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Free oscillations in a rotating semi-circular bay

by

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§1. Introduction.

Several authors have investigated the free oscillations of shallow rotating lakes (seas), bays and channels of different shapes. We mention the rectangular, circular, elliptic and semi-circular lake (Corkan and Doodson 1952, Van Dantzig 1958, Jeffreys 1924, Lamb 1932, Proudman 1928, Taylor 1922), and the rectangular bay (Van Dantzig 1958, Taylor 1922). The present investigation concerns the shallow semi-circular bay of constant depth. By means of LAUWERIER's theory of trigonometric series with prescribed phases, valid in the half-period interval, a determinantal equation is obtained, which implicitly determines the free frequencies ω of the semi-circular bay as a function of Ω, the apparent angular velocity at the site of the bay. An explicit formula is derived for a first approximation in $(\Omega/\omega)^2$. For comparison we have included LAMB's treatment of the rotating circular lake.

§ 2. General equations.

If friction is neglected and if no exterior forces come into play, the equations of motion and continuity read (Lamb 1932, p.318)

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \Omega\, v + gd\, \frac{\partial \zeta}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + \Omega\, u + gd\, \frac{\partial \zeta}{\partial y} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} &= 0 \end{aligned}\right\} \qquad 2.1$$

where u the x-component of the total current
v the y-component of the total current
ζ the elevation of the free surface above its equilibrium position
Ω the Coriolis coefficient
d the depth of the bay
g the acceleration of gravity.

In the following the bay will cover the area $y > 0$, $x^2+y^2 < a^2$. Along the diameter $y=0$ the ocean condition $\zeta=0$ holds and on the circumference $x^2+y^2=a^2$, $y > 0$ the radial component of the total current vanishes (coastal condition). We assume g, Ω and d to be constants. Then it is possible to introduce dimensionless quantities by means of

$$\left.\begin{aligned} (x, y, \zeta) &= a(x', y', \zeta') \\ (u, v) &= a\sqrt{gd}\ (u', v') \\ (t, 1/\Omega) &= a/\sqrt{gd}\ (t', 1/\Omega')\ . \end{aligned}\right\} \qquad 2.2$$

This amounts to putting $gd=1$ in 2.1 and $a=1$ in the coastal condition. The dimensionless form of 2.1, hence, is

$$\left.\begin{aligned} \frac{\partial \Omega}{\partial t} - \Omega\, v + \frac{\partial \zeta}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + \Omega\, u + \frac{\partial \zeta}{\partial y} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} &= 0 \end{aligned}\right\} \qquad 2.3$$

If, instead of Cartesian coordinates, we introduce polar coordinates

$$\left.\begin{aligned} x &= r\cos\varphi\,, \\ y &= r\sin\varphi\,, \end{aligned}\right\} \qquad 2.4$$

$$\left.\begin{aligned} u_r &= u\cos\varphi + v\sin\varphi\,, \\ u_\varphi &= -u\sin\varphi + v\cos\varphi\,, \end{aligned}\right\} \qquad 2.5$$

we find for 2.3

$$\left.\begin{aligned} &\frac{\partial u_r}{\partial t} - \Omega u_\varphi + \frac{\partial \zeta}{\partial r} = 0\,, \\ &\frac{\partial u_\varphi}{\partial t} + \Omega u_r + \frac{1}{r}\frac{\partial \zeta}{\partial \varphi} = 0\,, \\ &\frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\varphi}{\partial \varphi} + \frac{1}{r}u_r + \frac{\partial \zeta}{\partial t} = 0. \end{aligned}\right\} \qquad 2.6$$

For the semi-circular bay the boundary conditions read

$$\zeta = 0 \text{ for } r=0,\ \varphi=0 \text{ and } \varphi=\pi \quad (-1 < r < 1) \qquad 2.7$$

$$u_r = 0 \text{ for } r=1 \qquad 2.8$$

and for the circular lake

$$u_r = 0 \quad \text{for} \quad r = 1\ . \qquad 2.9$$

In stead of a boundary condition comes the requirement that u_r, u_y and ζ remain unaltered if we add to φ a multiple of 2π. Elimination of u_r and u_φ from either 2.3 or 2.6 yields

$$\Delta\zeta = \frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\zeta}{\partial\varphi^2} = \frac{\partial^2\zeta}{\partial t^2} + \Omega^2\zeta. \qquad 2.10$$

From 2.6 we also derive

$$\frac{\partial^2 u_r}{\partial t^2} + \Omega^2 u_r + \frac{\partial^2\zeta}{\partial r\,\partial t} + \frac{\Omega}{r}\frac{\partial\zeta}{\partial\varphi} = 0 \qquad 2.11$$

Assuming harmonic motion

$$\zeta = \hat{\zeta}\, e^{i\omega t}\,, \qquad 2.12$$

the equations 2.10 and 2.11 become

$$\Delta\zeta + \kappa^2\zeta = 0 \qquad 2.13$$

and

$$\kappa^2 u_r = i\omega \frac{\partial \zeta}{\partial r} + \frac{\Omega}{r} \frac{\partial \zeta}{\partial \varphi} \qquad 2.14$$

where

$$\kappa^2 = \omega^2 - \Omega^2 \quad . \qquad 2.15$$

§3. Free oscillation of the circular lake.

This case has been investigated by LAMB (Lamb 1932, p.320-324). His treatment, adapted to the present notation, is as follows.
The Helmholtz' equation

$$\Delta \zeta + \kappa^2 \zeta = 0$$

possesses the solution

$$\zeta = \sum_{m=-\infty}^{\infty} A_m J_m(\kappa r) e^{im\varphi} \qquad 3.1$$

which is invariant for the substitution $\varphi' = \varphi + 2k\pi$, k integer. The boundary condition $u_r=0$ for $r=1$ becomes, using 2.14,

$$\frac{\partial \zeta}{\partial r} - i \frac{\Omega}{\omega} \frac{\partial \zeta}{\partial \varphi} = 0 \quad \text{for } r=1 \quad . \qquad 3.2$$

Substitution of 3.2 in 3.1 yields

$$\sum_{m=-\infty}^{\infty} A_m \left[J_m'(\kappa) + \frac{\Omega}{\omega} m J_m(\kappa) \right] e^{im\varphi} = 0 \ . \qquad 3.3$$

This equation is satisfied if

$$\left.\begin{array}{l} A_m = 0, \quad m \neq n \\ \text{and} \\ \kappa J_n'(\kappa) + \frac{\Omega}{\omega} n J_n(\kappa) = 0 \quad . \end{array}\right\} \qquad 3.4$$

For every n this equation has an infinite series of roots. In particular we have for n=0

$$J_0'(\kappa) = 0$$

or

$$J_1(\kappa) = 0 \quad . \qquad 3.5$$

For $n \geqslant 1$ the free frequencies follow from

$$\frac{\kappa J_n'(\kappa)}{n J_n(\kappa)} = - \frac{\Omega}{\omega} \qquad 3.6$$

or

$$\frac{J_{n+1}(\kappa)}{J_{n-1}(\kappa)} = \frac{\omega + \Omega}{\omega - \Omega} \qquad 3.7$$

Introducing the quantities α and s_n by mean of

$$\Omega = \omega \tanh \tfrac{1}{2}\alpha\pi \qquad 3.8$$

$$s_n(\kappa) = J_{n+1}(\kappa)/J_{n-1}(\kappa) \qquad 3.9$$

we can write 3.7 in the form

$$s_n(\kappa) = \exp \alpha\pi \quad . \qquad 3.10$$

The righthand member of 3.7 is not necessarily negative. If ω and Ω have the same sign the free frequencies correspond to modes which rotate in the same direction as the lake, if ω and Ω have opposite signs, the corresponding modes rotate in a direction opposite to that of the lake.
For a given Ω and fixed n two unequal values of ω exist, with opposite signs, which determine two modes which rotate in opposite directions.
If $\Omega \rightarrow \infty$ and κ remains finite, it follows that also $\omega \longrightarrow \infty$ and $\Omega/\omega \longrightarrow \pm 1$. From 3.7 we then find that for $\Omega/\omega \longrightarrow 1$ or $\alpha \rightarrow \infty$ the corresponding values of κ follow from $J_{n-1}(\kappa)=0$ and that for $\Omega/\omega \longrightarrow -1$ or $\alpha \longrightarrow -\infty$ the corresponding values of κ follow from $J_{n+1}(\kappa)=0$.

§4. Free oscillations of the semi-circular bay.

We again have the equation

$$\Delta \zeta + \kappa^2 \zeta = 0 \;, \quad \kappa^2 = \omega^2 - \Omega^2 .$$

We may expand ζ for constant r in an absolutely convergent sine series

$$\zeta = \sum_{n=1}^{\infty} A_n \, J_n(\kappa r) \sin n\varphi \qquad 4.1$$

which series also satisfies the boundary conditions $\zeta=0$ for $\varphi=0$ and $\varphi=\pi$.

If we suppose ζ three times differentiable with respect to φ in the interval $0 \leq \varphi \leq \pi$ for constant r, it can easily be shown that $A_n J_n(\kappa r) = 0(n^{-3})$. Hence it is permitted to differentiate the series 4.1 with respect to φ. The boundary condition $u_r=0$ for r=1 then becomes, using 2.14,

$$\sum_{n=1}^{\infty} A_n J_n{}'(\kappa) \left[\sin n\varphi - i \frac{\Omega \, n J_n(\kappa)}{\omega \kappa J_n{}'(\kappa)} \cos n\varphi \right] = 0 \qquad 4.2$$

for all values of φ in the interval $(0,\pi)$.

This is a Fourier series with prescribed phases. These series were the subject of recent investigations (Hofsommer 1958, Lauwerier 1957, Veltkamp 1955).

We use the following result, due to LAUWERIER:

A function f(x) can, in general, be developed in a slowly convergent series with prescribed phases γ_n,

$$f(x) = \sum_{n=1}^{\infty} b_n(\sin nx + \gamma_n \cos nx)$$

if $\gamma_n = \gamma + 0(n^{-\theta})$, $\theta > 1$ for $n \rightarrow \infty$.

In 4.2 we have

$$\gamma_n = -i \frac{\Omega}{\omega} \frac{n}{\kappa} \frac{J_n(\kappa)}{J_n{}'(\kappa)} = i \frac{\Omega}{\omega} \left[1 + \kappa^2/2n(n+1) + 0(n^{-4}) \right]$$

so that the above condition is fulfilled with $\theta=2$.

For a given Ω there exists a set of coefficients A, satisfying 4.2 which are not all zero if ω belongs to the spectrum of eigenvalues of the problem. These eigenvalues are obviously functions of Ω. They follow from a determinantal relation in ω and Ω which will be derived by means of a method which is described in (Hofsommer 1958). We will outline the procedure, but for details we refer to that report.

We introduce the quantities α and s_n by means of

$$\Omega = \omega \tanh \tfrac{1}{2}\alpha\pi \qquad 4.3$$

$$s_n(\kappa) = J_{n+1}(\kappa)/J_{n-1}(\kappa) \quad . \qquad 4.4$$

Then 4.2 can be written as

$$\sum_{n=1}^{\infty} A_n \, J_{n-1}(\kappa) \left[\sin(n\varphi - \tfrac{1}{2}i\alpha\pi) - s_n(\kappa)\sin(n\varphi + \tfrac{1}{2}i\alpha\pi) \right] = 0. \qquad 4.5$$

LAUWERIER has derived explicit expressions for the functions biorthogonal to the set $\sin(n\varphi + \mu\pi)$ (Lauwerier 1957). Let $k_m(\varphi)$ be the set of functions biorthogonal to $\sin(n\varphi - \frac{1}{2}i\alpha\pi)$. Then we have

$$\frac{1}{\pi}\int_0^{\pi} k_m(\varphi)\sin(n\varphi - \tfrac{1}{2}i\alpha\pi)\,d\varphi = \delta_{m,n} \quad . \qquad 4.6$$

Furthermore we define

$$\frac{1}{\pi}\int_0^{\pi} k_m(\varphi)\sin(n\varphi + \tfrac{1}{2}i\alpha\pi)\,d\varphi = i^{m-n}\, d_{m,n}(\alpha). \qquad 4.7$$

The factor i^{m-n} is inserted in the righthand member in order that the $d_{m,n}$ become real if α is real. These coefficients $d_{m,n}(\alpha)$ are related to the $e_{m,n}(\alpha)$, introduced in (Hofsommer 1958) according to

$$e_{m,n}(i\alpha) = i^{m-n}\, d_{m,n}(\alpha). \qquad 4.8$$

By means of 4.6 and 4.7 we derive from 4.5, putting

$$A_n J_{n-1}(\kappa) = i^n B_n, \qquad 4.9$$

the set of equations

$$B_m - \sum_{n=1}^{\infty} B_n s_n(\kappa) d_{m,n}(\alpha) = 0. \qquad 4.10$$

This set only admits a solution if, formally,

$$\mathrm{Det}\left[\delta_{m,n}-s_n(\kappa)d_{m,n}(\alpha)\right] = 0. \qquad 4.11$$

This represents a certain relation between α and κ and hence also between Ω and ω.

The convergence of the infinite determinant 4.11 can be shown in the following way.
In 4.10 substitute $B_n=\sqrt{n/s_n}\, C_n$. We find

$$C_m - \sum_{n=1}^{\infty} C_n\sqrt{\frac{s_m s_n}{mn}}\, n\, d_{m,n}(\alpha) = 0 \qquad 4.12$$

and, instead of 4.11, we get the equivalent equation

$$\mathrm{Det}\left[\delta_{m,n} - \sqrt{\frac{s_m s_n}{mn}}\, n\, d_{m,n}(\alpha)\right] = 0 . \qquad 4.13$$

Consider the generating function for the $e_{m,n}$ (Hofsommer 1958 eq.2.1)

$$\cos(mx-\tfrac{1}{2}\beta\pi) = \sum_{n=0}^{\infty} e_{m,n}(\beta)\cos(nx+\tfrac{1}{2}\beta\pi),$$

or, with $\beta=i\alpha$,

$$\sin\left[mx-\tfrac{1}{2}(1+i\alpha)\pi\right] = -\sum_{n=0}^{\infty} e_{m,n}(i\alpha)\sin\left[nx+\tfrac{1}{2}(1+i\alpha)\pi\right].$$

It has been shown in (Lauwerier 1957, § 4) that, if in the development

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(nx+\mu\pi)$$

$\mathrm{Re}\,\mu=\tfrac{1}{2}$, the $a_n=O(n^{-2})$.
Hence

$$e_{m,n}(i\alpha) = O(n^{-2})$$

and

$$n\, e_{m,n}(i\alpha) = O(n^{-1}).$$

However, we have (Hofsommer 1958, eq.1.12)

$$(-1)^n\, n\, e_{m,n} = (-1)^m\, m\, e_{n,m}.$$

Hence also

$$n\, e_{m,n}(i\alpha) = O(m^{-1})$$

and

$$n\, d_{m,n}(\alpha) = O(m^{-1}n^{-1}) \;. \qquad 4.14$$

Moreover

$$s_n(\kappa) = \frac{J_{n+1}(\kappa)}{J_{n-1}(\kappa)} = O(n^{-2}). \qquad 4.15$$

Hence

$$\sqrt{\frac{s_m s_n}{mn}}\; n\, d_{m,n}(\alpha) = O(m^{-2\frac{1}{2}}n^{-2\frac{1}{2}}) \qquad 4.16$$

and the double sum $\sum\limits_{m,n} \sqrt{\frac{s_m s_n}{mn}}\; n\, d_{m,n}(\alpha)$ is convergent, which, according to H.v. KOCH (Hellinger und Toeplitz 1953, § 17) entails the convergence of the determinant 4.13.

Explicitly the upper left part of (4.11) reads

$$\begin{matrix} 1-s_1(1+2\alpha^2) & -\frac{4}{3}s_2\alpha(1+\alpha^2) & -\frac{2}{3}s_3\alpha^2(1+\alpha^2) \\ -\frac{8}{3}s_1\alpha(1+\alpha^2) & 1-s_2(1+4\alpha^2+2\alpha^4) & -\frac{8}{15}s_3\alpha(1+\alpha^2)(3+2\alpha^2) \\ -2s_1\alpha^2(1+\alpha^2) & -\frac{4}{5}s_2\alpha(1+\alpha^2)(3+2\alpha^2) & 1-\frac{1}{9}s_3(9+38\alpha^2+28\alpha^4+8\alpha^6). \end{matrix}$$

It follows easily from the recurrence relations 5.8 and 5.9 for the $d_{m,n}$ viz.

$$2\alpha\, d_{m,o} = (m+1)d_{m+1,o}+(m-1)d_{m-1,o} \qquad ,$$

$$m(d_{m+1,n}+d_{m-1,n})=(n+1)d_{m,n+1}+(n-1)d_{m-1,o} \; ,$$

$$d_{o,o} = 1 \qquad ,$$

that the $d_{m,n}$ and hence the $a_{m,n}$ are even functions of α if m+n is even and odd functions of α if m+n is odd. Then 4.11 determines κ as an even function of α or Ω/ω . Hence the free frequencies occur in pairs of equal magnitude but opposite sign, this in contrast to the free frequencies of the circular lake.

§ 5. Solutions for small values of Ω.

For $\Omega=0$ the solution of our problem is well-known. It also follows from eq.4.2. The free frequencies $\omega_{n,s}$ constitute a double infinite set of values, which follow from

$$J_n'(\omega_{n,s})=0, \quad n,s=1,2,3,\ldots ,$$

or

$$s_n(\omega_{n,s})=1, \quad n,s=1,2,3,\ldots . \qquad 5.1$$

The first index, n, refers to the order of the Bessel-function and the second index indicates the s^{th} zero of the derivative of J_n.

Let $\kappa_o=\omega_{n,s}$ be a solution of 5.1 and suppose that κ differs slightly from κ_o,

$$\kappa^2 = \kappa_o^2 - \varepsilon_{n,s}\,\alpha^2 . \qquad 5.2$$

Then we have

$$s_n(\kappa)=s_n(\kappa_o)-\frac{\varepsilon_{n,s}}{2\kappa_o}\,s_n'(\kappa_o)\alpha^2+O(\varepsilon_{n,s}^2)$$

$$= 1-n\,\varepsilon_{n,s}\left(\frac{1}{n^2}-\frac{1}{\kappa_o^2}\right)\alpha^2+O(\varepsilon_{n,s}^2) . \qquad 5.3$$

On the other hand we can put

$$s_n(\kappa) = 1 - \delta_{n,s}\alpha^2 + O(\alpha^4) . \qquad 5.4$$

It then follows from 5.3 and 5.4 that

$$\varepsilon_{n,s} = \delta_{n,s}/n\ (n^{-2}-\kappa_o^{-2}) + O(\alpha^4) . \qquad 5.5$$

Again, we have from 5.2

$$\omega_{n,s}^2 = \kappa_o^2 + \Omega^2 - \varepsilon_{n,s}\alpha^2 + O(\alpha^4).$$

However

$$\alpha = \frac{2}{\pi}\frac{\Omega}{\omega} + O\left[(\Omega/\omega)^3\right]. \qquad 5.6$$

Hence
$$\omega_{n,s}^2 = \kappa_o^2 + \left(1-\frac{4\varepsilon_{n,s}}{\pi^2\kappa_o^2}\right)\Omega^2 + O(\Omega^4)$$

and
$$\omega_{n,s}(\Omega) = \omega_{n,s}(0) + p_{n,s}\Omega^2+0(\Omega^4) \qquad 5.7$$

where
$$p_{n,s}=\tfrac{1}{2}\left[1-4n\,\delta_{n,s}/\pi^2\left\{\omega_{n,s}(0)-n^2\right\}\right]. \qquad 5.8$$

We shall now show that
$$\delta_{n,s}=D_n-\sum_{m=1}^{\infty}\frac{s_{m,s}\{\omega_{n,s}(0)\}}{1-s_{m,s}\{\omega_{n,s}(0)\}}\cdot\frac{16\ mn}{m^2-n^2}+0(\alpha^2) \qquad 5.9$$

A few values of D_n are $D_1=2$, $D_2=4$, $D_3=38/9$, $D_4=40/9$, $D_5=1018/225$, $D_6=1036/225$.
By means of 4.8 we borrow from (Hofsommer 1958) the following relations between the $d_{m,n}$

$$m(d_{m+1,n}+d_{m-1,n}) = (n+1)d_{m,n+1}+(n-1)d_{m,n-1} \qquad 5.10$$

$$2\alpha\, d_{m,o} = (m+1)d_{m+1,o}+(m-1)d_{m-1,o}. \qquad 5.11$$

Special values are $d_{o,o}=1$ $\quad d_{1,o}=2\alpha$ $\quad d_{2,o}=2\alpha^2$.
It follows from 5.11 and from $d_{o,o}=1$, that

$$\begin{aligned} d_{m,o} &= 0(\alpha) && m \text{ odd}, \\ d_{m,o} &= 0(\alpha^2) && m \text{ even},\ m\neq 0, \\ d_{o,o} &= 1 && . \end{aligned}$$

Formula 5.11 is seen to be a recurrence relation between four coefficients $d_{m,n}$ for which the sum of the subscripts is either only even or only odd. Since $d_{m,-1}=0$ (Hofsommer 1958) it is possible to derive all $d_{m,n}$ from the sequence $d_{m,o}$.
Then it follows from 5.10 that

$$\begin{aligned} d_{m,n} &= 0(\alpha) && m+n \text{ odd} \\ d_{m,n} &= 0(\alpha^2) && m+n \text{ even},\ m\neq n \\ d_{m,n} &= 1+0(\alpha^2) && m=n. \end{aligned}$$

Now let for some n, say n=p

$$s_p(\kappa) = 1-\alpha^2\delta_p$$

and substitute this in the determinant

$$\mathrm{Det}\left[\delta_{m,n}-s_n\, d_{m,n}\right] .$$

Again denoting the elements of this determinant by $a_{m,n}$ we have

$$\begin{aligned}
a_{m,n} &= O(\alpha) && m+n \text{ odd}\\
a_{m,n} &= O(\alpha^2) && m+n \text{ even, } m\neq n\\
a_{m,n} &= O(1) && m=n\neq p\\
a_{p,p} &= O(\alpha^2) .
\end{aligned}$$

It is seen that the row $a_{p,n}$ and the column $a_{m,p}$ are divisible by α (the element $a_{p,p}$ is divisible by α^2). Bringing this factor α^2 outside the determinant and subsequently putting $\alpha=0$ in the arising elements, we find non-zero elements only in the principle diagonal, in the p^{th} row and in the p^{th} column. More precisely we have

$$\begin{aligned}
a_{m,n} &= O(\alpha) && m+n \text{ odd, } m\neq p,\ n\neq p\\
a_{m,n} &= O(\alpha^2) && m+n \text{ even, } m\neq n,\ m\neq p,\ n\neq p\\
a_{m,n} &= O(1) && m=n\neq p\\
a_{p,n}/\alpha &= O(1) && p+n \text{ odd}\\
a_{m,p}/\alpha &= O(1) && m+p \text{ odd}\\
a_{p,n}/\alpha &= O(\alpha) && p+n \text{ even, } n\neq p\\
a_{m,p}/\alpha &= O(\alpha) && m+p \text{ even, } m\neq p\\
a_{p,p}/\alpha^2 &= O(1) .
\end{aligned}$$

With these simplifications the equation

$$\mathrm{Det}\ a_{m,n} = 0$$

takes the form

$$\sum_{n=1}^{\infty} \frac{s_p(\kappa)s_n(\kappa)d_{p,n}d_{n,p}}{\left[1-s_p(\kappa)d_{p,p}\right]\left[1-s_n(\kappa)d_{n,n}\right]} = 1+O(\alpha^2),\ n+p \text{ odd}. \qquad 5.12$$

Between the coefficients $s_n(\kappa)$ the following relation exists

$$s_{n+1}(\kappa) = \frac{4n(n+1)}{\kappa^2}\cdot\frac{s_n(\kappa)}{s_n(\kappa)+1} - 1 . \qquad 5.13$$

Putting $n=p$, $\kappa^2 = \omega_o^{\ 2}-\varepsilon_p\alpha^2$, $s_p(\kappa)=s_p(\kappa_o)-\delta_p\alpha^2$ it is seen that

$$s_{n+1}(\kappa) = s_{n+1}(\kappa_o) + O(\alpha^2).$$

In this way it is easily found that

$$s_n(\kappa) = s_n(\kappa_o) + O(\alpha^2). \qquad 5.14$$

for all values of n.
This means that, in 5.12, $s_n(\kappa)$ may be replaced by $s_n(\kappa_o)$. It then follows from 5.12 that

$$\delta_{p,s} = \left[\frac{d_{p,p}-1}{\alpha^2} + \sum_{n=1}^{\infty} \frac{s_n(\kappa_o)}{1-s_n(\kappa_o)} \frac{d_{p,n}\, d_{n,p}}{\alpha^2} \right]_{\alpha=0} + O(\alpha^2). \qquad 5.15$$

The factor $(d_{p,n}\, d_{n,p}/\alpha^2)_{\alpha=0}$ can be evaluated in the following way.
It follows from 5.11 that

$$(m+1)d_{m+1,0} = -(m-1)d_{m-1,0} + O(\alpha^3), \text{ m even.}$$

This recurrence relation is solved by

$$d_{m,0} = \frac{2}{m} (-1)^{\frac{1}{2}(m-1)}\alpha + O(\alpha^3), \quad \text{m odd} \qquad 5.16$$

taking into account that $d_{1,0}=2\alpha$.
Then the recurrence relation (5.10) is solved by

$$d_{m,n} = \frac{4m}{n^2-m^2} (-1)^{\frac{1}{2}(m-n)}\alpha + O(\alpha^3), \text{ m+n odd.} \qquad 5.17$$

Hence

$$d_{p,n}\, d_{n,p}/\alpha^2 = \frac{-16mn}{(m^2-n^2)^2} + O(\alpha^2). \qquad 5.18$$

Substituting this result in 5.15 and denoting $\left[(d_{p,p}-1)/\alpha^2\right]_{\alpha=0}$ by D_p gives 5.9. By means of 5.8 and 5.9 the following numerical values are obtained:

$$\omega_{1,1} = 1{,}841 + 0{,}398\,\Omega^2 + O(\Omega^4),$$
$$\omega_{1,2} = 5{,}331 + 0{,}337\,\Omega^2 + O(\Omega^4),$$
$$\omega_{2,1} = 3{,}054 + 0{,}144\,\Omega^2 + O(\Omega^4),$$
$$\omega_{3,1} = 4{,}201 + 0{,}301\,\Omega^2 + O(\Omega^4),$$
$$\omega_{4,1} = 5{,}317 + 0{,}164\,\Omega^2 + O(\Omega^4).$$

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