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On the expansion of a function in a Fourier series with prescribed phases

by

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§ 1. Introduction

In this report[1] we shall consider the following expansion of a given real function

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx - \tfrac{1}{2}\alpha_n \pi) \qquad 1.1$$

in the halfperiod interval $0 \leqslant x \leqslant \pi$. The α_n are given constants, which will satisfy the conditions

$$\alpha_n = \alpha + O(n^{-\theta}), \quad \theta > 1$$

for $n \to \infty$ and

$$-1 < \operatorname{Re} \alpha < 1 .$$

We shall also consider the expansion

$$f(x) = \sum_{n=0}^{\infty} a_n \sin(nx - \tfrac{1}{2}\alpha_n \pi) \qquad 1.2$$

which differs from 1.1 in having a constant term.

These expansions have been investigated by H.A. LAUWERIER[2] and G.W. VELTKAMP. The former obtained the following results. Expansion 1.1 is generally possible if it is required that $b_n \to \infty$ for $n \to \infty$. The coefficients b_n appear to be of subharmonic order $-1+|\operatorname{Re}\alpha|$ for $n \to \infty$, expansion 1.2 is generally possible if it is required that $\sum_0^{\infty} |a_n| < \infty$. The coefficients a_n appear to be of hyperharmonic order $-1-|\operatorname{Re}\alpha|$ for $n \to \infty$.

In the simple special case $\alpha_n = \alpha$ it is possible to determine the expansion coefficients in the following way.

To the set

$$\sin(nx - \tfrac{1}{2}\alpha\pi), \quad n \geqslant 1 \; ^{3)}$$

the following set of biorthogonal functions is associated

$$k_m(x) = 2 \tan^{\alpha} \tfrac{1}{2}x \sum_{k=1}^{m} e_{m-k} \sin kx, \quad m \geqslant 1 \qquad 1.3$$

where the e_k are defined by the generating function

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1) Research carried out under the direction of Prof.Dr D. van Dantzig.

2) H.A. Lauwerier, On certain trigonometrical expansions. Report TW43 (1957) Mathematisch Centrum.

3) In Lauwerier's notation is $\alpha = -2\mu$.

$$\left(\frac{1+s}{1-s}\right)^{\alpha} = \sum_{0}^{\infty} e_k s^k \quad . \qquad 1.4$$

The first few coefficients are

$$e_0 = 1 \quad e_1 = 2\alpha \quad e_2 = 2\alpha^2 \ .$$

We also note the relation

$$\frac{2}{\pi}\int_0^{\pi} \tan^{-\alpha}\tfrac{1}{2}x \,\sin kx\, dx = -\sec\tfrac{1}{2}\alpha\pi\, e_k, \ k \geq 1. \qquad 1.5$$

In view of the orthogonality relation

$$\frac{1}{\pi}\int_0^{\pi} k_m(x)\sin(nx-\tfrac{1}{2}\alpha\pi)dx = \delta_{m,n} \ , \quad m,n \geq 1 \qquad 1.6$$

the coefficients b_n can be written as

$$b_n = \frac{1}{\pi}\int_0^{\pi} f(x)\ k_n(x)dx \ , \qquad n \geq 1 \ . \qquad 1.7$$

The more general expansion 1.1 can be reduced to a set of linear equations in the following way. Since the set

$$\sin(nx-\tfrac{1}{2}\alpha_n\pi), \quad n \geq 1 \ ,$$

is asymptotically orthogonal to the set $k_m(x)$, $m \geq 1$, it seems appropriate to determine the coefficients b_n from

$$\sum_{n=1}^{\infty} c_{m,n} b_n = \frac{1}{\pi}\int_0^{\pi} f(x)k_m(x)dx \ , \quad m \geq 1 \ , \qquad 1.8$$

where

$$c_{m,n} = \frac{1}{\pi}\int_0^{\pi} k_m(x)\sin(nx-\tfrac{1}{2}\alpha_n\pi)dx \ , \ m,n \geq 1. \qquad 1.9$$

It will be shown in §4 that the $c_{m,n}$ can be expressed in terms of coefficients

$$e_{m,n} = \frac{1}{\pi}\int_0^{\pi} k_m(x)\sin(nx+\tfrac{1}{2}\alpha\pi)dx, \quad m \geq 1,\ n \text{ integer}. \qquad 1.10$$

Comparison with 1.5 shows that

$$e_{m,n} = -\delta_{m,-n} \text{ for } n \leq -1. \qquad 1.11$$

For these coefficients the following relations will be proved

$$(-1)^n n\ e_{m,n} = (-1)^m m\ e_{n,m} \ , \qquad 1.12$$

$$m(e_{m+1,n}-e_{m-1,n}) + (n+1)e_{m,n+1}-(n-1)e_{m,n-1}=0, \qquad 1.13$$

$$e_{m+1,n}+e_{m-1,n}-e_{m,n+1}-e_{m,n-1}= e_{m,o}e_{1,n} , \qquad 1.14$$

$$e_{m,o} = e_m , \qquad 1.15$$

$$e_{m,n} = m \sum_{k=0}^{n} (-1)^k \frac{(m+k-1)!}{k!(n-k)!(m-n+k)!} e_{m-n+2k} . \qquad 1.16$$

Since the coefficients $e_{m,n}$ and e_n depend on α, we shall sometimes write $e_{m,n}(\alpha)$ and $e_n(\alpha)$. For coefficients with opposite α we shall derive

$$e_{m,n}(-\alpha) = (-1)^{m+n} e_{m,n}(\alpha) . \qquad 1.17$$

The rapidly convergent expansion 1.2 may be treated in a similar way. We first consider the case $\alpha_n = \alpha$. Then to the set

$$\sin(nx-\tfrac{1}{2}\alpha\pi) , \quad n \geq 0,$$

the following set of biorthogonal functions is associated

$$h_m(x) = -\tan^{\alpha-1} \tfrac{1}{2}x \sum_{k=0}^{m} \varepsilon_k e_{m-k}(\alpha-1)\cos kx, \; m \geq 0 \qquad 1.18$$

if $0 < \alpha < 1$ and

$$h_m(x) = \tan^{\alpha+1}\tfrac{1}{2}x \sum_{k=0}^{m} k\, e_{m-k}(\alpha+1)\cos kx, \; m \geq 0 \qquad 1.19$$

if $-1 < \alpha < 0$. ($\varepsilon_o=1$, $\varepsilon_k=2$ $(k \geq 1)$).
In view of the orthogonality relation

$$\frac{1}{\pi} \int_0^{\pi} h_m(x)\sin(nx-\tfrac{1}{2}\alpha\pi)dx = \delta_{m,n}, \quad m,n \geq 0, \qquad 1.20$$

the expansion coefficients a_n can be written as

$$a_n = \frac{1}{\pi} \int_0^{\pi} f(x)\, h_n(x)dx , \qquad n \geq 0 . \qquad 1.21$$

In the case of the more general expansion 1.2 the expansion coefficients again can be determined from a set of linear equations viz.

$$\sum_{n=0}^{\infty} d_{m,n}\, a_n = \frac{1}{\pi} \int_0^{\pi} f(x)h_m(x)dx, \quad m \geq 1, \qquad 1.22$$

where

$$d_{m,n} = \frac{1}{\pi}\int_0^\pi h_m(x)\sin(nx-\tfrac{1}{2}\alpha_n\pi)dx, \qquad m \geq 0 . \qquad 1.23$$

As in the previous case the $d_{m,n}$ can be expressed in terms of coefficients

$$f_{m,n} = \frac{1}{\pi}\int_0^\pi h_m(x)\sin(nx+\tfrac{1}{2}\alpha\pi)dx, \quad m \geq 0 , \qquad 1.24$$

but these coefficients will be shown to be related to the $e_{m,n}$ according to

$$f_{m,n}(\alpha) = -e_{n,m}(1-\alpha), \quad 1 > \alpha > 0 , \qquad 1.25$$
$$f_{m,n}(\alpha) = -e_{n,m}(-1-\alpha), \quad -1 < \alpha < 0 .$$

§ 2. The coefficients $e_{m,n}$.

In this section we prove formulae 1.12 - 1.17. We first derive a generating function for the $e_{m,n}$. According to 1.6 and 1.10 we have

$$\frac{1}{\pi}\int_0^\pi k_m(x)\sin(nx-\tfrac{1}{2}\alpha\pi)dx = \delta_{m,n} , \qquad m,n \geq 1,$$
$$\frac{1}{\pi}\int_0^\pi k_m(x)\sin(nx+\tfrac{1}{2}\alpha\pi)dx = e_{m,n}, \qquad m \geq 1.$$

Addition and substraction yields

$$\frac{2}{\pi}\int_0^\pi k_m(x)\sin nx\, dx = \sec\tfrac{1}{2}\alpha\pi\,(e_{m,n}+\delta_{m,n}), \quad m,n \geq 1,$$
$$\frac{\varepsilon_n}{\pi}\int_0^\pi k_m(x)\cos nx\, dx = \csc\tfrac{1}{2}\alpha\pi\,(e_{m,n}-\delta_{m,n}), \quad m \geq 1, n \geq 0.$$

Hence we have

$$k_m(x) = \sec\tfrac{1}{2}\alpha\pi\sum_{n=1}^{\infty}(e_{m,n}+\delta_{m,n})\sin nx, \quad m \geq 1,$$
$$= \csc\tfrac{1}{2}\alpha\pi\sum_{n=0}^{\infty}(e_{m,n}-\delta_{m,n})\cos nx, \quad m \geq 1.$$

From this formula we find the generating function [4]

$$\cos(mx-\tfrac{1}{2}\alpha\pi) = \sum_{n=0}^{\infty} e_{m,n}\cos(nx+\tfrac{1}{2}\alpha\pi) . \qquad 2.1$$

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4) This generating function is due to a suggestion of H.A. Lauwerier.

This formula has been obtained for $m \geq 1$. By means of this expression the $e_{m,n}$ can also be defined for $m \leq 0$, $n \geq 0$. Then we have

$$e_{m,n} = \delta_{-m,n} \ , \ m \leq n, \ n \geq 0. \qquad 2.2$$

If, in 2.1 we replace x by $\pi - x$, we find

$$\cos(mx+\tfrac{1}{2}\alpha\pi) = \sum_{n=0}^{\infty} (-1)^{m+n} e_{m,n} \cos(nx-\tfrac{1}{2}\alpha\pi) \ . \qquad 2.3$$

Comparison of 2.1 with 2.3 immediately yields 1.17.
Relation 1.14 can be proved as follows

$$\sum_{n=0}^{\infty} (e_{m+1,n}+e_{m-1,n}-e_{m,n+1}-e_{m,n-1})\cos(nx+\tfrac{1}{2}\alpha\pi)$$
$$= \sum_{n=0}^{\infty} (e_{m+1,n}+e_{m-1,n})\cos(nx+\tfrac{1}{2}\alpha\pi)$$
$$- \sum_{n=0}^{\infty} e_{m,n}\cos\left\{(n-1)x+\tfrac{1}{2}\alpha\pi\right\} - \sum_{n=0}^{\infty} e_{m,n}\cos\left\{(n+1)x+\tfrac{1}{2}\alpha\pi\right\}$$
$$= \cos\left\{(m+1)x-\tfrac{1}{2}\alpha\pi\right\} + \cos\left\{(m-1)x-\tfrac{1}{2}\alpha\pi\right\}$$
$$-2\cos x \sum_{n=0}^{\infty} e_{m,n}\cos(nx+\tfrac{1}{2}\alpha\pi)+e_{m,o}\cos(x-\tfrac{1}{2}\alpha\pi)$$
$$= e_{m,o} \sum_{n=0}^{\infty} e_{1,n}\cos(nx+\tfrac{1}{2}\alpha\pi)$$

because of $e_{m,-1}=0$ (comp.1.10).
Identification of corresponding coefficients yields 1.14.
Differentiation of 2.1 and 2.3 yields

$$m\sin(mx-\tfrac{1}{2}\alpha\pi) = \sum_{n=1}^{\infty} n e_{m,n}\sin(nx+\tfrac{1}{2}\alpha\pi) \qquad 2.4$$

$$m\sin(mx+\tfrac{1}{2}\alpha\pi) = \sum_{n=1}^{\infty} n(-1)^{m+n} e_{m,n}\sin(nx-\tfrac{1}{2}\alpha\pi) \ . \qquad 2.5$$

Multiply 2.5 with $k_p(x)/\pi$ and integrate from 0 to π. We find

$$m\, e_{p,m} = n(-1)^{m+n} e_{m,n}\, \delta_{p,n} = p(-1)^{m+p} e_{m,p}$$

in accordance with 1.12.
Relation 1.13 can be proved by means of 2.1 and 2.4. We have

$$\sum_{n=0}^{\infty} m(e_{m+1,n}-e_{m-1,n})\cos(nx+\tfrac{1}{2}\alpha\pi)$$
$$= m\cos\left\{(m+1)x-\tfrac{1}{2}\alpha\pi\right\} - m\cos\left\{(m-1)x-\tfrac{1}{2}\alpha\pi\right\}$$

$$= -2m \sin x \sin(mx-\tfrac{1}{2}\alpha\pi)$$
$$= -2 \sin x \sum_{n=1}^{\infty} n\, e_{m,n} \sin(nx+\tfrac{1}{2}\alpha\pi)$$
$$= \sum_{n=1}^{\infty} n\, e_{m,n} \cos\left\{(n+1)x+\tfrac{1}{2}\alpha\pi\right\} - \sum_{n=1}^{\infty} n\, e_{m,n} \cos\left\{(n-1)x+\tfrac{1}{2}\alpha\pi\right\}$$
$$= \sum_{n=0}^{\infty} \left\{(n-1)e_{m,n-1}-(n+1)e_{m,n+1}\right\} \cos(nx+\tfrac{1}{2}\alpha\pi),$$

because of $e_{m,-1}=0$.
Identification of corresponding coefficients yields 1.13.

For the proof of 1.15 we first show

$$\sum_{0}^{m} e_{m-k}(\alpha)e_k(-\alpha) = \delta_{m,o} . \qquad 2.6$$

Indeed we have, because of 1.4,

$$1 = \left(\frac{1+s}{1-s}\right)^{\alpha}\left(\frac{1+s}{1-s}\right)^{-\alpha}$$
$$= \sum_{m=0}^{\infty} e_m(\alpha)s^m \sum_{k=0}^{\infty} e_k(-\alpha)s^k$$
$$= \sum_{m=0}^{\infty} s^m \sum_{k=0}^{m} e_{m-k}(\alpha)e_k(-\alpha).$$

Using 1.10, 1.3, 1.5 and 2.6 we have

$$e_{m,o} = \sin \tfrac{1}{2}\alpha\pi \; \frac{1}{\pi} \int_{o}^{\pi} k_m(x)dx$$
$$= \sin \tfrac{1}{2}\alpha\pi \sum_{k=1}^{m} e_{m-k}(\alpha) \frac{2}{\pi} \int_{o}^{\pi} \tan^{\alpha} \tfrac{1}{2}x \sin kx\, dx$$
$$= - \sum_{k=1}^{m} e_{m-k}(\alpha)e_k(-\alpha)$$
$$= e_m ,$$

because $e_o=1$.

We shall prove 1.16 by induction. By substitution it is verified, that 1.16 satisfies the recurrence relation 1.13. Moreover for n=0 1.16 becomes identical with 1.15 which we already know to be true. For n=-1 we find from 1.16 $e_{m,-1}=0$ in accordance with 1.11. Hence, since 1.16 is valid for two values of n, it must be valid for all values of n.

We conclude this section with a remark concerning the practical evaluation of the $e_{m,n}$. Let it be required to evaluate a square array of $e_{m,n}$, $m,n=1...N$. We start with the column $e_{m,o}=e_m$. For the e_m Lauwerier proved the recurrence relation

$$me_m = 2\alpha e_{m-1} + (m-2)e_{m-2} \qquad 2.7$$

which admits the evaluation of all e_m from $e_o=1$ and $e_1=2\alpha$. Then the $e_{m,1}$ can be evaluated by aid of 1.13, except $e_{N,1}$. However, from 1.13 and 1.14 we obtain

$$(m+n+1)e_{m,n+1}=2m\ e_{m-1,n}-(m-n+1)e_{m,n-1}-m\ e_{m,o}e_m \qquad 2.8$$

and this formula can be used for evaluating $e_{N,1}$.
In general we evaluate $e_{m,n}$, $m \geqslant n$, $m < N$ from two preceding columns by aid of 1.13. For m n we use 1.12 and $e_{N,n}$ is found by aid of 2.8.
A table of $e_{m,n}$, $m,n=0(1)6$ is added to this report.

§ 3. The coefficients $f_{m,n}$

According to 1.20 and 1.22 we have

$$\frac{1}{\pi}\int_0^\pi h_m(x)\sin(nx-\tfrac{1}{2}\alpha\pi)dx = \delta_{m,n}\ , \quad m,n \geqslant 0,$$

$$\frac{1}{\pi}\int_0^\pi h_m(x)\sin(nx+\tfrac{1}{2}\alpha\pi)dx = f_{m,n}, \quad m \geqslant 0.$$

In a similar way as before we find

$$\cos(mx-\tfrac{1}{2}\alpha\pi) = \sum_{n=0}^{\infty} f_{m,n}\cos(nx+\tfrac{1}{2}\alpha\pi), \quad m \geqslant 0 \qquad 3.1$$

Furthermore we have, because of 2.5 and 1.12,

$$\sin(mx+\tfrac{1}{2}\alpha\pi) = \sum_{n=1}^{\infty} \frac{n}{m}(-1)^{m+n} e_{m,n}(\alpha)\sin(nx-\tfrac{1}{2}\alpha\pi)$$

$$= \sum_{n=1}^{\infty} e_{n,m}(\alpha)\sin(nx-\tfrac{1}{2}\alpha\pi) \qquad 3.2$$

If $0 < \alpha < 1$ we put $\alpha = 1-\alpha'$. Substitution in 3.2 gives, if the primes are dropped,

$$\cos(mx-\tfrac{1}{2}\alpha\pi) = -\sum_{n=1}^{\infty} e_{n,m}(1-\alpha)\cos(nx+\tfrac{1}{2}\alpha\pi)\ . \qquad 3.3$$

Comparison of 3.1 and 3.3 yields

$$f_{m,n}(\alpha) = -e_{n,m}(1-\alpha) \ , \quad 1 > \alpha > 0.$$

If $-1 < \alpha < 0$ we put $\alpha = -1-\alpha'$ and find in the same way

$$f_{m,n}(\alpha) = -e_{n,m}(-1-\alpha) \qquad 0 > \alpha > -1 \ .$$

Hence the relations 1.25 are proved

§ 4. The general expansions

We shall show how the expansion coefficients b_n in

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx - \tfrac{1}{2}\alpha_n \pi)$$

can be determined.

Let $\alpha_n \longrightarrow \alpha$ for $n \longrightarrow \infty$. By means of the trigonometrical identity

$$\sin(\varphi-\beta) = \frac{\sin(\beta+\gamma)}{\sin 2\gamma}\left[\sin(\varphi-\gamma) + \frac{\sin(\beta-\gamma)}{\sin(\beta+\gamma)}\sin(\varphi+\gamma)\right],$$

this can be written in the form

$$f(x) = \sum_{n=1}^{\infty} b_n' \left[\sin(nx-\tfrac{1}{2}\alpha\pi) + s_n \sin(nx+\tfrac{1}{2}\alpha\pi)\right], \tag{4.1}$$

where

$$b_n' = \frac{\sin \frac{1}{2}(\alpha+\alpha_n)}{\sin \alpha\pi} b_n \ , \tag{4.2}$$

$$s_n = \frac{\sin \frac{1}{2}(\alpha-\alpha_n)}{\sin \frac{1}{2}(\alpha+\alpha_n)} \ . \tag{4.3}$$

Proceeding as indicated in the introduction we find

$$\sum_{n=1}^{\infty} \left\{ \delta_{m,n} + s_n e_{m,n}(\alpha) \right\} b_n' = \frac{1}{\pi}\int_0^{\pi} f(x) k_m(x) dx. \tag{4.4}$$

In the same way we can derive for the rapidly convergent expansion 1.2 the sets of linear equations

$$\sum_{n=0}^{\infty} \left\{ \delta_{m,n} - s_n e_{n,m}(1-\alpha) \right\} a_n' = \frac{1}{\pi}\int_0^{\pi} f(x) h_m(x) dx, \quad 0 < \alpha < 1, \tag{4.5}$$

$$\sum_{n=0}^{\infty} \left\{ \delta_{m,n} - s_n e_{n,m}(-1-\alpha) \right\} a_n' = \frac{1}{\pi}\int_0^{\pi} f(x) h_m(x) dx, \quad -1 < \alpha < 0, \tag{4.6}$$

where

$$a'_n = \frac{\sin \frac{1}{2}(\alpha+\alpha_n)}{\sin \alpha\pi} a_n .$$

If for f(x) a Fourier expansion valid in the half-period interval $0 < x < \pi$ is known, we can evaluate the right hand members of 4.4, 4.5 or 4.6 in a similar way. Let e.g.

$$f(x) = \sum_{n=1}^{\infty} p_n \sin(nx - \tfrac{1}{2}\varphi_n \pi)$$

then

$$\frac{1}{\pi}\int_0^{\pi} f(x) k_m(x) dx = \sum_{n=1}^{\infty} p'_n \left\{\delta_{m,n} + q_n e_{m,n}(\alpha)\right\}$$

where

$$p'_n = \frac{\sin \frac{1}{2}(\alpha+\varphi_n)}{\sin \alpha\pi} p_n ,$$

$$q_n = \frac{\sin \frac{1}{2}(\alpha-\varphi_n)\pi}{\sin \frac{1}{2}(\alpha+\varphi_n)\pi} .$$

Table of $e_{m,n}(\alpha)$

$e_{00}= 1$
$e_{10}= 2\alpha$
$e_{20}= 2\alpha^2$
$e_{30}= \frac{2}{3}\alpha(1+2\alpha^2)$
$e_{40}= \frac{2}{3}\alpha(2+\alpha^2)$
$e_{50}= \frac{2}{15}\alpha(3+10\alpha^2+2\alpha^4)$
$e_{60}= \frac{2}{45}\alpha^2(23+20\alpha^2+2\alpha^4)$

$e_{01}= 0$
$e_{11}= 1-2\alpha^2$
$e_{21}= \frac{8}{3}\alpha(1-\alpha^2)$
$e_{31}= 2\alpha^2(1-\alpha^2)$
$e_{41}= \frac{16}{15}\alpha(1-\alpha^2)(1+\alpha^2)$
$e_{51}= \frac{2}{9}\alpha^2(1-\alpha^2)(7+2\alpha^2)$
$e_{61}= \frac{8}{105}\alpha(1-\alpha^2)(9+16\alpha^2+2\alpha^4)$

$e_{02}= 0$
$e_{12}= -\frac{4}{3}\alpha(1-\alpha^2)$
$e_{22}= 1-2\alpha^2(2-\alpha^2)$
$e_{32}= \frac{4}{5}\alpha(1-\alpha^2)(3-2\alpha^2)$
$e_{42}= \frac{8}{9}\alpha^2(1-\alpha^2)^2$
$e_{52}= \frac{4}{21}\alpha(1-\alpha^2)(5-2\alpha^2-2\alpha^4)$
$e_{62}= \frac{2}{15}\alpha^2(1-\alpha^2)^2(6+\alpha^2)$

$e_{03}= 0$
$e_{13}= \frac{2}{3}\alpha^2(1-\alpha^2)$
$e_{23}= -\frac{8}{15}\alpha(1-\alpha^2)(3-2\alpha^2)$
$e_{33}= 1-\frac{2}{9}\alpha^2(19-14\alpha^2+4\alpha^4)$
$e_{43}= \frac{16}{63}\alpha(1-\alpha^2)(9-5\alpha^2+2\alpha^4)$
$e_{53}= \frac{2}{9}\alpha^2(1-\alpha^2)(3-\alpha^2+\alpha^4)$
$e_{63}= \frac{8}{405}\alpha(1-\alpha^2)(45-14\alpha^2+10\alpha^4+4\alpha^6)$

$e_{04}= 0$
$e_{14}= -\frac{4}{15}\alpha(1-\alpha^2)(1+\alpha^2)$
$e_{24}= \frac{4}{9}\alpha^2(1-\alpha^2)^2$
$e_{34}= -\frac{4}{21}\alpha(1-\alpha^2)(9-5\alpha^2+2\alpha^4)$
$e_{44}= 1-\frac{2}{9}\alpha^2(20-14\alpha^2+4\alpha^4-\alpha^6)$
$e_{54}= \frac{4}{81}\alpha(1-\alpha^2)(45-29\alpha^2+4\alpha^4-2\alpha^6)$
$e_{64}= \frac{4}{225}\alpha^2(1-\alpha^2)^2(27+4\alpha^2+2\alpha^4)$

$$e_{05} = 0$$
$$e_{15} = \frac{2}{45}\alpha^2(1-\alpha^2)(7+2\alpha^2)$$
$$e_{25} = -\frac{8}{105}\alpha(1-\alpha^2)(5-2\alpha^2-2\alpha^4)$$
$$e_{35} = \frac{2}{15}\alpha^2(1-\alpha^2)(3-\alpha^2+\alpha^4)$$
$$e_{45} = -\frac{16}{405}\alpha(1-\alpha^2)(45-29\alpha^2+4\alpha^4-2\alpha^6)$$
$$e_{55} = 1-\frac{2}{225}\alpha^2(509-390\alpha^2+112\alpha^4-10\alpha^6+4\alpha^8)$$
$$e_{65} = \frac{8}{2475}\alpha(1-\alpha^2)(675-414\alpha^2+136\alpha^4+4\alpha^6+4\alpha^8)$$

$$e_{06} = 0$$
$$e_{16} = -\frac{4}{315}\alpha(1-\alpha^2)(9+16\alpha^2+2\alpha^4)$$
$$e_{26} = \frac{2}{45}\alpha^2(1-\alpha^2)(6-5\alpha^2-\alpha^4)$$
$$e_{36} = -\frac{4}{405}\alpha(1-\alpha^2)(45-14\alpha^2+10\alpha^4+4\alpha^6)$$
$$e_{46} = \frac{8}{675}\alpha^2(1-\alpha^2)(27+4\alpha^2+2\alpha^4)$$
$$e_{56} = -\frac{4}{1485}\alpha(1-\alpha^2)(675-414\alpha^2+136\alpha^4+4\alpha^6+4\alpha^8)$$
$$e_{66} = 1-\frac{2}{2025}\alpha^2(4662-3499\alpha^2+1096\alpha^4-222\alpha^6-8\alpha^8-4\alpha^{10}).$$