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Free oscillations of a fluid in a rotating rectangular basin

by

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§ 1. Introduction

The present study concerns the free tidal oscillations in a rotating rectangular sea. This problem is of importance for the investigation into the behaviour of the free waves of the North Sea basin. The North Sea has roughly a rectangular shape bordered by three coasts and open at one end. (Cf. D. van Dantzig, 1954). The depth varies only a little and has the average (harmonic) value of roughly 65 m. At the open end the sea goes over into the ocean which has a depth of some 1000 m. So it may not be unrealistic to consider the mathematical model of an open rectangular sea of constant depth bordering on an infinitely deep ocean.

Tidal oscillations in rotating rectangular seas have been studied by a number of, mostly British, investigators. However, they considered only the case of a closed sea surrounded by four coasts.

Rayleigh (1903,1909) considered the free oscillations when the rotation (Ω) is small but some of his results were in error. Taylor (1922) gave the first complete solution for a closed rectangular sea. Jeffreys (1925) criticized Taylor's conclusions and pointed out that a double infinity of eigenvalues was implied in Taylor's solution and that there might be modes moving round the basin in both directions. Lamb (1932) derived by a different method the approximations to the lowest eigenvalues when Ω is small. In particular he obtained for a square sea the formula

$$\omega - \omega_0 = \pm \frac{4}{\pi^2} \Omega .$$

Goldsbrough (1931) gave an approximate solution for the free oscillations in a rotating rectangular sea. In particular the case of a square sea was investigated and Lamb's formula was confirmed. Proudman (1933) re-examined Rayleigh's investigation and showed that by a correct application of Rayleigh's principle Lamb's formula can be obtained. Corkan and Doodson (1952) considered free oscillations in a rotating square sea. By the use of iteration methods a number of numerical cases were treated.

The flood disaster of February 1953 stimulated further research in this field. Van Dantzig (1954) gave a review of the results obtained at the Mathematical Centre. At that time some preliminary results concerning the free oscillations in an open rotating rectangle were obtained. More recent work is reviewed by Van Dantzig (1956,1958). In the subsequent reports of the Mathematical Centre some aspects were treated which are of importance also in the case of the free motions. Lauwerier (1955,1957b) and Veltkamp investigated a type of trigonometrical expansion which appears here in § 6. Veltkamp (1956a) discussed at length the Kelvin and Poincaré waves which appear here in §§3,4 and 6. Veltkamp (1956b) also considered in detail the nature of the singularity which is introduced at the confluence of the coast and ocean boundary. Lauwerier (1957a,1958) considered the free motions in an infinite strip with a coast and ocean boundary and in a semi-infinite strip with three coasts. The latter case is not explicitly dealt with here but can easily be obtained as a limiting case of the rectangular basin of § 4 or also of § 6.

Summary. The free motions and the free periods depend on the Coriolis coefficient Ω and the coefficient of friction λ. Although later on only the case of real Ω (and $\lambda=0$) is mentioned explicitly, the results are valid for complex Ω also, so that by the transformation mentioned the case $\lambda\neq0$ (and real Ω) also is covered. In § 2 it is shown following Veltkamp (1956a) that by a complex transformation the case $\lambda\neq0$ can be reduced to the case $\lambda=0$, which therefore is the only one that need be considered. For $\Omega=0$ the problem is elementary and in the rectangular cases of § 4 and § 6 a discrete spectrum of eigenvalues is obtained. For $\Omega\neq0$ only those eigenvalues are considered which for $\Omega\to0$ tend to this discrete spectrum. A possible continuous spectrum is left out of consideration here.

In § 3 the -well-known- case of an infinite channel is treated. In § 4 the problem is solved for a rectangular lake by making use of the familiar technique of operators in Hilbert space. The formal solution obtained in this section is developed for small Ω in § 5. As a specialization the case of a square rotating sea is considered. The results obtained in this section include those obtained by the

authors mentioned above.

In § 6 the case of a rectangular bay is dealt with. The results obtained in this section are all new. Again an expansion of the eigenvalues for small Ω is derived. For the lowest eigenvalue we obtain for a bay which is twice as long as it is wide

$$\frac{\omega}{\omega_0} = 1 - 2.01\,\Omega^2 + 0(\Omega^4)$$

so that a slight rotation tends to increase the free period, which is in contrast to the case of a rectangular lake. In the same dimensions as above we find in the latter case

$$\frac{\omega}{\omega_0} = 1 + 0.30\,\Omega^2 + 0(\Omega^4)\ .$$

The treatment in this paper is not in all respects satisfactory since the answers obtained are valid for small Ω only.

By using expansions of Fredholm type for the resolvents this rectriction could have - but has not - been avoided. Therefore the results obtained here are applicable to a laboratory model of a rectangular basin rotating with not too large velocity, rather than to the North Sea where Ω is rather large. In view of the considerable complications occurring here the determination of the orthogonal functions into which an arbitrary elevation pattern might be expanded has not been carried out.

I am greatly indebted to Dr H.A. Lauwerier and Dr D.J. Hofsommer for carefully studying the text and improving it on several points.

§2. Exposition of the problems

We consider a plane sheet of water of constant density 1 and at equilibrium of depth d, rotating with angular velocity $\frac{1}{2}\Omega$. We assume quantities of higher order than the first in the velocities and their derivatives to be negligible, and friction to be representable by a force parallel to and proportional with the velocity, and the pressure distribution in a vertical column to be the same as in the stationary state, so that the equations in three dimensions can be integrated over the depth of the sea. Let x and y be orthogonal Cartesian coordinates in the plane; ζ the height of the free surface above its equilibrium level; u and v the components of the current i.e. the velocity integrated over the depth (H. Lamb denotes our d, n and v by h, hu and hv respectively); W_1 and W_2 the components of the exterior force; λ the coefficient of friction; and g the acceleration of gravity.

Then the equations of motion and of continuity, integrated over the depth, are

$$(2.1)\qquad \begin{aligned} \lambda u - \Omega v + \frac{\partial u}{\partial t} + gd\,\frac{\partial \zeta}{\partial x} &= W_1 \\ \Omega u + \lambda v + \frac{\partial v}{\partial t} + gd\,\frac{\partial \zeta}{\partial y} &= W_2 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} &= 0 . \end{aligned}$$

We assume g, d, λ and Ω to be positive constants. We may then choose the propagation velocity $\sqrt{gd}$ of free waves in the absence of friction and rotation as the unit of velocity, so that gd=1. In the case of a basin having roughly the mean depth of the North Sea, this unit of velocity is about 100 km/h (corresponding for $g = 9{,}81\ m/sec^2$ with $d = 78{,}7$ m). We further assume the basin either to be bounded on all sides by impenetrable coasts (i.e. the influence of rivers, estuaries, flat shores etc. to be negligible), or partly by such coasts and partly by an infinitely deep ocean. Along a coast the normal component of the velocity vanishes; along an ocean the height of the sea level is constant, i.e. $\zeta = 0$.

We want to study the free (damped) oscillations of a rectangular basin only. Then $W_1 = W_2 = 0$, and u, v and ζ are the real parts of quantities proportional with $e^{i\omega t}$, where ω will be complex with positive imaginary part for $\lambda > 0$. Denoting the proportionality factors by the same symbols, (2.1) passes into

$$
(2.2) \qquad
\begin{aligned}
&(i\omega+\lambda)u - \Omega v + \frac{\partial \zeta}{\partial x} = 0 \\
&\Omega u + (i\omega+\lambda)v + \frac{\partial \zeta}{\partial y} = 0 \\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\omega\zeta = 0 .
\end{aligned}
$$

By means of a complex change of scale these equations may be reduced to a system having the same type (and the same boundary conditions), but without friction (Veltkamp, TW 37). Therefore we put

$$
(2.3) \qquad
\begin{aligned}
&\omega' \overset{\text{def}}{=\!=} \omega\left(1 - \frac{i\lambda}{\omega}\right)^{\frac{1}{2}} \\
&\Omega' \overset{\text{def}}{=\!=} \Omega\left(1 - \frac{i\lambda}{\omega}\right)^{-\frac{1}{2}} \\
&\zeta' \overset{\text{def}}{=\!=} \zeta\left(1 - \frac{i\lambda}{\omega}\right)^{-\frac{1}{2}}
\end{aligned}
$$

thereby obtaining a system of the form (2.2), with $u'=u$, $v'=v$ and $\lambda'=0$. If the free frequencies for this system under given boundary conditions have been obtained in the form $\omega' = \varphi(\Omega')$ then ω is obtained by solving the equation

$$
(2.4) \qquad \omega\left(1 - \frac{i\lambda}{\omega}\right)^{\frac{1}{2}} = \varphi\left(\Omega\left(1 - \frac{i\lambda}{\omega}\right)^{-\frac{1}{2}}\right).
$$

If $\lambda \ll |\omega|$ we obtain the first order approximation

$$
(2.5) \qquad \omega = \varphi(\Omega) + \tfrac{1}{2} i\lambda\left[1 + \Omega\varphi'(\Omega)/\varphi(\Omega)\right] + o(\lambda^2/\omega^2) .
$$

Hence the periods remain unaltered to a first approximation, but the oscillations are damped, the damping factor
$\exp -\frac{1}{2}\lambda\left[1+\Omega\varphi'(\Omega)/\varphi(\Omega)\right] t$ being computable as soon as $\varphi(\Omega)$ is known for <u>real</u> Ω only.
Similarly, if $\zeta' = \psi(\Omega') = \psi(\Omega', x, y)$ then

$$
(2.6) \qquad \zeta = \psi(\Omega) - i\frac{\lambda}{2\omega}\left[\psi(\Omega) - \Omega\psi'(\Omega)\right] + o(\lambda^2/\omega^2) .
$$

Hence the amplitude of ζ for $\lambda > 0$ is obtained to a first approximation from that for $\lambda = 0$ by multiplication with the damping facto

$$\exp - \frac{\lambda}{2\omega}\left[\psi(\Omega) - \Omega\psi'(\Omega)\right]t \tag{2.7}$$

It is therefore sufficient to consider the frictionless case only.

By dropping the primes, or also by putting $\lambda = 0$ in equations (2.2) these simplify to

$$\begin{aligned} i\omega u - \Omega v + \frac{\partial \zeta}{\partial x} &= 0 \\ \Omega u + i\omega v + \frac{\partial \zeta}{\partial y} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\omega\zeta &= 0\,. \end{aligned} \tag{2.8}$$

It follows that

$$\begin{aligned} \Delta u - \kappa^2 u &= 0 \\ \Delta v - \kappa^2 v &= 0 \\ \Delta \zeta - \kappa^2 \zeta &= 0 \end{aligned} \tag{2.9}$$

where $\Delta \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and

$$\kappa^2 \stackrel{\text{def}}{=} \Omega^2 - \omega^2\,. \tag{2.10}$$

Moreover we notice the relations

$$\left\{ \begin{aligned} u &= -\kappa^{-2}\left(i\omega \frac{\partial \zeta}{\partial x} + \Omega \frac{\partial \zeta}{\partial y}\right) \\ v &= \kappa^{-2}\left(\Omega \frac{\partial \zeta}{\partial x} - i\omega \frac{\partial \zeta}{\partial y}\right) \end{aligned} \right. \tag{2.11}$$

$$\left\{ \begin{aligned} \frac{\partial^2 \zeta}{\partial y^2} + \omega^2 \zeta &= i\omega \frac{\partial u}{\partial x} - \Omega \frac{\partial u}{\partial y} \\ \frac{\partial^2 v}{\partial y^2} + \omega^2 v &= i\Omega\omega u - \frac{\partial^2 u}{\partial x \partial y} \end{aligned} \right. \tag{2.12}$$

§3. Infinite channel

For convenience we reformulate the long known solutions for an infinite straight channel of (constant) width 2a. We choose a/π as the unit of length, so that $a/\pi\sqrt{gd}$ becomes the unit of time. If the channel has the mean width and depth of the North Sea, the unit of length becomes roughly 125 km, and the unit of time accordingly 1,25 h. At the latitude of the North Sea Ω is roughly $0{,}44\ h^{-1}$, i.e. 0,55 in our dimensionless units.

Choosing one coast of the channel as the axis of y, so that the two coasts become $x=0$, $x=\pi$, the boundary conditions are

$$u=0 \quad \text{for} \quad x=0 \text{ and } x=\pi \ .$$

We consider solutions (u,v,ζ) only which are (for almost every y) quadratically integrable (shortly: $\in L_2$) over $0<x<\pi$ so that the theories of Fourier series and of Hilbert space can be applied. Every $u\in L_2$ on this interval has a development

$$u = \sum_{-\infty}^{+\infty} u_n(y)\ e^{inx} \ .$$

The first differential equation (2.9) requires $u_n(y) :: e^{\pm \nu_n y}$ where

$$(3.1) \qquad \nu_n \overset{\text{def}}{=\!=} \sqrt{n^2+k^2} \qquad (k^2 = \Omega^2 - \omega^2) \ .$$

Hence, if ε denotes a variable, running through the two values +1 and -1 (or also: through the two signs + and -), summation over which is denoted by a small Σ, we have, inserting a factor $-\frac{1}{2}i(n^2+\Omega^2)$

$$u = -\tfrac{1}{2}i\,\varepsilon\sum C_n^{\varepsilon}\,(n^2+\Omega^2)\ e^{inx+\varepsilon\nu_n y} \ .$$

In order that the boundary conditions be satisfied, it is necessary and sufficient that

$$(3.2) \qquad C_{-n}^{\varepsilon} = -C_n^{\varepsilon}$$

i.e. that the C_n^{ε} are odd functions of n. Hence u is a linear combination with coefficients $(n^2+\Omega^2)C_n^{\varepsilon}$ of the elementary solutions

$$e^{\varepsilon\nu_n y} \sin nx$$

which, together with the corresponding v and ζ, obtained from (2.11) by substituting for these unknown linear combinations of the $e^{inx-\varepsilon\nu_n y}$, are called "Poincaré waves". From (2.11) however, we see that we still have to add the solutions of the corresponding homogeneous equations, together with u=0, the socalled "Kelvin waves". These can also be obtained formally from the Poincaré waves by substitution of $-i\varepsilon\Omega$ for n, $i\omega$ for ν_n. So we obtain the general solution for the infinite channel with a slight change of irrelevent constant factors:

$$u = -\tfrac{1}{2}i \,\Sigma \sum C_n^{\varepsilon}(n^2+\Omega^2)\, e^{inx-\varepsilon\nu_n y}$$

(3.3) $$v = \tfrac{1}{2}\, \Sigma \sum C_n^{\varepsilon}(\varepsilon n\nu_n+\Omega\omega)\, e^{inx-\varepsilon\nu_n y} - \Sigma C^{\varepsilon}\, e^{\varepsilon(\Omega x - i\omega y - \frac{1}{2}\pi\Omega)}$$

$$\zeta = \tfrac{1}{2}i\, \Sigma \sum C_n^{\varepsilon}(n\omega-\varepsilon\Omega\nu_n)\, e^{inx-\varepsilon\nu_n y} - \Sigma\varepsilon C^{\varepsilon}\, e^{\varepsilon(\Omega x - i\omega y - \frac{1}{2}\pi\Omega)}$$

where $\sum$ denotes a summation over all integers n and Σ one over the two values $\varepsilon = +1$ and $\varepsilon = -1$, or also over the two signs $\varepsilon = +$ and $\varepsilon = -$, whereas the conditions (3.2) must be satisfied. We notice that $\nu_{-n}=\nu_n$ so that the Poincaré wave parts of ν and ζ contain a term ($C_n^{\varepsilon}\Omega\omega$, $C_n^{\varepsilon}\varepsilon\Omega\nu_n$ respectively) which is an odd and one ($\varepsilon C_n^{\varepsilon} n\nu_n$, $C_n^{\varepsilon} n\omega$ respectively) which is an even function of n. By means of (3.2) the equations (3.3) become

$$u = \Sigma\sum_1^{\infty} C_n^{\varepsilon}(n^2+\Omega^2) \sin nx\; e^{-\varepsilon\nu_n y}$$

(3.4) $$v = \Sigma\sum_1^{\infty} C_n^{\varepsilon}(\varepsilon n\nu_n \cos nx + i\Omega\omega \sin nx)\, e^{-\varepsilon\nu_n y} + - \Sigma C^{\varepsilon}\, e^{\varepsilon(\Omega x - i\omega y - \frac{1}{2}\pi\Omega)}$$

$$\zeta = \Sigma\sum_1^{\infty} C_n^{\varepsilon}(in\omega \cos nx + \varepsilon\Omega\nu_n \sin nx)\, e^{-\varepsilon\nu_n y} + - \Sigma\varepsilon C^{\varepsilon}\, e^{\varepsilon(\Omega x - i\omega y - \frac{1}{2}\pi\Omega)}\,.$$

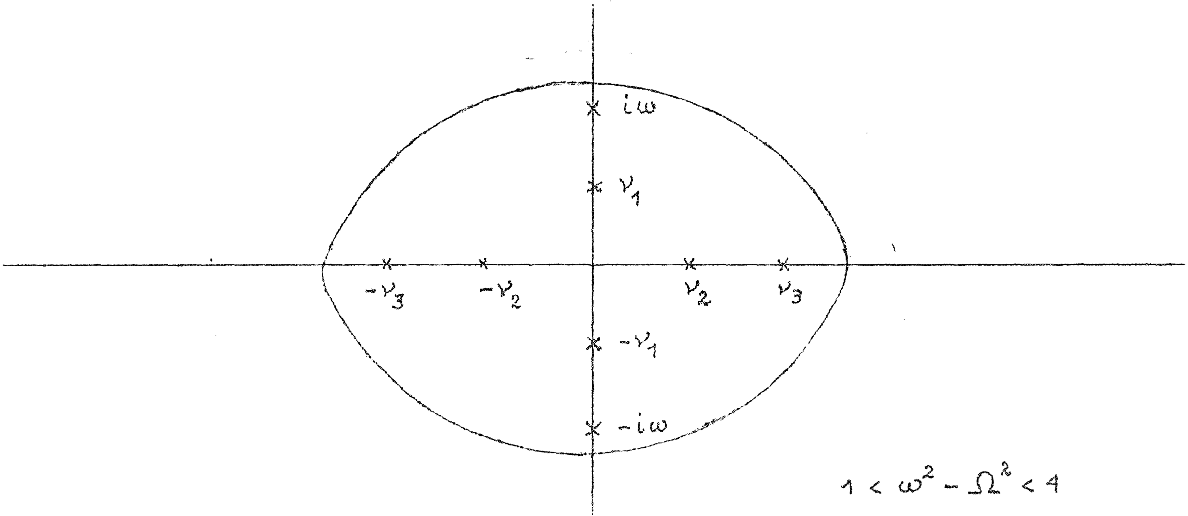
The fact that the two Kelvin waves appear to be analogous with a pair of Poincaré waves could be exploited by treating them all on the same footing by using the calculus of residus instead of the calculus of series. This can e.g. be done in a complex ν-plane, with $n=n(\nu)= \pm(\nu^2+\omega^2-\Omega^2)^{\frac{1}{2}}$. The twovaluedness of $n(\nu)$ is not serious, as only even functions of it enter into the integrands. With respect to an increasing sequence of domains asymptotically including all zeros of the denominators (or also a single domain $|\mathrm{Im}\,\nu|<a$, where $a>|\omega|$), we obtain instead of (3.4), with a function $C(\nu)$, holomorphic in the (limiting) domain,

$$u = \mathcal{E}\, \frac{\sin n(\nu)\ x\ C(\nu)\ e^{-\nu y}}{\tan \pi n(\nu)}$$

(3.5) $$v = \mathcal{E}\, \frac{\nu n(\nu) \cos n(\nu)x + i\omega\Omega \sin n(\nu)x}{(\nu^2+\omega^2)\tan\pi n(\nu)}\; C(\nu)\, e^{-\nu y}$$

$$\zeta = \mathcal{E} \frac{i\omega\, n(\nu)\, \cos n(\nu)x + \Omega\nu \sin n(\nu)x}{(\nu^2+\omega^2)\, \tan \pi n(\nu)}\, C(\nu)\, e^{-\nu y}$$

where $\mathcal{E}$ denotes the sum of the residuals:



We shall, however, not make use of this calculus. Formal developments (3.4) will only be considered as solutions of our problem if u and v belong (for almost every y) to the class L_2 of quadratically integrable functions of x on $0 < x < \pi$. The same then is true for $\partial \zeta / \partial x$ and $\partial \zeta / \partial y$, and a fortiori for ζ itself. The condition is equivalent with saying that the sequences $n^2\, C_n^{\varepsilon}\, e^{-\varepsilon \nu_n y}$ belong for both ε and all y with $0 < y < b$ to the class l_2 of quadratically summable sequences, or also:

$$\sum n^4 \left| C_n^{\varepsilon} \right|^2 e^{2b|\mathrm{Re}\,\nu_n|} < \infty$$

The series occurring in (3.4) and later need not be convergent everywhere.

§4. Rectangular lake

We now pass to a rectangular lake (i.e. a basin bounded completely by impenetrable coasts) of length 2b, which we may consider as a channel $0 < x < \pi$, bounded by dams at y=b and y=-b. Hence we have to submit the solutions (3.2) - (3.4) to the new boundary conditions

$$v=0 \quad \text{for} \quad y=b \quad \text{and for} \quad y=-b\ .$$

Explicitly we obtain

$$\Sigma\sum_1^\infty C_n^\varepsilon\, e^{-\varepsilon\nu_n b}(\varepsilon n\nu_n \cos nx + i\omega\Omega \sin nx) = \Sigma C^\varepsilon e^{\varepsilon(\Omega x - i\omega b - \frac{1}{2}\pi\Omega)}$$

(4.1)

$$\Sigma\sum_1^\infty C_n^\varepsilon\, e^{\varepsilon\nu_n b}(\varepsilon n\nu_n \cos nx + i\omega\Omega \sin nx) = \Sigma C^\varepsilon e^{\varepsilon(\Omega x + i\omega b - \frac{1}{2}\pi\Omega)}$$

or, by taking sums and differences of corresponding members, and putting

(4.2) $$d_n^+ \overset{\text{def}}{=} \Sigma\varepsilon\, C_n^\varepsilon \qquad d_n^- \overset{\text{def}}{=} -i\,\Sigma\, C_n^\varepsilon$$

$$\sum_1^\infty \cosh \nu_n b(n\nu_n d_n^+ \cos nx - \omega\Omega\, d_n^- \sin nx) =$$

$$\cos\omega b\,\Sigma\, C^\varepsilon\, e^{\varepsilon\Omega(x-\frac{1}{2}\pi)}$$

(4.3)

$$\sum_1^\infty \sinh \nu_n b(n\nu_n d_n^- \cos nx + \omega\Omega\, d_n^+ \sin nx) =$$

$$\sin\omega b\,\Sigma\,\varepsilon\, C^\varepsilon\, e^{\varepsilon\Omega(x-\frac{1}{2}\pi)}.$$

We notice that the number i does not occur explicitly anymore, although it may occur implicitly in some ν_n, viz. if $\omega^2 > n^2+\Omega^2$.

The problem is: to determine all values of ω, for which non vanishing systems (d_n^+, d_n^-, C^+, C^-) exist, and the corresponding solution (u, v, ρ).

For $\Omega = 0$ the solution is trivial. As the functions cos nx for $n \geqslant 0$ are all independent, it is necessary and sufficient that all coefficients vanish, i.e.

$$(\cos\omega b)\Sigma C^\varepsilon = 0 \qquad (\sin\omega b)\Sigma\,\varepsilon\, C^\varepsilon = 0$$

$$d_n^+\, \nu_n \cosh\nu_n b = 0 \qquad d_n^-\, \nu_n \sinh\nu_n b = 0\ .$$

Hence either $\sin 2\omega b = 0$ or $\sin 2\nu_n b = 0$ for some n. The latter condition gives $2\nu_n b = i\pi k$ for some integer k, i.e.

(4.4) $$\omega = \pm\,(n^2 + \frac{1}{4}\pi^2 k^2 b^{-2})^{\frac{1}{2}}.$$

The solutions of $\sin 2\omega b = 0$ also have the form (4.4) with n=0.

The solutions belonging to an ω of the form (4.4) need not be unique, as more than one pair (n,k) may give the same ω, which then is a multiple free frequence. We shall not write down the solutions explicitly.

Returning to the general case $\Omega \neq 0$ we define

(4.5)
$$f_n^+ \overset{\text{def}}{=\!=} n\,\nu_n d_n^+ \cosh \nu_n b$$
$$f_n^- \overset{\text{def}}{=\!=} n\,\nu_n d_n^- \sinh \nu_n b$$

(4.6)
$$a^+(x) \overset{\text{def}}{=\!=} (\cos \omega b) \Sigma\, C^{\varepsilon}\, e^{\varepsilon\Omega(x-\frac{1}{2}\pi)}$$
$$a^-(x) \overset{\text{def}}{=\!=} (\sin \omega b) \Sigma\, \varepsilon\, C^{\varepsilon}\, e^{\varepsilon\Omega(x-\frac{1}{2}\pi)}$$

so that (4.3) becomes

(4.7)
$$\sum_1^\infty f_n^+ \cos nx - \omega\Omega \sum_1^\infty \frac{\operatorname{ctnh} \nu_n b}{n\nu_n} f_n^- \sin nx = a^+(x)$$
$$\sum_1^\infty f_n^- \cos nx + \omega\Omega \sum_1^\infty \frac{\tanh \nu_n b}{n\nu_n} f_n^+ \sin nx = a^-(x)\ .$$

The condition about quadratic integrability mentioned implies that $f_n^{\pm} \in l^2$.

For any pair of real functions $\varphi, \psi \in L_2$ (i.e. quadratically integrable, always on $0 < x < \pi$) we define the "inner product"

(4.8)
$$(\varphi, \psi) \overset{\text{def}}{=\!=} \frac{2}{\pi} \int_0^\pi \varphi(x)\, \overline{\psi(x)}\, dx$$

(notice the factor 2 [1]), and for any φ the norm

(4.9)
$$\|\varphi\| \overset{\text{def}}{=\!=} (\varphi, \varphi)^{\frac{1}{2}}\ .$$

The cosine development of a function φ on $0 < x < \pi$ is

(4.10)
$$\varphi(x) = \Sigma_* \varphi_m \cos mx$$

if the series converges, where

---

1) It is possible to drop this factor 2 and use $\sum_{-\infty}^{+\infty}$ instead of Σ_* throughout by defining the coefficients in an appropriate way for negative n also. Advantages and disadvantages of this change of formalism as well as of replacing the interval $0, \pi$ by $-\frac{1}{2}\pi$, $\frac{1}{2}\pi$ or $-\pi$, $+\pi$ are approximately in balance.

(4.11) $$\varphi_m \overset{\text{def}}{=} (\cos mx, \varphi(x))$$

whereas the symbol Σ_* denotes a summation over all <u>positive</u> integers m together with <u>half</u> the value of the summand for m=0. From (4.8) - (4.11) it follows that

$$(\varphi, \psi) = \Sigma_* \varphi_m \overline{\psi_m}$$

in particular

$$\|\varphi\|^2 = \Sigma_* |\varphi_m|^2 .$$

In particular for

(4.12) $$I_{m,n} \overset{\text{def}}{=} (\cos mx, \cos nx) \qquad (m \geq 0, n \geq 0)$$

we have

(4.13) $$I_{m,n} = \begin{cases} 1 & \text{if } m = n \neq 0 \\ 2 & \text{if } m = n = 0 \\ 0 & \text{if } m \neq n \end{cases}$$

and for all φ

$$\varphi_m = \Sigma_* I_{mn} \varphi_n .$$

Also, for

(4.14) $$\Gamma_{m,n} \overset{\text{def}}{=} (\cos mx, \sin nx)$$

we have, $\Gamma_{mn} = 0$ if m-n is even, and $\Gamma_{mn} = 4n/\pi(n^2-m^2)$ if n-m is odd, i.e.

(4.15) $$\Gamma_{mn} = \frac{4}{\pi} \frac{n}{n^2-m^2} \, p(n-m)$$

where

(4.16) $$p(k) \overset{\text{def}}{=} \tfrac{1}{2}\left[1-(-1)^k\right] = \begin{cases} 0 \text{ if } k \text{ is even} \\ 1 \text{ if } k \text{ is odd} \end{cases} .$$

In particular

(4.17) $$\Gamma_{mo} = 0 \qquad \Gamma_{on} = \frac{4p(n)}{\pi n} .$$

Hence

(4.18) $$\sin nx = \Sigma_* \cos mx \; \Gamma_{mn} \qquad (0 < x < \pi) .$$

Moreover we have

$$\cos mx = \sum_* \Gamma_{mn} \sin nx \qquad (0<x<\pi) \tag{4.19}$$

where, of course, the symbol $\sum_*$ might be replaced by $\sum_1^\infty$ or $\sum_0^\infty$.

We notice the identities

$$\sum_{n*} \Gamma_{ln} \Gamma_{mn} = I_{ln} \tag{4.20}$$

but

$$\sum_{m*} \Gamma_{mk} \Gamma_{mn} = I^o_{kn} \underset{=}{\overset{\text{def}}{=}} \begin{cases} I_{kn} & \text{if} \quad k \neq 0 \\ 0 & \text{if} \quad k = 0 \end{cases} \tag{4.21}$$

i.e. $I^o_{kn} = I_{kn}(1-\delta_{ko})$. Moreover $\Gamma_{nm} = -\frac{m}{n}\Gamma_{mn}$ unless $n=0$.

Building the inner products of both members of equations (4.7) with cos mx we obtain for $m>0$

$$\begin{aligned} f_m^+ - \omega\Omega \sum_1^\infty \Gamma_{mn} \frac{\operatorname{ctnh} \nu_n b}{n\nu_n} f_n^- &= (\cos mx,\ a^+(x)) \\ f_m^- + \omega\Omega \sum_1^\infty \Gamma_{mn} \frac{\tanh \nu_n b}{n\nu_n} f_n^+ &= (\cos mx,\ a^-(x)) \ . \end{aligned} \tag{4.22}$$

For $m=0$ (4.22) remains valid provided we define

$$f_o^+ \underset{=}{\overset{\text{def}}{=}} 0 \qquad f_o^- \underset{=}{\overset{\text{def}}{=}} 0 \ . \tag{4.23}$$

Defining for $m \geq 0$

$$a_m^\pm \underset{=}{\overset{\text{def}}{=}} (\cos mx,\ a^\pm(x)) \tag{4.24}$$

we have, because of $(\cos mx,\ e^{\lambda x}) = \frac{2\lambda}{\pi(m^2+\lambda^2)}\left[(-1)^m e^{\pi\lambda} - 1\right]$

$$\begin{aligned} a_m^+ &= \frac{2\Omega}{\pi(m^2+\Omega^2)} \cos\omega b \,\Sigma\, \varepsilon\, C^\varepsilon \left[(-1)^m e^{\varepsilon\pi\Omega} - 1\right] e^{-\frac{1}{2}\varepsilon\Omega\pi} = \\ &= \begin{cases} \dfrac{4\Omega}{\pi(m^2+\Omega^2)} \ b^+ \sinh \tfrac{1}{2}\pi\Omega \cos\omega b & m \text{ even} \\[2ex] -\dfrac{4\Omega}{\pi(m^2+\Omega^2)} \ b^- \cosh \tfrac{1}{2}\pi\Omega \cos\omega b & m \text{ odd} \end{cases} \end{aligned} \tag{4.25}$$

$$a_m^- = \frac{2\Omega}{\pi(m^2+\Omega^2)} \sin\omega b \Sigma\; C^\varepsilon\left[(-1)^m\, e^{\varepsilon\pi\Omega} - 1\right] e^{-\frac{1}{2}\varepsilon\Omega\pi}$$

$$= \begin{cases} \dfrac{4\Omega}{\pi(m^2+\Omega^2)}\; b^+ \sinh\; \tfrac{1}{2}\pi\Omega \sin\omega b & m \text{ even} \\[2ex] -\dfrac{4\Omega}{\pi(m^2+\Omega^2)}\; b^- \cosh\; \tfrac{1}{2}\pi\Omega \sin\omega b & m \text{ odd} \end{cases}$$

where

$$(4.26) \qquad b^+ \stackrel{\text{def}}{=} \Sigma\, c^\varepsilon \qquad b^- \stackrel{\text{def}}{=} \Sigma\,\varepsilon\, c^\varepsilon .$$

Because of (4.25) the $a_n^{\pm}$ are $\in l^1$ (i.e. absolutely summable), hence a fortiori $\in l^2$ (quadratically summable).

Defining moreover for $n > 0$

$$(4.27) \qquad K_{mn}^+ \stackrel{\text{def}}{=} \Gamma_{mn} \frac{\operatorname{ctnh}\nu_n b}{n\nu_n} \qquad K_{mn}^- \stackrel{\text{def}}{=} \Gamma_{mn} \frac{\tanh\nu_n b}{n\nu_n}$$

and $K_{mo}^{\pm} \stackrel{\text{def}}{=} 0$, then equations (4.22) become

$$(4.28) \qquad \begin{aligned} f_m^+ - \omega\Omega \textstyle\sum_* K_{mn}^+ f_n^- &= a_m^+ \\ f_m^- + \omega\Omega \textstyle\sum_* K_{mn}^- f_n^+ &= a_m^- . \end{aligned}$$

We notice that $K^{\pm}$ have finite norms $\| K^{\pm} \|$:

$$\| K^{\pm} \|^2 \stackrel{\text{def}}{=} \sum_{m\,*} \sum_{n\,*} |K^{\pm}|^2 = \sum_1^\infty{}_n \frac{|\tan\nu_n b|^{\mp 2}}{n^2|\nu_n|^2} \sum_{m\,*} \Gamma_{mn}^2$$

$$= \sum_1^\infty \frac{|\tan\nu_n b|^{\mp 2}}{n^2|\nu_n|^2} < \infty$$

(by 4.21), as for $n \to \infty$ $\tanh\nu_n b \to 1$ and $\nu_n/n \to 1$.

Denoting vectors and matrices in Hilbert space, by the same symbols as their components with the suffix(es) omitted, equations (4.28) may be written shortly [1])

---

1) Using the same notation the identities (4.20), (4.21) take the simple forms $\Gamma\Gamma' = I$, $\Gamma'\Gamma = I^0$, where Γ' denotes the transpose of the matrix Γ.

$$f^+ - \omega\Omega K^+ f^- = a^+$$
(4.29)
$$f^- + \omega\Omega K^- f^+ = a^-$$

We have now to find the non trivial solutions (f^+, f^-, c^+, c^-) of (4.29) with (4.25), to subject these to the conditions (4.23), and finally to solve (4.5) for the $d_n^{\pm}$. This latter solution is only possible and unique if $\sinh 2\nu_n b \neq 0$. This condition is not necessarily satisfied, but it will, for given Ω, only be violated for special values of b (and vice versa). This occurs if and only if, analogous with (4.4)

(4.30)
$$\omega^2 = \Omega^2 + n^2 + \pi^2 k^2/4b^2 \quad .$$

For this reason, we shall assume $\sinh 2\nu_n b \neq 0$ for all $n \geqslant 1$, so that solution of (4.29), (4.25), (4.23) is sufficient.

Eliminating in (4.29) either f^- or f^+ we obtain

$$f^+ + \omega^2\Omega^2 K^+K^- f^+ = h^+ \overset{\text{def}}{=} a^+ + \omega\Omega K^+ a^-$$
(4.30)
$$f^- + \omega^2\Omega^2 K^-K^+ f^- = h^- \overset{\text{def}}{=} a^- - \omega\Omega K^- a^+ \quad .$$

The system (4.29) is equivalent with the one obtained by replacing <u>one</u> of its (vector) equations, say the first one, by the corresponding one of (4.30).

As $K^{\pm}$, hence also $K^{\pm}K^{\mp}$ have finite norms, the matriced $I + \omega^2\Omega^2 K^{\pm}K^{\mp}$ have, for sufficiently small $\omega\Omega$, unique twosided reciprocals $R^{\pm}$, obtainable for sufficiently small $\omega^2\Omega^2$ by a "Neumann development"

$$R^+ \overset{\text{def}}{=} (I+\omega^2\Omega^2 K^+K^-)^{-1} = \sum_0^\infty (-\omega^2\Omega^2)^n (K^+K^-)^n$$ 1)2)
(4.31)
$$R^- \overset{\text{def}}{=} (I+\omega^2\Omega^2 K^-K^+)^{-1} = \sum_0^\infty (-\omega^2\Omega^2)^n (K^-K^+)^n .$$

---

1) The fact that in $R^{\pm}$ (cf. also (5.4) below) two different operators alternate is not essential. By using instead of (4.27)

$$K'^{\pm} \overset{\text{def}}{=} \pm(-1)^n \Gamma_{mn}(n\nu_n)^{-1}(\tanh\nu_n b)^{\mp(-1)^n} = (-1)^n K_{mn}^{\pm(-1)^n}$$
$$f_n'^{\pm} \overset{\text{def}}{=} f_n^{\pm(-1)^n}$$

we obtain seperate equations for f'^+ and f'^-, each depending on the corresponding operator K'^+, K'^- only.

2) See page 14.

We notice that

$$(4.32)\qquad \begin{aligned} R^- &= I - \omega^2\Omega^2 K^- R^+ K^+, & K^+ R^- &= R^+ K^+ \\ R^+ &= I - \omega^2\Omega^2 K^+ R^- K^-, & K^- R^+ &= R^- K^- \end{aligned}$$

By means of (4.31), (4.30) can be solved:

$$(4.33)\qquad \begin{aligned} f^+ &= R^+ h^+ \\ F^- &= R^- h^- \end{aligned}$$

and subjected to the conditions (4.23) i.e.

$$(4.34)\qquad \begin{aligned} \textstyle\sum_* R^+_{om} h_m^+ &= 0 \\ \textstyle\sum_* R^-_{om} h_m^- &= 0 \ . \end{aligned}$$

We remind that Γ_{mn}, $K^{\pm}_{mn}$ are only $\neq 0$ if m and n have unequal parity. Hence $R^{\pm}_{om} = 0$ unless m is even, and only $h^{\pm}_m$ with <u>even</u> m occur in (4.34). These depend by (4.30) on $a^{\pm}_m$ with even m and $a^{\mp}_n$ with odd n, hence by (4.25) only on b^+ or on b^-. Hence we obtain two sets of solutions, one with $b^+ \neq 0$, $b^- = 0$ the other with $b^+ = 0$, $b^- \neq 0$.

Defining finally

$$(4.35)\qquad \rho^{\pm} \stackrel{\text{def}}{=} \Omega^2 \textstyle\sum_* R^{\pm}_{om} (m^2+\Omega^2)^{-1}$$

$$(4.36)\qquad \sigma^{\pm} \stackrel{\text{def}}{=} \Omega^2 \textstyle\sum_* (R^{\pm} K^{\pm})_{on} (n^2+\Omega^2)^{-1}$$

we obtain from (4.34), (4.30) and (4.25) either $b^- = 0$, $b^+ \neq 0$ and

$$(4.37)\qquad \rho^+ \sin h\tfrac{1}{2}\pi\Omega \cos\omega b - \omega\Omega\sigma^+ \cos h\tfrac{1}{2}\pi\Omega \sin\omega b = 0$$

or $b^+ = 0$, $b^- \neq 0$

$$(4.38)\qquad \rho^- \sin h\tfrac{1}{2}\pi\Omega \sin\omega b + \omega\Omega\sigma^- \cos h\tfrac{1}{2}\pi\Omega \cos\omega b = 0$$

---

2) If $\omega^2\Omega^2 > \| K^+K^- \|^{-1}$ the developments into power series (4.31) need not exist anymore. The reciprocals $R^{\pm}$, however, still exist unless $-(\omega\Omega)^{-2}$ is a latent root of K^+K^- (and, equivalently, of K^-K^+). If it is a root of multiplicity r, equations (4.30) require for their solvability r linear relations for h^+ and r for h^-; then f^+ as well as f^- is determined except for r constants. Hence in this case also each of the conditions (4.23) implies $(r+1)-r = 1$ linear relation between b^+ and b^-. We shall not go into their explicit form, which requires the spectral development of $K^{\pm} K^{\mp}$.

§5. Development for small Ω

Each of the transcendental equations (4.37), (4.38) for ω, given Ω and b, is extremely complicated. For, ω occurs a) implicitly in the $K^{\pm}_{mn}$ through the argument $\nu_n = (n^2+\Omega^2-\omega^2)^{\frac{1}{2}}$ b) implicitly in the $R^{\pm}_{om}$ through the argument $\omega\Omega$ c) explicitly through the factor $\omega^{\pm 1}\tan\omega b$. The dependence on Ω is even more and that on b almost equally complicated.

We can, however, develop $\rho^{\pm}$ and $\sigma^{\pm}$ in a series of powers of Ω^2, from which approximations to ω for small Ω^2 can easily be derived (we remind that in ordinary units Ω has to be replaced by $\Omega a/\pi\sqrt{gd}$). We shall do this for the first equation (4.37) only.

As R^+_{om} vanishes unless m is even, and $R^+_{oo} = I_{oo} = 2$ we have from (4.35)

$$\rho^+ = 1 - \sum_{s=1}^{\infty} (-\Omega^2)^s \sum_{m=2}^{\infty} m^{-2s} R^+_{om} \quad . \tag{5.1}$$

Moreover for $m \neq 0$

$$R^+_{om} = \sum_{r=1}^{\infty} (-\omega^2\Omega^2)^r \left[(K^+K^-)^r\right]_{om} \tag{5.2}$$

whence

$$\rho^+ = 1 - \omega^2\Omega^4 \sum_{r=0}^{\infty} (-\Omega^2)^r \sum_{h=0}^{r} \omega^{2r} M^{2h+2}_{r-h+1} \tag{5.3}$$

where

$$M^{2h}_{l} = \sum_{n=1}^{\infty} n^{-2l} \left[(K^+K^-)^h\right]_{on}$$

(5.4)

$$M^{2h+1}_{l} = \sum_{n=1}^{\infty} n^{-2l} \left[(K^+K^-)^h K^+\right]_{on}$$

and similarly, as $(R^+K^+)_{on}$ vanishes unless n is odd,

$$\sigma^+ = \Omega^2 \sum_{r=0}^{\infty} (-\Omega^2)^r \sum_{0}^{r} \omega^{2h} M^{2h+1}_{r-h+1} \quad . \tag{5.5}$$

In order to obtain a second order approximation (in Ω^2) it is sufficient to take

$$\rho^+ = 1 + 0\,(\Omega^4)$$

$$\sigma^+ = \Omega^2 M^1_1 - \Omega^4(M^1_2 + \omega^2 M^3_1) + 0\,(\Omega^6) \; .$$

We obtain

$$(5.6)\qquad \frac{\mathrm{ctn}\,\omega b}{\omega} = \frac{\Omega}{\tanh\frac{1}{2}\pi\Omega}\ \frac{\sigma^+}{\rho^+}$$

$$= \frac{2}{\pi}\left[\Omega^2 M_1^1 - \Omega^4\left(-\frac{1}{12}\pi^2 M_1^1 + M_2^1 + \omega^2 M_1^3\right)\right. + O(\Omega^6).$$

But

$$(5.7)\qquad M_1^1 = \sum_1^\infty n^{-21} K_{on}^+ = \frac{4}{\pi}\sum_1^\infty \frac{p(n)}{n^{21+2}\ \nu_n \tanh\nu_n b} =$$

$$= \frac{4}{\pi}\sum_1^\infty \frac{p(n)}{n^{21+2}\ \nu_n^{\ o}\ \tanh\nu_n^{\ o} b} - \frac{2}{\pi}\Omega^2 \sum_1^\infty \frac{p(n)}{n^{21+2}\ \nu_n^{\ o3}}\ \times$$

$$\times\left\{\frac{1}{\tanh\nu_n^{\ o}b} + \frac{\nu_n^{\ o}}{\sinh^2\nu_n^{\ o}b}\right\} + O(\Omega^4)$$

with $\nu_n^{\ o} = \sqrt{n^2-\omega_o^{\ 2}}$, and assuming $\nu_n^{\ o} \neq 0$ for all n, whereas M_1^3 can be taken with the arguments $\nu_n^{\ o}$ instead of ν_n. A further simplification can be obtained if b is large (we may always assume $b \geq a = \frac{1}{2}\pi$) this will be done below seperately. Patience only is required to obtain a few higher order approximations. If a first order approximation only is asked, we obtain, putting $\delta \overset{\text{def}}{=\!=} \omega_o - \omega$ where $\cos\omega_o b = 0$ so that $\omega^{-1}\ \mathrm{ctn}\,\omega b = \omega_o^{\ -1}\delta b + O(\Omega^2)$

$$(5.8)\qquad \frac{\delta}{\omega_o} = \frac{8\Omega^2}{\pi^2}\sum_1^\infty \frac{p(n)}{n^4\ \nu_n^{\ o}\ b\ \tanh\nu_n^{\ o}b} + O(\Omega^4).$$

The series converges rather rapidly, as n runs through the odd numbers only, and the general term is $O(n^{-5})$.

For the lowest frequency in particular $\omega_o b = \frac{1}{2}\pi$ so that

$$\nu_n^{\ o}\ b = \sqrt{n^2b^2 - \frac{1}{4}\pi^2}\ .$$

If, more in particular, the length 2b of the lake is twice its width $2a=\pi$, so that $b=\pi$, $\omega_o=\frac{1}{2}$, then (5.8) becomes

$$\frac{\delta}{\omega_o} = \frac{16}{\pi^3}\Omega^2\left(\frac{1}{\sqrt{3}\ \tanh\frac{1}{2}\pi\sqrt{3}} + \frac{1}{81\sqrt{35}\ \tanh\frac{1}{2}\pi\sqrt{35}} + \frac{1}{625\sqrt{99}\tanh\frac{1}{2}\pi\sqrt{99}} + \ldots\right)$$

$$= 0{,}302\,\Omega^2.$$

Clearly the hyperbolic tangents rapidly approach 1. This fact may be used for the higher terms in Ω^2 also, if b is sufficiently large. We shall give only a first order approximation in $\beta = e^{-2\nu_1 b}$. Then $e^{-2\nu_n b} = O(\beta^3)$ if $n \geq 3$. Moreover

$$
(5.9) \qquad
\begin{aligned}
\operatorname{ctnh}\nu_1 b - 1 &= \frac{e^{-\nu_1 b}}{\sinh \nu_1 b} = 2\beta + O(\beta^2) \\
1 - \tanh\nu_1 b &= \frac{e^{-\nu_1 b}}{\cosh \nu_1 b} = 2\beta + O(\beta^2) .
\end{aligned}
$$

Hence, putting [1)]

$$
(5.10) \qquad K_{mn} \overset{\text{def}}{=} \begin{cases} \Gamma_{mn} (n\nu_n)^{-1} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases}
$$

we have

$$
(5.11) \qquad
\begin{aligned}
K^{+}_{m1} &= K_{m1}(1+2\beta) + O(\beta^2) \\
K^{-}_{m1} &= K_{m1}(1-2\beta) + O(\beta^2) \\
K^{\pm}_{mn} &= K_{mn} + O(\beta^3) \qquad \text{if } n > 1 .
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\left[(K^+ K^-)^h\right]_{mn} = \\
&(K^{2h})_{mn}(1-2\beta\delta_{n1}) + 2\beta \sum_{1}^{2h-1} (-1)^{j-1}(K^j)_{m1}(K^{2h-j})_{1n} + \\
&\qquad + O(\beta^2)
\end{aligned}
$$

$$
(5.12) \qquad
\begin{aligned}
&\left[(K^+ K^-)^h K^+\right]_{mn} = \\
&(K^{2h+1})_{mn}(1+2\beta\delta_{n1}) + 2\beta \sum_{1}^{2h} (-1)^{j-1}(K^j)_{m1}(K^{2h+1-j})_{1n} + \\
&\qquad + O(\beta^2) .
\end{aligned}
$$

We omit the substitution of these values into (5.2) - (5.5) and in the equations derived from them. We use them only to obtain the second order approximation for δ if $\omega_0 = \pi/2b$ and if b is large.

Finally we consider the case, dismissed above, where one of the $\nu_n^{\,o}$, $\nu_s^{\,o}$ say, vanishes. This means that $\omega_0 = s$ (or -s) (s odd) is a degenerate free frequency. The simplest case, for which

---

1) We remind that we have assumed $\nu_n \neq 0$ for all n.

a first order approximation is known since long (cf. Lamb, Proudman, Corkan and Doodson) is the lowest frequency for a square lake: $b=\frac{1}{2}\pi$, $\omega_0=1$.

In the case $\nu_s^{\,0}=0$ the argument up to (5.6) remains valid, but (5.7) remains true for the terms $n \neq s$ only, whereas

$$(5.13) \qquad \nu_s^{\,2} = s^2+\Omega^2-(\omega_0-\delta)^2 = 2\,\omega_0\delta+\Omega^2-\delta^2 \, .$$

Hence M_1^1 becomes of the order ν_s^{-2}, i.e. δ^{-1}, and $\delta = O(\Omega)$ instead of $\delta = O(\Omega^2)$. The term n=s in the first sum in the right hand member of (5.7) becomes

$$(5.14) \qquad \frac{4}{\pi}\,\frac{1}{s^{2l+2}\,\nu_s\,\tanh\nu_s b} = \frac{4}{\pi}\,\frac{1}{s^{2l+2}b}\,\frac{1+\frac{2}{3}\omega_0\,\delta\,b^2}{2\,\omega_0\delta+\Omega^2-\delta^2}\,(1+O(\Omega^4)) \, .$$

Hence to a first approximation we have by (5.6)

$$\frac{\delta b}{\omega_0} = \frac{\delta b}{s} = \frac{2\Omega^2}{\pi}\,\frac{2}{\pi b s^5\delta}$$

or

$$(5.15) \qquad \delta = \frac{2}{\pi s^2 b} + O(\Omega^2) \, .$$

For the particular case of the square lake ($s=1$, $b=\frac{1}{2}\pi$) this becomes

$$\delta = \frac{4\Omega}{\pi^2} + O(\Omega^2)$$

equivalent with Lamb's value.

Using (5.14) for obtaining the second order approximation we obtain with (5.6), (5.4)

$$(5.16) \quad \delta = \frac{2\Omega}{\pi b s^2}\left[1-\frac{\pi\Omega b s}{8}\left(1+\frac{4}{\pi^2 b^2 s^4}-\frac{16}{3\pi^2 s^2}\right)+\frac{2\Omega s^3}{\pi}\sum_{n\neq s}\frac{p(n)}{n^4\nu_n^0\tanh\nu_n^0 b}\right]+O(\Omega^3).$$

In the special case of the lowest frequency in a square lake this becomes

$$(5.17) \qquad \delta = \frac{4\Omega}{\pi^2}\left[1-\frac{\Omega}{4}\left(\frac{\pi^2}{4}+\frac{4}{\pi^2}-\frac{4}{3}\right)+\frac{2\Omega}{\pi}\sum_{3}^{\infty}\frac{p(n)}{n^4\nu_n^0\tanh\frac{1}{2}\nu_n^0\pi}\right]+O(\Omega^3)$$

with $\nu_n^{\,0} = \sqrt{n^2-1}$ or, numerically

$$\delta = 0{,}4053\,\Omega\,(1 - 0{,}3570\,\Omega)\;.$$

Hence, e.g. for $\Omega = 0{,}55$ the decrease of the frequency is almost 20 % overrated by taking the first approximation only.

§6. The rectangular bay

Under the same assumptions as in section 2 we consider a rectangular bay $0 < x < \pi$, $0 < y < b$, bounded by impenetrable coasts $x=0$, $0 < y < b$; $y=0$, $0 < x < \pi$; $x=\pi$, $0 < y < b$ and by an ocean $y > b$, considered as being infinitely deep. Along the latter ζ may be assumed to be constantly zero.

In order to determine its free oscillations we start again with the solution (3.4) for the infinite channel, on which we have to impose the conditions

$$\begin{aligned} v &= 0 \quad \text{for} \quad y = 0,\ 0 < x < \pi \\ \zeta &= 0 \quad \text{for} \quad y = b,\ 0 < x < \pi \;. \end{aligned}$$

Explicitly these are

$$(6.1)\qquad \begin{aligned} &\Sigma\sum c_n^{\varepsilon}(\varepsilon n\nu_n \cos nx + i\omega\Omega \sin nx) = \Sigma\, c^{\varepsilon} e^{\varepsilon\Omega(x-\frac{1}{2}\pi)} \\ &\Sigma\sum c_n^{\varepsilon}\, e^{-\varepsilon\nu_n b}(i\omega n \cos nx + \varepsilon\Omega\nu_n \sin nx) = \Sigma\,\varepsilon\, c^{\varepsilon} \times \\ &\qquad\qquad \times\, e^{\varepsilon\Omega(x-\frac{1}{2}\pi) - i\varepsilon\omega b}\,. \end{aligned}$$

We define (cf. also (4.26))

$$(6.2)\qquad b^{+} \stackrel{\text{def}}{=} \Sigma\, c^{\varepsilon} \qquad b^{-} \stackrel{\text{def}}{=} \Sigma\,\varepsilon\, c^{\varepsilon}$$

$$(6.3)\qquad \begin{cases} \gamma(x) \stackrel{\text{def}}{=} \cosh\ \Omega\,(x-\frac{1}{2}\pi) \\ \sigma(x) \stackrel{\text{def}}{=} \sinh\ \Omega\,(x-\frac{1}{2}\pi) \end{cases}$$

$$(6.4)\qquad \begin{cases} \varphi(x) \stackrel{\text{def}}{=} \Sigma\, c^{\varepsilon}\, e^{\varepsilon\Omega\,(x-\frac{1}{2}\pi)} \\ \psi(x) \stackrel{\text{def}}{=} \Sigma\,\varepsilon\, c^{\varepsilon}\, e^{\varepsilon\Omega\,(x-\frac{1}{2}\pi)} \end{cases}$$

and for $n > 0$

$$(6.5)\qquad \begin{aligned} f_n &\stackrel{\text{def}}{=} n\nu_n\, \Sigma\,\varepsilon\, c_n^{\varepsilon} \\ g_n &\stackrel{\text{def}}{=} i\omega n\, \Sigma\, c_n^{\varepsilon}\, e^{-\varepsilon\nu_n b} \end{aligned}$$

We assume that $\cos\omega b \neq 0$ and $n\nu_n \cosh\nu_n b \neq 0$ for all $n \geq 1$. This for given b is true for almost all Ω and vice versa, provided ω is a free frequency. Then the C_n^{ε} can be solved from (6.5). We obtain

$$C_n^{\varepsilon} = \frac{1}{2i\omega\, n\nu_n \cosh \nu_n b} \left(\nu_n g_n + \varepsilon\, i\omega\, e^{\varepsilon \nu_n b} f_n\right) \tag{6.6}$$

whereas

$$C^{\varepsilon} = \tfrac{1}{2}(b^{+} + \varepsilon\, b^{-}) \tag{6.7}$$

and

$$\begin{cases} \varphi(x) = \gamma(x)\, b^{+} + \sigma(x)\, b^{-} \\ \psi(x) = \sigma(x)\, b^{+} + \gamma(x)\, b^{-}. \end{cases} \tag{6.8}$$

The expressions for (u, v, ζ) become

$$u = \sum_{1}^{\infty} \frac{(n^2+\Omega^2)\sin nx}{i\omega\, n\nu_n \cosh \nu_n b}\left[i\omega f_n \sinh \nu_n(b-y) + \nu_n g_n \cosh \nu_n y\right]$$

$$v = \sum_{1}^{\infty} \frac{1}{i\omega \cosh \nu_n b}\Big[\cos nx \left\{ i\omega f_n \cosh \nu_n(b-y) - \nu_n g_n \times \right.$$
$$\left. \times \sinh \nu_n y\right\} +$$
$$+ \frac{i\omega\Omega}{n\nu_n} \sin nx \left\{ i\omega f_n \sinh \nu_n(b-y) + \nu_n g_n \cosh \nu_n y\right\}\Big] + \tag{6.9}$$
$$- \varphi(x) \cos\omega y + i\psi(x) \sin\omega y$$

$$\zeta = \sum_{1}^{\infty} \frac{1}{i\omega\, n\nu_n \cosh \nu_n b}\Big[i\omega n \cos nx \left\{ i\omega f_n \sinh \nu_n(b-y) + \right.$$
$$\left. \nu_n g_n \cosh \nu_n y\right\} +$$
$$+ \Omega\nu_n \sin nx \left\{ i\omega f_n \cosh \nu_n(b-y) - \nu_n g_n \sinh \nu_n y\right\}\Big] +$$
$$- \psi(x) \cos\omega y + i\varphi(x) \sin\omega y.$$

The conditions (6.1) become

$$\sum_1^\infty f_n\left(\cos nx + \frac{i\omega\Omega}{n\nu_n}\tanh \nu_n b \sin nx\right) + \Omega \sum_1^\infty \frac{g_n \sin nx}{n \cosh \nu_n b} = \varphi(x)$$

(6.10)
$$\sum_1^\infty g_n\left(\cos nx + \frac{i\nu_n\Omega}{n\omega}\tanh \nu_n b \sin nx\right) + \Omega \sum_1^\infty \frac{f_n \sin nx}{n \cosh \nu_n b} = \psi_*(x)$$

$$\psi_*(x) \overset{\text{def}}{=\!=} \psi(x) \cos\omega b - i\varphi(x) \sin\omega b \quad .$$

For $\Omega = 0$ the problem again is trivial. As the cos nx for $n \geq 0$ all are independent, we obtain from (6.1), using (6.5) and (6.2):

(6.11)
$$\Sigma C^\varepsilon = \Sigma \varepsilon C^\varepsilon e^{-i\varepsilon\omega b} = 0$$

$$\nu_n \Sigma \varepsilon C_n^\varepsilon = \omega \Sigma C_n^\varepsilon e^{-\varepsilon\nu_n b} = 0$$

for all n. Hence also

(6.12)
$$\cos\omega b \, \Sigma \varepsilon C^\varepsilon = 0$$

$$\omega \cosh \nu_n b \, \Sigma C_n^\varepsilon = 0 \quad .$$

The only non trivial solutions (i.e. such that (u, v, ζ) do not all vanish identically) are obtained from

(6.13)
$$\cos\omega b = 0 \quad ; \quad \omega = (k+\tfrac{1}{2})\pi b^{-1} \quad ; \quad C^- = C^+ \quad (\text{e.g.} = \tfrac{1}{2})$$

$$u = 0 \quad ; \quad v = -i \sin\omega y \quad ; \quad \zeta = \cos\omega y$$

and for some positive integers from

$$\cosh \nu_s b = 0 \; ; \omega^2 = s^2 + (k+\tfrac{1}{2})^2 \pi^2 b^{-2} \; ; \; C_s^+ = C_s^- \; \left(\text{e.g.} = \frac{1}{2is}\right)$$

(6.14)
$$u = -is \sin sx \cosh \nu_s y$$

$$v = i\nu_s \cos sx \sinh \nu_s y$$

$$\zeta = \omega \cos sx \cosh \nu_s y \quad .$$

Formally the former mode of oscillation (6.13) can be obtained from the latter (apart from a constant factor) by admitting s to take the value 0 also (hence arbitrary integer values).

Returning to the general case, we take the cosine coefficients of both members of equation (6.10). We have, using with integer $m \geq 0$

$$(e^{\lambda x-\frac{1}{2}\lambda\pi},\ \cos mx) = \frac{2\lambda\{(-1)^m e^{\frac{1}{2}\lambda\pi} - e^{-\frac{1}{2}\lambda\pi}\}}{\pi(\lambda^2+m^2)} \tag{6.15}$$

$$\gamma_m \overset{\text{def}}{=} (\cosh \Omega(x-\tfrac{1}{2}\pi),\ \cos mx) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \dfrac{4\Omega \sinh \frac{1}{2}\pi\Omega}{\pi(\Omega^2+m^2)} & \text{if } m \text{ is even} \end{cases}$$

(6.16)

$$\sigma_m \overset{\text{def}}{=} (\sinh \Omega(x-\tfrac{1}{2}\pi),\ \cos mx) = \begin{cases} 0 & \text{if } m \text{ is even} \\ -\dfrac{4\Omega \cosh \frac{1}{2}\pi\Omega}{\pi(\Omega^2+m^2)} & \text{if } m \text{ is odd} \end{cases}$$

hence

$$\gamma_m = \frac{\Omega^2}{\Omega^2+m^2}\,\gamma_0 \qquad (m \text{ even})$$

$$\sigma_m = \frac{\Omega^2+1}{\Omega^2+m^2}\,\sigma_1 \qquad (m \text{ odd})$$

and, by (6.8)

$$\varphi_m \overset{\text{def}}{=} \gamma_m b^+ + \sigma_m b^-$$

(6.17)

$$\psi_m \overset{\text{def}}{=} \sigma_m b^+ + \gamma_m b^- .$$

Moreover, defining for $m \geq 0$, $n > 0$

$$K^-_{mn} \overset{\text{def}}{=} \Gamma_{mn} \frac{\tanh \nu_n b}{n\nu_n}$$

$$H_{mn} \overset{\text{def}}{=} \Gamma_{mn} \frac{\nu_n}{n} \tanh \nu_n b \tag{6.18}$$

$$N_{mn} \overset{\text{def}}{=} \Gamma_{mn} \frac{1}{n \cosh \nu_n b} .$$

We obtain for $m \geq 0$

$$f_m + i\omega\Omega \sum_1^\infty K^-_{mn} f_n + \Omega \sum_1^\infty N_{mn} g_n = \varphi_m$$

(6.19)

$$g_m + \frac{i\Omega}{\omega} \sum_1^\infty H_{mn} g_n + \Omega \sum_1^\infty N_{mn} f_n = \psi_{*m} ,$$

provided we put

$$f_0 \overset{\text{def}}{=} (1,f) = 0 \quad , \quad g_0 \overset{\text{def}}{=} (1,g) = 0 \tag{6.20}$$

We extend the definitions (6.18) to n=0 by requiring $\Gamma_{mn}/n = -\Gamma_{nm}/m$ and $\Gamma_{mn}=0$ if $m=N(\nu)$ to remain valid for n=0 also:

$$K^{-}_{mo} \stackrel{def}{=} -\frac{4p(m)}{m^2}\frac{\tanh \varkappa b}{\varkappa}$$

(6.18)$_o$
$$H_{mo} \stackrel{def}{=} -\frac{4p(m)}{m^2}\varkappa \tanh \varkappa b$$

$$N_{mo} \stackrel{def}{=} -\frac{4p(m)}{m^2}\frac{1}{\cosh \varkappa b}$$

as $\nu_o = x$.

Then the (m,0)-components of K^-, H, N remain finite for all $m \geq 0$ and vanish for m = 0. Hence the $\sum_1^\infty$ in (6.19) may, because of (6.20) be replaced by $\sum_*$ and (6.19) may be written in operatorform:

(6.21)
$$f + i\omega\Omega K^- f + \Omega N g = \varphi$$
$$g + \frac{i\Omega}{\omega} H g + \Omega N f = \psi_* .$$

The main difference in comparison with the case of a lake is the following. As we saw in section 4, K^- has finite norm, and the same holds for N as the sums of squares of the coefficients of Γ_{mn} in these matrices converge, but not for H as $\sum |\nu_n n^{-1} \tanh \nu_n b|^2 = \infty$. Nevertheless H is bounded, i.e. if $\psi \in L_2$, then $H\psi \in L_2$ and $\|H\psi\| / \|\psi\|$ is bounded. In fact,

$$\|H\psi\|^2 = \sum_* |n^{-1} \nu_n \tanh \nu_n b \ \psi_n|^2 \leq (1+|\varkappa|^2)\ C^2 \|\psi\|^2$$

where C is an upper limit for all $|\tanh \nu_n b|$. This exists, as we have supposed $\cosh \nu_n b \neq 0$ for all n, so that, if $C' = \max |\tanh \nu_n b|$ over all $n^2 \leq \Omega^2 - \omega^2$, C = max (1,C') satisfies the required condition. It follows that for all sufficiently small Ω/ω (viz. $|\Omega/\omega| \leq C(1+|\varkappa^2|)^{\frac{1}{2}}$) the operator

(6.22)
$$Q^+ \stackrel{def}{=} (I + \frac{i\Omega}{\omega} H)^{-1}$$

exists, and that

(6.23)
$$Q^+ = \sum_0^\infty (-\frac{i\Omega}{\omega})^n H^n \quad .$$

The condition for existence of inverses for operators of the type $I + \gamma H$ have been investigated on two interesting reports by

LAUWERIER and G.W. VELTKAMP respectively, who used different methods. For the special case $H = \Gamma$ Lauwerier obtained interesting explicit solutions to which he reduced the general case where $H_{mn} = \Gamma_{mn} + O(n^{-2})$.

Putting

$$h \stackrel{\text{def}}{=} g + \frac{i\Omega}{\omega} H g \tag{6.24}$$

we have by (6.22)

$$g = Q^{+} h . \tag{6.25}$$

With the abbreviation

$$\begin{aligned} Q^{-} &\stackrel{\text{def}}{=} (I + i\omega\Omega K^{-})^{-1} \\ &= \sum_{0}^{\infty} (-i\omega\Omega)^{n} (K^{-})^{n} \end{aligned} \tag{6.26}$$

(6.20) leads to

$$\begin{aligned} h + \Omega N f &= \psi_{*} \\ f &= Q^{-} (\varphi - \Omega N Q^{+} h) \end{aligned} \tag{6.27}$$

whence

$$\begin{aligned} (I-\Omega^{2}Q^{-}NQ^{+}N)f &= \chi^{+} \stackrel{\text{def}}{=} Q^{-} (\varphi - \Omega N Q^{+}\psi_{*}) \\ (I-\Omega^{2}NQ^{-}NQ^{+})h &= \psi_{*} - \Omega N Q^{-}\varphi . \end{aligned} \tag{6.28}$$

Defining the operator

$$\begin{aligned} S^{+} &\stackrel{\text{def}}{=} (I-\Omega^{2}Q^{-}NQ^{+}N)^{-1} \\ &= \sum_{0}^{\infty}\Omega^{2n} (Q^{-}NQ^{+}N)^{n} \end{aligned} \tag{6.29}$$

we obtain from (6.28), (6.27), (6.26)

$$\begin{aligned} f &= S^{+}\chi^{+} \\ g &= Q^{+} (\psi_{*} - \Omega N S^{+}\chi) . \end{aligned} \tag{6.30}$$

We have also

$$g = S^{-} \chi^{-} \tag{6.31}$$

with

$$S^{-} \stackrel{\text{def}}{=} Q^{+}(I-\Omega^{2}NQ^{-}NQ^{+})^{-1}(Q^{+})^{-1} = (I-\Omega^{2}Q^{+}NQ^{-}N)^{-1} \tag{6.32}$$

and

$$(6.33)\qquad \chi^- = Q^+(\psi_* - \Omega N Q^- \varphi)\ .$$

Between S^- and S^+ the identities

$$(6.34)\qquad \begin{aligned} S^{\mp} &= I - \Omega^2\, Q^{\pm}\, N\, S^{\pm}\, Q^{\mp}\, N \\ S^{\mp}\ Q^{\pm} N &= Q^{\pm}\, N\, S^{\pm} \end{aligned}$$

exist.

We notice that i does not occur explicitly anymore, although it still does so implicitly, viz. through $i\omega\Omega\, K^-$ in Q^- and through $i\,\frac{\Omega}{\omega} H$ in Q^+, hence also in $S^{\pm}$.

The conditions (6.21) yield two homogeneous linear equations for the two Kelvin coefficients C^{ε}, or, equivalently for the b^{ε}. The free frequency condition is the vanishing of the determinant of these equations. Denoting by f^{ε}, g^{ε} the coefficient of b^{ε} in f, g respectively, we find by (6.8), (6.10), (6.28) into (6.30), (6.31)

$$(6.35)\qquad \begin{aligned} f^+ &= S^+Q^-\left[(I + i\Omega \sin\omega b\ NQ^+)\gamma - \Omega\cos\omega b\ NQ^+\sigma\right] \\ f^- &= S^+Q^-\left[(I + i\Omega \sin\omega b\ NQ^+)\sigma - \Omega\cos\omega b\ NQ^+\gamma\right] \end{aligned}$$

$$(6.36)\qquad \begin{aligned} g^+ &= S^-Q^+\left[\cos\omega b\,\sigma - (i\ \sin\omega b I + \Omega N Q^-)\gamma\right] \\ g^- &= S^-Q^+\left[\cos\omega b\,\gamma - (i\ \sin\omega b I + \Omega N Q^-)\sigma\right]\ . \end{aligned}$$

The 0-components of these sequences must satisfy the <u>free frequency condition</u>

$$(6.37)\qquad \Delta \stackrel{\text{def}}{=} \begin{vmatrix} f_o^+ & g_o^+ \\ f_o^- & f_o^- \end{vmatrix} = 0$$

or, explicitly

$$(6.38)\qquad \begin{vmatrix} (S^+Q^-\gamma)_0 - \Omega(S^+Q^-NQ^+\{\sigma\cos\omega b - i\gamma\sin\omega b\})_0 & (S^-Q^+\{\sigma\cos\omega b - i\gamma\sin\omega b\})_0 + \Omega(S^-Q^+NQ^-\gamma)_0 \\ (S^+Q^-\sigma)_0 - \Omega(S^+Q^-NQ^+\{\gamma\cos\omega b - i\sigma\sin\omega b\})_0 & (S^-Q^+\{\gamma\cos\omega b - i\sigma\sin\omega b\})_0 + \Omega(S^-Q^+NQ^-\sigma)_0 \end{vmatrix} = 0$$

This is the required equation for the free frequencies ω. If it is satisfied, $b^{\varepsilon} \neq 0$ exist, such that f and g, defined by (6.35)

(6.36) satisfy (6.20) and (u,v,φ) defined by (6.7) are the corresponding modes of oscillation.

Although equation (6.38) still contains implicitly as well as explicitly imaginary quantities, it is actually real, as can be seen as follows. First, γ_m and σ_m vanish unless m is even and odd respectively. Then the operator Γ, hence also K^-, N, H and products of odd order of these quantities change the parity of the suffix, whereas even order products preserve it. Finally, taking the inner product with the constant 1 means taking the component with suffix 0. Hence γ and σ occur ultimately only accompanied by even and odd order products respectively. Now K^- and H occur in R^-, L and S always with a factor i. As the left hand member of (6.20) can be written as a real function of Q^ε, S^ε, γ, $i\sigma$, iN, it follows that all imaginary quantities cancel.

§7. Long bay low frequencies

Because of the complicated nature of the equation (6.38) we shall discuss in some more detail only approximate solutions under the conditions 1. that the length b of the bay is large, 2. that Ω is sufficiently small. More precisely, we shall discuss the zero and first order approcimation with respect to

$$\beta \overset{\text{def}}{=\!=} e^{-\nu_1 b} \tag{7.1}$$

assuming that ν_1 is real, i.e. $\omega^2 < 1+\Omega^2$. Then all $e^{-\nu_n b}$ $(n>1)$ are $O(\beta^2)$. In the case of the North Sea we have roughly $b=2a=2\pi$, $\Omega = 0{,}55$. The lowest frequencies tend for $\Omega \to 0$ to the roots of $\cos \omega_o b=0$, i.e. $\omega_o=(k+\frac{1}{2})\pi/b$. Hence for k=0 and k=1, $\omega_o=1/4$ and $\omega_o=3/4$ respectively. So β is of the order of (roughly) $e^{-2\pi}=0{,}002$ and $e^{-\pi}=0{,}043$ respectively. So for $\omega_o= 1/4$ the zero order and for $\omega_o= 3/4$ the first order approximation may be expected to be quite good. For the frequencies which for $\Omega \to 0$ tend to the roots of $\cosh \nu_s^o b = 0$ the approximation is no longer valid.

Now by (6.18) (cf. also (5.11))

$$\begin{aligned} K_{mn} &\overset{\text{def}}{=\!=} \Gamma_{mn} \frac{1}{n\nu_n} = K^-_{mn} + O(\beta^2) \\ H^o_{mn} &\overset{\text{def}}{=\!=} \Gamma_{mn} \frac{\nu_n}{n} = H_{mn} + O(\beta^2) \end{aligned} \tag{7.2}$$

$$N_{mn} = \begin{cases} 2\beta\Gamma_{m1} + O(\beta^3) & \text{if } n = 1 \\ O(\beta^2) & \text{if } n > 1 \end{cases} .$$

In particular all terms containing two factors N, notably the second term between brackets in (6.32) can be neglected.

Moreover, introducing the real parts $R^{\pm}$ of $Q^{\pm}$

$$(7.3)\qquad \begin{aligned} R^{+} &\stackrel{\text{def}}{=} (I - \frac{\Omega^2}{\omega^2} H^2)^{-1} = \sum_{0}^{\infty} (\frac{\Omega^2}{\omega^2})^n H^{2n} \\ R^{-} &\stackrel{\text{def}}{=} (I - \omega^2\Omega^2 (K^{-})^2)^{-1} = \sum_{0}^{\infty} (\omega^2\Omega^2)^n (K^{-})^{2n} \end{aligned}$$

we have

$$(7.4)\qquad \begin{aligned} Q^{+} &= R^{+} - i\frac{\Omega}{\omega} R^{+}H = R^{+} - i\frac{\Omega}{\omega} H R^{+} \\ Q^{-} &= R^{-} - i\omega\Omega R^{-}K^{-} = R^{-} - i\omega\Omega K^{-}R^{-} . \end{aligned}$$

Hence, because of the parity relations mentioned,

$$(7.5)\qquad \begin{aligned} (Q^{\pm}\gamma)_o &= (R^{\pm}\gamma)_o \\ (Q^{+}\sigma)_o &= -i\Omega\omega^{-1}(R^{+}H\sigma)_o \\ (Q^{-}\sigma)_o &= -i\Omega\omega(R^{-}K^{-}\sigma)_o \end{aligned}$$

$$(7.6)\qquad \begin{aligned} (Q^{+}NQ^{-}\gamma)_o &= i\omega\Omega(R^{+}NK^{-}R^{-}\gamma)_o - i\Omega\omega^{-1}(R^{+}HNR^{-}\gamma)_o \\ (Q^{-}NQ^{+}\gamma)_o &= -i\omega\Omega(R^{-}K^{-}NR^{+}\gamma)_o - i\Omega\omega^{-1}(R^{-}NHR^{+}\gamma)_o \\ (Q^{+}NQ^{-}\sigma)_o &= (R^{+}NR^{-}\sigma)_o - \Omega^2(R^{+}HNK^{-}R^{-}\sigma)_o \\ (Q^{-}NQ^{+}\sigma)_o &= (R^{-}NR^{+}\sigma)_o - \Omega^2(R^{-}K^{-}NHR^{+}\sigma)_o . \end{aligned}$$

Noticing that for arbitrary operators

$$(UNV)_{mn} = 2\beta(U\Gamma)_{m1} V_{1n} (1+O(\beta))$$

and that

$$(R^{\pm} HN)_{o1} = (R^{\pm} K^{-}N)_{o1} = 0$$

(7.6) becomes, omitting further the $O(\beta^2)$ correction,

$$(7.7)\qquad \begin{aligned} (Q^{+}NQ^{-}\gamma)_o &= -2i\Omega\omega\beta(R^{+}\Gamma)_{o1} (K^{-}R^{-}\gamma)_1 \\ (Q^{-}NQ^{+}\gamma)_o &= -2i\Omega\omega^{-1}\beta(R^{-}\Gamma)_{o1} (HR^{+}\gamma)_1 \\ (Q^{+}NQ^{-}\sigma)_o &= 2\beta(R^{+}\Gamma)_o (R^{-}\sigma)_1 \\ (Q^{-}NQ^{+}\sigma)_o &= 2\beta(R^{-}\Gamma)_o (R^{+}\sigma)_1 . \end{aligned}$$

Consequently the zero order approximation of (6.38) becomes

$$\Delta_o = \begin{vmatrix} (Q^-\gamma)_o & (Q^+\sigma)_o\cos\omega b - i(Q^+\gamma)_o\sin\omega b \\ (Q^-\sigma)_o & (Q^+\gamma)_o\cos\omega b - i(Q^+\sigma)_o\sin\omega b \end{vmatrix}$$

(7.8)

$$= \begin{vmatrix} (R^-\gamma)_o & -i\Omega\omega^{-1}(R^+H\sigma)_o\cos\omega b - i(R^+\gamma)_o\sin\omega b \\ -i\omega\Omega(R^-K^-\sigma)_o & (R^+\gamma)_o\cos\omega b - \Omega\omega^{-1}(R^+H\sigma)_o\sin\omega b \end{vmatrix}$$

or

(7.9) $$\Delta_o = \Delta_{oc}\cos\omega b - \Delta_{os}\sin\omega b$$

with

$$\Delta_{oc} \overset{def}{=} \begin{vmatrix} (R^-\gamma)_o & \Omega(R^+H\sigma)_o \\ -\Omega(R^-K^-\sigma)_o & (R^+\gamma)_o \end{vmatrix}$$

(7.10)

$$\Delta_{os} \overset{def}{=} \begin{vmatrix} (R^-\gamma)_o & \omega(R^+\gamma)_o \\ \Omega(R^-K^-\sigma)_o & \Omega\omega^{-1}(R^+H\sigma)_o \end{vmatrix} .$$

The first order correction is

$$\Delta_1 = -\Omega\begin{vmatrix} (Q^-NQ^+\sigma)_o\cos\omega b - i(Q^-NQ^+\gamma)_o\sin\omega b & (Q^+\sigma)_o\cos\omega b - i(Q^+\gamma)_o\sin\omega b \\ (Q^-NQ^+\gamma)_o\cos\omega b - i(Q^-NQ^+\sigma)_o\sin\omega b & (Q^+\gamma)_o\cos\omega b - i(Q^+\sigma)_o\sin\omega b \end{vmatrix}$$

$$+\Omega\begin{vmatrix} (Q^-\gamma)_o & (Q^+NQ^-\gamma)_o \\ (Q^-\sigma)_o & (Q^+NQ^-\sigma)_o \end{vmatrix}$$

$$= -2\Omega\beta(R^-\Gamma)_{o1}\begin{vmatrix} (R^+\sigma)_1\cos\omega b - \Omega\omega^{-1}(R^+H\gamma)_1\sin\omega b & -i\Omega\omega^{-1}(R^+H\sigma)_o\cos\omega b - i(R^+\gamma)_o\sin\omega b \\ -i\Omega\omega^{-1}(R^+H\gamma)_1\cos\omega b - i(R^+\sigma)_1\sin\omega b & (R^+\gamma)_o\cos\omega b - \Omega\omega^{-1}(R^+H\sigma)_o\sin\omega b \end{vmatrix}$$

$$+ 2\Omega\beta(R^+\Gamma)_{o1}\begin{vmatrix} (R^-\gamma)_o & -i\Omega\omega(R^-K^-\gamma)_1 \\ -i\Omega\omega(R^-K^-\sigma)_o & (R^-\sigma)_1 \end{vmatrix}$$

i.e.

$$(7.11)\qquad \Delta_1 = 2\Omega\beta(R^+\Gamma)_{01}\begin{vmatrix} (R^-\gamma)_0 & \Omega\omega(R^-K^-\gamma)_1 \\ -\Omega\omega(R^-K^-\sigma)_0 & (R^-\sigma)_1 \end{vmatrix}$$

$$-2\Omega\beta(R^-\Gamma)_{01}\begin{vmatrix} (R^+\sigma)_1 & \Omega\omega^{-1}(R^+H\sigma)_0 \\ -\Omega\omega^{-1}(R^+H\gamma)_1 & (R^+\gamma)_0 \end{vmatrix} .$$

Our approximate equation having the form

$$(7.12)\qquad \Delta_{oc}\cos\omega b - \Delta_{os}\sin\omega b + \Delta_1 = 0$$

where $\Delta_1 = O(\beta)$, the zero order approximate solution is

$$(7.13)\qquad \omega = \frac{1}{b}\,\text{arc ctn}\,\frac{\Delta_{os}}{\Delta_{oc}} + O(\beta)$$

and the first order one

$$(7.14)\qquad \omega = \frac{1}{b}\left[\text{arc ctn}\,\frac{\Delta_{os}}{\Delta_{oc}} + \Delta_1(\Delta_{oc}^2+\Delta_{os}^2)^{-\frac{1}{2}}\right] + O(\beta^2) \quad .$$

In order to get more explicit results we restrict ourselves to the case where Ω is small and where the oscillation passes for $\Omega\rightarrow 0$ into one of the simple one-dimensional ones (6.13), having a frequency $\omega_o = (k+\frac{1}{2})\pi/b$ satisfying $\cos\omega_o b = 0$. We assume that there is no degeneration, i.e. that for this value ω_o of ω and $\Omega = 0$ all $\nu_n = \nu_n^o$ are $\neq 0$. We shall give the approximation up to $O(\Omega^4)$ inclusive, treating β and Ω^2 as being of equal order, and later explicitize further the first order approximation which is $O(\Omega^2)$. We notice that $\gamma_o = 4\sinh\frac{1}{2}\pi\Omega/\pi\Omega$ is $O(1)$ in Ω (viz. $2+O(\Omega^2)$), that all γ_m for $m>1$ are $O(\Omega^2)$ and all σ_n are $O(\Omega)$. Consequently for $n\geq 1$

$$(H^{2n}\gamma)_o = \sum\nolimits_*(H^{2n})_{om}\gamma_m = O(\Omega^2)$$

as $(H^{2n})_{oo} = \sum\nolimits_*(H^{2n-1})_{ok}H_{ko} = 0$ because of $H_{ko}=0$ for all k. Similarly $((K^-)^{2n}\gamma)_o = O(\Omega^2)$ for $n\geq 1$, and $(H^{2n+1}\sigma)_o = O(\Omega)$, $((K^-)^{2n+1}\sigma)_o = O(\Omega)$. Hence

$$(R^+\gamma)_o = \sum_o^\infty\left(-\frac{\Omega^2}{\omega^2}\right)^n(H^{2n}\gamma)_o = \gamma_o + O(\Omega^4)$$

(not only $O(\Omega^2)$)

$$(R^+\gamma)_o = \gamma_o - \frac{\Omega^2}{\omega^2}(H^2\gamma)_o + O(\Omega^6)$$

and similarly

$$(R^- \gamma)_o = \gamma_o - \Omega^2 \omega^2((K^-)^2 \gamma)_o + O(\Omega^6) .$$

Further

$$(R^+ H \sigma)_o = (H\sigma)_o + O(\Omega)$$

$$(R^- K^- \sigma)_o = (K^- \sigma)_o + O(\Omega^3) .$$

Consequently by (7.10)

$$\Delta_{oc} = \begin{vmatrix} \gamma_o - \omega^2\Omega^2((K^-)^2\gamma)_o & \Omega(H\sigma)_o \\ -\Omega(K^-\sigma)_o & \gamma_o + \Omega^2\omega^{-2}(H^2\gamma)_o \end{vmatrix} + O(\Omega^6) \tag{7.15}$$

$$= \gamma_o^2 + O(\Omega^4)$$

$$\Delta_{os} = \begin{vmatrix} \gamma_o & \omega\gamma_o \\ (K^-\sigma)_o - \omega^2\Omega^2((K^-)^3\sigma)_o & \omega^{-1}(H\sigma)_o - \omega^{-3}\Omega^2(H^3\sigma)_o \end{vmatrix} + O(\Omega^6) = \tag{7.16}$$

$$= \Omega\gamma_o\left[\omega^{-1}(H\sigma)_o - \omega(K^-\sigma)_o\right] +$$

$$- \Omega^3 \gamma_o \left[\omega^{-3}(H^3\sigma)_o - \omega^3((K^-)^3\sigma)_o\right] + O(\Omega^6)$$

$$\Delta_1 = O(\Omega^4 \beta^2) . \tag{7.17}$$

Hence by (7.13) the first order approximation is, putting

$$\omega = \omega_o - \delta \tag{7.18}$$

whence $\operatorname{ctn}\omega b + \tan\delta b = \delta b + O(\delta^3)$ and $\delta = \delta_1 + O(\Omega^4, \beta)$ with

$$\begin{aligned} \delta_1 &= b^{-1}\Delta_{os}/\Delta_{oc} + O(\beta) \\ &= \tfrac{1}{2}\Omega^2 b^{-1}\left[(\omega\Omega)^{-1}(H\sigma)_o - \omega\Omega^{-1}(K^-\sigma)_o\right] \end{aligned} \tag{7.19}$$

and the second order approximation $\delta = \delta_2 + O(\Omega^6, \beta^2)$ with

$$\begin{aligned} \delta_2 &= b^{-1}\left[\Delta_{os}/\Delta_{oc} - \tfrac{1}{2}\Delta_1(\Delta_{os}^2 + \Delta_{oc}^2)^{-\frac{1}{2}}\right] + O(\beta^2) \\ &= \delta_1\left(1 - \frac{\pi^2\Omega^2}{12}\right) - \tfrac{1}{2}\Omega^3 b^{-1}\left[\omega^{-3}(H^3\sigma)_o - \omega^3(K^-)^3\sigma)_o\right] \end{aligned} \tag{7.20}$$

Now, writing again ν_n° for $\sqrt{n^2-\omega_o^2}$

$$(H\sigma)_o = \sum_1^\infty \Gamma_{on} \frac{\nu_n \tanh \nu_n b}{n} \sigma_n = \frac{16}{\pi^2}\Omega \cosh \tfrac{1}{2}\pi\Omega \times$$

$$\times \sum_1^\infty \frac{p(n)\, \nu_n \tanh \nu_n b}{n^2(n^2+\Omega^2)} =$$

$$= -\frac{16\Omega}{\pi^2} \sum_1^\infty \frac{p(n)\, \nu_n^{\circ}}{n^4} + O(\Omega^2, \beta^2)$$

(7.21)

$$(K\sigma)_o = \sum_1^\infty \Gamma_{on} \frac{\tanh \nu_n b}{n\nu_n} \sigma_n = -\frac{16\Omega}{\pi^2} \sum_1^\infty \frac{p(n)}{n^4 \nu_n^{\circ}} +$$

$$+ O(\Omega^2, \beta^2)$$

so that (7.19) becomes

$$\frac{\delta_1}{\omega_o} = -\frac{8\Omega^2}{\pi^2 b\, \omega_o^2} \sum_1^\infty \frac{(n^2-2\omega_o^2)p_n}{n^4 \nu_n^{\circ}} . \qquad (7.22)$$

We shall not explicitize (7.20), where e.g. the difference between ω^2 and ω_o^2 implicit in ν_n has to be taken into account.

For the special case $b=2\pi$, $\omega_o=1/4$ (lowest ocsillation of North Sea-like bay) (7.22) becomes

$$\frac{\delta_1}{\omega_o} = -\frac{32\Omega^2}{\pi^3}\left(\frac{7}{\sqrt{15}} + \frac{71}{81\sqrt{143}} + \frac{199}{625\sqrt{399}} \cdots\cdots\right) = -2.01\Omega^2.$$

Hence ω is for $\Omega = 0{,}55$ about 63% larger than it would be for $\Omega = 0$. Actually we have to this approximation $\omega = 0{,}42$ instead of $\omega = 0{,}25$. In ordinary units the oscillation time $2\pi\omega^{-1}$ becomes to this approximation 15 instead of 25 hours. It must, however, be reminded that the neglect of the terms $O(\Omega^2)$ is decidedly too rough for this application.

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