$\varphi = A \exp\left\{\Omega(x-\tfrac{\pi}{2}) - py\right\} + B \exp\left\{-\Omega(x-\tfrac{\pi}{2}) + py\right\} + \\ + \sum_{1}^{\infty} C_k(\frac{\Omega}{k}\sin kx + \frac{p}{\nu_k}\cos kx)e^{-\nu_k y} + \sum_{1}^{N} C_k'(\frac{\Omega}{k}\sin kx - \frac{p}{\nu_k}\cos kx)e^{\nu_k y}, \quad 2.5$$

where

$$\nu_k = (\Omega^2+k^2+p^2)^{\frac{1}{2}}$$

with $\arg \nu_k = \pi/2$ for $\omega > \sqrt{k^2+\Omega^2}$,

and where

$$N = \left[\, (\omega^2-\Omega^2)^{\frac{1}{2}} \,\right].$$

The expressions 2.3, 2.4, 2.5 satisfy the differential equations, are bounded as $y \to \infty$ and satisfy the boundary conditions at $x=0$ and $x=\pi$ for arbitrary values of A, B, C_k, C_k'. The boundary condition at $y=0$ gives

$$Ae^{\Omega(x-\frac{\pi}{2})} - Be^{-\Omega(x-\frac{\pi}{2})} + \sum_{1}^{\infty} C_k(\cos kx + \frac{p\Omega}{k\nu_k}\sin kx) + \\ + \sum_{1}^{N} C_k'(\cos kx - \frac{p\Omega}{k\nu_k}\sin kx) = 0. \quad 2.6$$

This condition is of the form

$$f(x) = \sum_{1}^{\infty} C_k (\cos kx + \gamma_n \sin kx), \quad 0 < x < \pi, \quad 2.7$$

where $\gamma_k = O(k^{-2})$.

The expansion 2.7 is possible if

$$\int_{0}^{\pi} h(x)\, f(x) dx = 0 \quad 2.8$$

where $h(x)$, apart from a constant factor, is uniquely determined by the set $\cos kx + \gamma_k \sin kx$, $k=1,2...$. For all k we have

$$\int_0^{\pi} h(x)(\cos kx + \gamma_k \sin kx)dx = 0 . \qquad 2.9$$

In this way a linear condition is imposed upon the coefficients A,B and C'_k, k=1,2...N.
This leads to the following result.

For $\omega < \sqrt{1+\Omega^2}$ there is a single free motion.

For $\omega > \sqrt{1+\Omega^2}$ there are N+1 independent free motions, where $N = [(\omega^2-\Omega^2)^{\frac{1}{2}}]$.
The N+1 independent solutions may be specified by:

the case $C'_k = 0$ for k=1,2...N

the N cases $C'_k = \delta_{jk}$, j=1,2...N and A=B.

The transformation

$$x \longrightarrow \pi - x \ , \ \omega \longrightarrow -\omega \ , \ v \longrightarrow -v$$

does not alter the differential equations 2.1 and the boundary conditions.
In view of the uniqueness of its solutions belonging to a certain class they may be normalised in such a way that

$$u(\pi-x) = u^*(x) \ , \ v(\pi-x) = -v^*(x).$$

This gives

$$(-1)^{k-1} C_k = C_k^*$$

$$(-1)^{k-1} C'_k = C'^*_k \qquad 1 \leq k \leq N \qquad 2.10$$

and $$A = B^* .$$

<u>The case $0 < \omega < \sqrt{1+\Omega^2}$</u>

In this case there is a single free motion. We may put $|A| = |B| = 1$ so that

$$A = \exp -i\alpha \ , \quad B = \exp i\alpha$$

with a real α .
The condition 2.6 becomes

$$\sum_1^\infty C_k(\cos kx + \frac{p\Omega}{k\nu_k}\sin kx) = 2\sinh\left\{\Omega(\tfrac{\pi}{2}-x)+i\alpha\right\} . \qquad 2.11$$

The first relation of 2.10 shows that C_k is real for odd indices and purely imaginary for even indices.

The orthogonal function h(x) satisfying

$$\int_0^\pi h(x)(\cos kx + i\,\frac{\omega\Omega}{k\nu_k}\sin kx)dx = 0 \qquad 2.12$$

for all $k \geqslant 1$ may be normalised so that

$$h(\pi-x) = h^*(x) .$$

Then 2.11 gives at once

$$\alpha = \int_0^\pi e^{\Omega(x-\frac{\pi}{2})} \arg h(x)dx. \qquad 2.13$$

We may put $h(x)=a(x)+ib(x)$, where $a(x)$ and $b(x)$ are real functions. In view of the symmetry relation $h(\pi-x)=h^*(x)$ we have $a(\pi-x)=a(x)$ and $b(\pi-x)=-b(x)$. The functions $a(x)$ and $b(x)$ may be determined by their cosine expansions

$$a(x) = \sum{}^{o} a_k \cos kx \qquad b(x) = \sum{}' b_k \cos kx \qquad 2.14$$

where $\sum^o$ denotes summation over even indices $k=0,2,4\ldots$ and $\sum'$ over odd indices $k=1,3,5\ldots$.

We may take $a_o=1$ which means that h(x) is normalised by

$$\frac{1}{\pi}\int_0^\pi h(x)dx = 1.$$

If the expressions 2.14 are substituted into 2.12 we obtain the following system of linear equations

$$\begin{cases} a_k = \dfrac{4\omega\Omega}{\pi\nu_k}\sum{}'\varepsilon_{kl}\,b_l & k=2,4\ldots \\ b_k = -\dfrac{4\omega\Omega}{\pi\nu_k}\sum{}^{o}\varepsilon_{kl}\,a_l & k=1,3\ldots , \end{cases} \qquad 2.15$$

where

$$\varepsilon_{kl} \overset{\text{def}}{=} \tfrac{1}{2}\int_0^\pi \frac{\sin kx}{k}\cos lx\,dx = \begin{cases} 0 & k\pm l \text{ even} \\ \dfrac{1}{k^2-l^2} & k\pm l \text{ odd} \end{cases} .$$

The coefficients a_k and b_k may be solved from 2.15 by means of successive approximations.

From 2.13 we obtain

$$\alpha = \arg \left\{ \sinh \frac{\Omega\pi}{2} \sum{}^{o} \frac{a_k}{k^2+\Omega^2} - i \cosh \frac{\Omega\pi}{2} \sum{}' \frac{b_k}{k^2+\Omega^2} \right\}. \quad 2.16$$

We may put $C_k=iD_k$ for even k and $C_k=E_k$ for odd k so that D_k and E_k are real. After separation of the real and imaginary part the relation 2.11 becomes

$$\left\{ \begin{array}{l} \sum{}' E_k \cos kx - \omega\Omega \sum{}^{o} D_k \frac{\sin kx}{k\nu_k} = 2\cos\alpha \ \sinh(\frac{\pi}{2}-x) \\ \sum{}^{o} D_k \cos kx + \omega\Omega \sum{}' E_k \frac{\sin kx}{k\nu_k} = 2\sin\alpha \ \cosh(\frac{\pi}{2}-x) \ . \end{array} \right. \quad 2.17$$

If the relations 2.17 are multiplied by cos lx, l=0,1,2... and integrated between 0 and π we obtain

$$\sin\alpha = \frac{\omega\Omega^2}{2\sinh\frac{\Omega\pi}{2}} \sum{}' \frac{E_k}{k^2\nu_k} , \quad 2.18$$

$$D_l = \frac{8\Omega \sin\alpha \sinh\frac{\Omega\pi}{2}}{\pi(l^2+\Omega^2)} - \frac{4\omega\Omega}{\pi} \sum{}' \frac{E_k}{(k^2-l^2)\nu_k} , \quad 2.19$$

$$E_l = \frac{8\Omega\cos\alpha \cosh\frac{\Omega\pi}{2}}{\pi(l^2+\Omega^2)} + \frac{4\omega\Omega}{\pi} \sum{}^{o} \frac{D_k}{(k^2-l^2)\nu_k} . \quad 2.20$$

From this system of linear equations α , D_l and E_l may be derived by successive approximation.

$\omega\Omega$ small

If $\omega\Omega$ is small we obtain from 2.18, 2.19, 2.20 the following approximation

$$\alpha = \frac{4\omega\Omega^3}{\pi\tanh\frac{\Omega\pi}{2}} \sum{}' \frac{1}{k^2(k^2+\Omega^2)\nu_k} + O(\omega^3\Omega^3) \quad 2.21$$

$$D_l = \frac{-32l^2\omega\Omega^2\cosh\frac{\Omega\pi}{2}}{\pi^2(l^2+\Omega^2)} \sum{}' \frac{1}{k^2(k^2-l^2)\nu_k} + O(\omega^3\Omega^3) \quad 2.22$$

$$E_l = \frac{8\Omega\cosh\frac{\Omega\pi}{2}}{\pi(l^2+\Omega^2)} + O(\omega^2\Omega^3) \quad 2.23$$

The value of α may also be derived from 2.15 and 2.16. From this the same approximation 2.23 is obtained.

$\Omega \to 0$

If $0 < \omega < \sqrt{1+\Omega^2}$ and $\Omega \to 0$ we have from 2.21, 2.22 and 2.23

$$D_1 = O(\Omega^2) , \quad \alpha = O(\Omega^2), \text{ and}$$

$$E_1 = \frac{8\Omega}{\pi l^2} + O(\Omega^3) . \qquad 2.24$$

Substitution into 2.3, 2.4, 2.5 gives

$$u = \frac{8\Omega}{\pi} \sum{}' \frac{1}{k\sqrt{k^2-\omega^2}} \sin kx \, e^{-y\sqrt{k^2-\omega^2}} + O(\Omega^3) , \qquad 2.25$$

$$v = -2i \sin \omega y + \Omega \Big\{ (2x-\pi)\cos \omega y + \frac{8}{\pi} \sum{}' \frac{1}{k^2} \cos kx \, e^{-y\sqrt{k^2-\omega^2}} \Big\} + O(\Omega^2) , \qquad 2.26$$

$$\varphi = 2 \cos \omega y + i\Omega \Big\{ (\pi-2x)\sin \omega y + \frac{8\omega}{\pi} \sum{}' \frac{1}{k^2\sqrt{k^2-\omega^2}} \cos kx \, e^{-y\sqrt{k^2-\omega^2}} \Big\} + O(\Omega^2). \qquad 2.27$$

We observe that the Ω-term of u is symmetrical with respect to the median $x=\pi/2$. The Ω-terms of v and φ are antisymmetrical.

For $\Omega \to 0$ the solution tends to

$$u = 0, \quad v = -2i \sin \omega y , \quad \varphi = 2 \cos \omega y.$$

A numerical case

We shall consider the case $\omega=0.1$, $\Omega^2=0.5$. Starting from the zeroth approximation

$$\alpha = 0 \qquad D_2=D_4=D_6=\ldots=0$$

we obtain from 2.20 values of $E_1, E_3 \ldots$ which, when substituted into 2.18 and 2.19 give new values for $\alpha, D_2, D_4 \ldots$ etc. according to the following table

	0	1	2
α	0	0.0308	0.0308
D_2	0	0.064	0.064
D_4	0	0.016	0.016
D_6	0	0.007	0.007
E_1	2.020	2.020	2.020
E_3	0.319	0.318	0.318
E_5	0.119	0.119	0.119

Thus after only a single iteration surprisingly accurate values are obtained. We may also say that in this case the formulae 2.21, 2.22, 2.23, without the O-terms, give very accurate results.

§ 3. The case $\lambda \neq 0$

We repeat the differential equations

$$\begin{aligned} (p+\lambda)u - \Omega v + \frac{\partial \zeta}{\partial x} &= 0 \\ (p+\lambda)v + \Omega u + \frac{\partial \zeta}{\partial y} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + p\zeta &= 0 , \end{aligned} \qquad 3.1$$

with $u = 0$ for $x = 0$ and $x = \pi$

$v = 0$ for $y = 0$.

If $\Omega = 0$, assuming $\lambda < 2$, the free oscillations are easily found to be

$$\left\{ \begin{aligned} u &= k \sin kx \cos y \sqrt{\omega^2 + \frac{\lambda^2}{4} - k^2} \\ v &= \sqrt{\omega^2 + \frac{\lambda^2}{4} - k^2} \cos kx \sin y \sqrt{\omega^2 + \frac{\lambda^2}{4} - k^2} \end{aligned} \right. \qquad 3.2$$

with $p = -\frac{\lambda}{2} \pm \omega i$ and $k = 0, 1 \ldots \left[(\omega^2 + \frac{\lambda^2}{4})^{\frac{1}{2}} \right]$.

The motion 3.2 is in general ellipsoidal. For k=0 we obtain however

$$u = 0 \quad , \qquad v = \sin y \sqrt{\omega^2 + \frac{\lambda^2}{4}} .$$

For $\omega = \sqrt{k^2 - \frac{\lambda^2}{4}}$ we have

$$v = 0 \quad , \qquad u = \sin kx \quad , \quad k = 1, 2 \ldots .$$

We consider now the case $\Omega \neq 0$.

The general solution of 3.1 may be written as follows

$$u = \sum_1^\infty \frac{s^2+k^2}{k\nu_k} C_k \sin kx\, e^{-\nu_k y} - \sum_1^N \frac{s^2+k^2}{k\nu_k} C_k' \sin kx\, e^{\nu_k y} \qquad 3.3$$

$$v = sA \exp\left\{ s(x-\tfrac{\pi}{2}) - y\sqrt{p^2+\lambda p}\right\} - sB \exp\left\{-s(x-\tfrac{\pi}{2}) + y\sqrt{p^2+\lambda p}\right\} +$$
$$+ \sum_1^\infty C_k(\cos kx + \frac{p\Omega}{k\nu_k}\sin kx) e^{-\nu_k y} + \sum_1^N C_k'(\cos kx - \frac{p\Omega}{k\nu_k}\sin kx)e^{\nu_k y}. \qquad 3.4$$

$$\zeta = \Omega A \exp\left\{s(x-\tfrac{\pi}{2}) - y\sqrt{p^2+\lambda p}\right\} + \Omega B \exp\left\{-s(x-\tfrac{\pi}{2}) + y\sqrt{p^2+\lambda p}\right\} +$$
$$+ \sum_1^\infty C_k(\frac{\Omega}{k}\sin kx + \frac{p+\lambda}{\nu_k}\cos kx)e^{-\nu_k y} + \sum_1^N C_k'(\frac{\Omega}{k}\sin kx - \frac{p+\lambda}{\nu_k}\cos kx)e^{\nu_k y}, \qquad 3.5$$

where

$$\nu_k = \sqrt{k^2+q^2}\ , \qquad \text{Re}\ \nu_k \geq 0\ ,$$
$$q^2 = p(p+\lambda)+s^2$$
$$s^2 = \frac{p\Omega^2}{p+\lambda}\ , \qquad \text{Re}\ s \geq 0.$$

The coefficients C_k' vanish if ν_k is not a pure imaginary.
The coefficient B vanishes if $\sqrt{p^2+\lambda p}$ is not a pure imaginary.
The expressions 3.3, 3.4, 3.5 satisfy the equations 3.1 and the boundary conditions at $x=0$ and $x=\pi$.
The condition at $y=0$ becomes of the form 2.7 or

$$sA\, e^{s(x-\frac{\pi}{2})} - sB\, e^{-s(x-\frac{\pi}{2})} + \sum_1^\infty C_k(\cos kx + \frac{p\Omega}{k\nu_k}\sin kx) +$$
$$+ \sum_1^N C_k'(\cos kx - \frac{p\Omega}{k\nu_k}\sin kx) = 0. \qquad 3.6$$

The conclusions are similar to those in the preceding section. We may distinguish the following cases

<u>a</u> p real, and $-\lambda < p < 0$.

We have $N = [\,|q|\,]$, $\arg \nu_k = \pi/2$, $\arg s = \pi/2$, $\arg \sqrt{p^2+\lambda p} = \pi/2$.
There are N+1 independent free motions which all have the character of a non-oscillatory damping.

<u>b</u> $p = -\frac{\lambda}{2} + \omega i$, ω real.

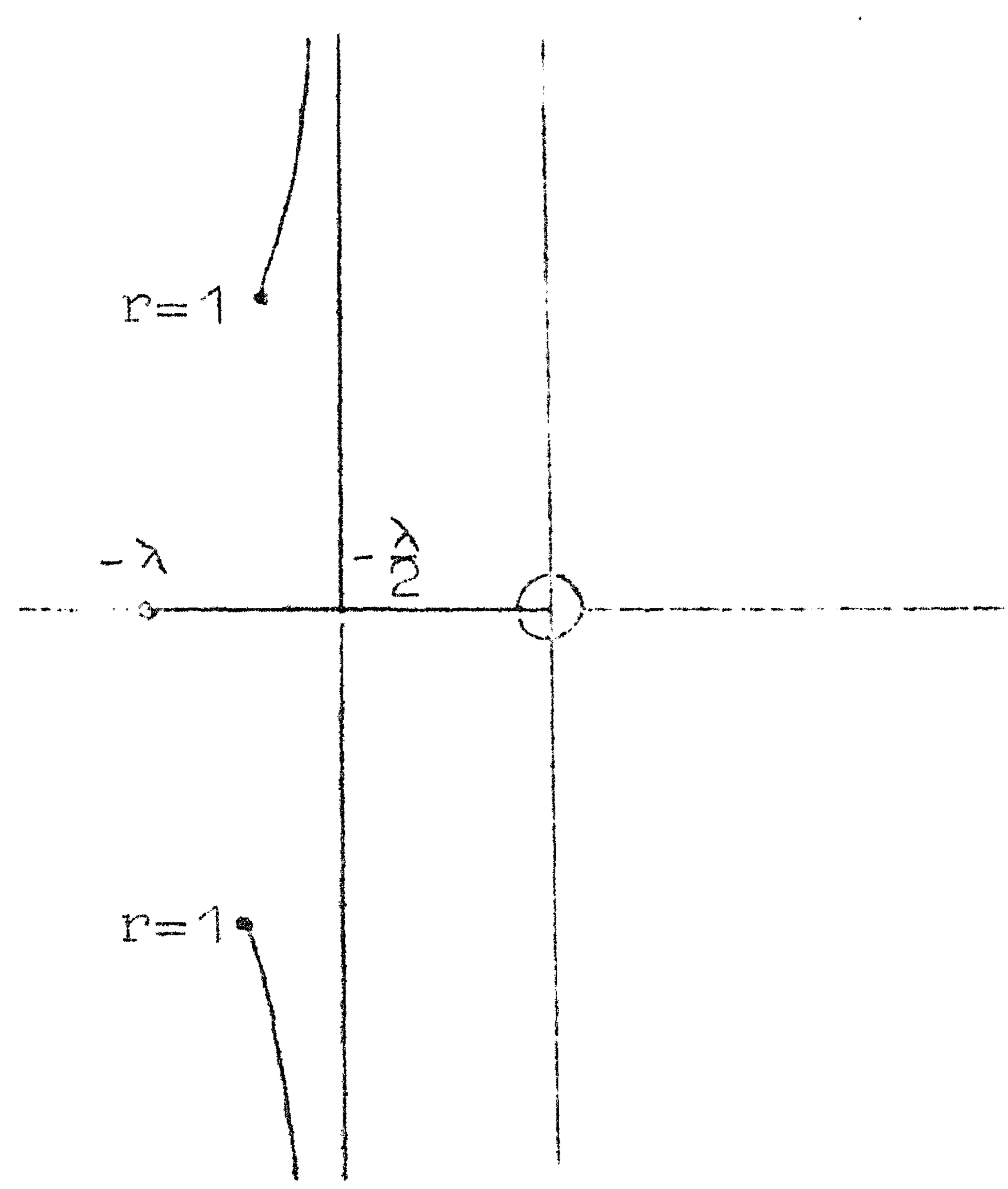
We may take $\omega > 0$. The coefficients C_k' vanish. There is only a single free motion of the oscillatory damping type.

<u>c</u> q^2 real and negative , p not real.

The equation $\frac{p}{p+\lambda}\left\{(p+\lambda)^2+\Omega^2\right\} = -r^2$, where r is real and positive has for any r three roots, one real and two conjugate complex with $-\lambda < \mathrm{Re}\, p < -\frac{\lambda}{2}$. For complex p the coefficient B vanishes. We have, however, N= [r] .
Hence there are N independent free motions of the oscillatory damping type, assuming $r \geq 1$ of course.
For r large the values of p become

$$p = -\frac{\lambda}{2} - \frac{\lambda\Omega^2}{2r^2}\left(1-\frac{\Omega^2}{r^2}+\frac{\Omega^2(\lambda^2+\Omega^2)}{r^4}\ldots\right) \pm$$
$$\pm\, ir\left(1+\frac{4\Omega^2-\lambda^2}{8r^2}-\frac{8\Omega^4+24\lambda^2\Omega^2+\lambda^4}{16r^4}\ldots\right) .$$

For $r=1$, $\lambda^2=0.02$, $\Omega^2=0.5$ we obtain $p=-0.094 \pm i\ 1.22$.
The possible p values yielding free motions are for the three cases sketched below.



<u>p Ω small</u>

We shall now consider the case <u>a</u> or <u>b</u> with pΩ small so that the coefficients C_k' vanish.
The condition at y=0 becomes

$$sA \exp s(x-\tfrac{\pi}{2}) - sB \exp s(\tfrac{\pi}{2}-x) + \sum_1^\infty C_k\left(\cos kx+\frac{p\Omega}{k\nu_k}\sin kx\right)=0. \qquad 3.7$$

Putting $sA=e^{-\alpha i}$, $sB=e^{\alpha i}$ we obtain in a similar way as in the preceding section

$$i\ \mathrm{tg}\,\alpha = \frac{ps\,\Omega}{2\sinh\frac{s\pi}{2}} \sum{}' \frac{c_k}{k^2 \nu_k} ,$$

$$l \text{ even} \quad c_l = i \sin\alpha \frac{8s \sinh\frac{s\pi}{2}}{\pi(l^2+s^2)} - \frac{4}{\pi} p\Omega \sum{}' \frac{c_k}{(k^2-l^2)\nu_k} , \qquad 3.9$$

$$l \text{ odd} \quad c_l = \cos\alpha \frac{8s \cosh\frac{s\pi}{2}}{\pi(l^2+s^2)} - \frac{4}{\pi} p\Omega \sum{}^{o} \frac{c_k}{(k^2-l^2)\nu_k} . \qquad 3.10$$

From this the following approximations may be obtained

$$i\alpha = \frac{4ps^2\Omega}{\pi \tanh\frac{s\pi}{2}} \sum{}' \frac{1}{k^2(k^2+s^2)\nu_k} + O(p^3\Omega^3) , \qquad 3.11$$

$$l \text{ even} \quad c_l = - \frac{32l^2 ps\,\Omega \cosh\frac{s\pi}{2}}{\pi^2(l^2+s^2)} \sum{}' \frac{1}{k^2(k^2-l^2)\nu_k} + O(p^3\Omega^3), \qquad 3.12$$

$$l \text{ odd} \quad c_l = \frac{8s \cosh\frac{s\pi}{2}}{\pi(l^2+s^2)} + O(p^2 s\,\Omega^2) . \qquad 3.13$$

$\Omega \to 0$

The formulae 3.11, 3.12, 3.13 give the first-order approximations

$$\alpha = O(\Omega^2) \qquad c_{even} = O(\Omega^2) \qquad 3.14$$
$$c_{odd} = \frac{8s}{\pi l^2} + O(\Omega^2)$$

From 3.14 we obtain for u, v, ζ

$$u = \frac{8s}{\pi} \sum{}' \frac{1}{k\sqrt{k^2+p^2+\lambda p}} \sin kx \exp -y\sqrt{k^2+p^2+\lambda p} + O(\Omega^3), \qquad 3.15$$

$$v = -2\sinh y\sqrt{p^2+\lambda p} + s\Big\{ (2x-\pi)\cosh y\sqrt{p^2+\lambda p} + \frac{8}{\pi}\sum{}' \frac{1}{k^2}\cos kx \exp -y\sqrt{k^2+p^2+\lambda p} \Big\} + O(\Omega^2), \qquad 3.16$$

$$\zeta = 2\sqrt{\frac{p+\lambda}{p}} \cosh y\sqrt{p^2+\lambda p} + \Omega\Big\{ (\pi-2x)\sinh y\sqrt{p^2+\lambda p} + \frac{8}{\pi}\sqrt{p^2+\lambda p} \sum{}' \frac{1}{k^2\sqrt{k^2+p^2+\lambda p}} \cos kx \exp -y\sqrt{k^2+p^2+\lambda p} \Big\} + O(\Omega^2). \qquad 3.17$$

<u>The numerical case</u> $p = -\frac{\lambda}{2} + \omega i$, $\lambda^2 = 0.02$, $\Omega^2 = 0.5$, $\omega = 0.1$.
We find by a similar iteration process as in the preceding section the following values x)

$$\begin{aligned}
\alpha &= 0.025 + 0.035\, i \\
c_1 &= 1.403 + 1.245\, i \\
c_2 &= -0.071 + 0.012\, i \\
c_3 &= 0.126 + 0.223\, i \\
c_4 &= -0.016 + 0.001\, i \\
c_5 &= 0.043 + 0.083\, i \\
c_6 &= -0.007 + 0.000\, i
\end{aligned}$$

x) The computations have been carried out by the Computation Department of the Mathematical Centre.