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Report
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# A uniform windfield on a rotating sea in presence of a semi-infinite barrier

by

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## § 1. Introduction.

The work described in this paper*) arises from certain aspects of an investigation into the influence of storms upon the North Sea. In this paper the stress is laid upon the method rather than upon numerical results.

The problem is to find a solution of the equation of Helmholtz in the x,y plane with a semi-infinite barrier at the positive x-axis. At the barrier a derivative of given constant obliqueness is prescribed.

This problem is somewhat related to Sommerfeld's well-known diffraction problem. A similar problem has been treated recently by J. Crease [1] who reduced his problem to a singular integral equation which could be solved by means of the familiar Wiener-Hopf technique.

However, we consider a problem which is essentially of a non-stationary kind. A more direct way of solving will be developed which has proved to be successful also in more complicated problems. The method consists of the reduction to a Hilbert problem [2] for sectionally holomorphic functions. Throughout this paper a frequent use is made of Laplace transformation.

By way of illustration the elevation of the sea at the barrier due to a stepfunction windfield will be calculated. In particular the influence of the rotation of the earth and of the bottom friction will be studied.

Let the (x,y)-plane represent a sea of constant depth with a barrier at the positive x-axis. We shall assume that at t=0 everything is at rest and that for $t > 0$ the elevation of the sea $\zeta(x,y,t)$ is influenced by a uniform windfield depending only upon the time t.

The equations of horizontal motion are

$$\begin{cases} (\frac{\partial}{\partial t} + \lambda)u - \Omega v + \frac{\partial \zeta}{\partial x} = U \\ (\frac{\partial}{\partial t} + \lambda)v + \Omega u + \frac{\partial \zeta}{\partial y} = V , \end{cases} \qquad 1.1$$

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*) Research carried out under the direction of Prof.Dr D.van Dantzi

1) J. Crease. Journal of fluid mechanics, vol.1, p.86-96 (1956).

2) Muskhelishvili. Singular integral equations. Groningen (1953), p.

where u,v are the components of velocity, U,V the components of the surface stress due to the wind, $\lambda$ an effective friction coefficient accounting for the bottom friction, and $\Omega$ the Coriolis parameter ; $\sqrt{gh}$ , where h is the constant depth, is taken as the unit of velocity. To this we may add the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0 \ . \qquad 1.2$$

At the barrier we have

$$y=0, \quad x > 0 \qquad\qquad v=0. \qquad 1.3$$

If upon 1.1 and 1.2 Laplace transformation is applied

$$\bar{\zeta}(x,y,p) = \int_0^\infty e^{-pt}\, \zeta(x,y,t)dt \qquad 1.4$$

etc.
we obtain for $\bar{\zeta}$ a Helmholtz equation

$$(\Delta - q^2)\bar{\zeta} = 0, \qquad 1.5$$

where

$$q^2 = p(p+\lambda) + \frac{p\Omega^2}{p+\lambda}, \qquad 1.6$$

with the oblique boundary condition

$$y=0,\ x > 0 \qquad \frac{\partial \bar{\zeta}}{\partial y} - \mathrm{tg}\,\gamma \frac{\partial \bar{\zeta}}{\partial x} = \bar{V} - \mathrm{tg}\,\gamma\, \bar{U} , \qquad 1.7$$

where

$$\mathrm{tg}\,\gamma = \frac{\Omega}{p+\lambda} . \qquad 1.8$$

## § 2. Solution of the problem.

We shall provisionally take q=1 , $\bar{V} - \mathrm{tg}\,\gamma\,\bar{U}=1$.
Then the solution of 1.5 with 1.7 will be represented in the following form

$$\bar{\zeta}(x,y,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp - \left\{ ixz + |y|\sqrt{1+z^2} \right\} \left\{ F(z) + \frac{y}{|y|} G(z) \right\} dz. \qquad 2.1$$

The continuity of $\frac{\partial \bar{\zeta}}{\partial y} - \mathrm{tg}\,\gamma \frac{\partial \bar{\zeta}}{\partial x}$ at the full x-axis gives at once

$$(iz\ \mathrm{tg}\,\gamma - \sqrt{1+z^2})(F+G) = (iz\ \mathrm{tg}\,\gamma + \sqrt{1+z^2})(F-G)$$

so that

$$F(z) = \frac{iz\ tg\gamma}{\sqrt{1+z^2}}\ G(z)\ . \qquad 2.2$$

The continuity of $\bar{f}$ at y=0, x < 0 gives

$$\int_{-\infty}^{\infty} e^{-ixz}\ G(z)dz=0 \quad \text{for} \quad x<0\ . \qquad 2.3$$

This is true if G(z) is holomorphic in the upper halfplane Im z > 0, free from singularities and vanishing at infinity. We shall write $G(z)=\phi^+(z)$. The boundary condition at y=0, x > 0 gives

$$\int_{-\infty}^{\infty} e^{-ixz} \left\{ \frac{z^2+\cos^2\gamma}{\sqrt{1+z^2}}\ G(z) + \frac{i}{z+\varepsilon i} \right\} dz = 0 \quad \text{for} \quad x>0, \qquad 2.4$$

where $\varepsilon \rightarrow +0$. Similarly we write

$$\phi^-(z) \overset{\text{def}}{=} \frac{z^2+\cos^2\gamma}{\sqrt{1+z^2}}\ G(z) + \frac{i}{z+\varepsilon i}$$

where $\phi^-(z)$ is holomorphic etc. in the lower halfplane Im z < 0. At the real axis we have

$$\frac{z^2+\cos^2\gamma}{\sqrt{1+z^2}}\ \phi^+(z) - \phi^-(z) = \frac{-i}{z+\varepsilon i} \qquad 2.5$$

which is the Hilbert problem mentioned in the introduction. This problem can be solved in a straightforward way. In view of the factorisation

$$\frac{z^2+\cos^2\gamma}{\sqrt{1+z^2}} = \frac{z+i\ \cos\gamma}{\sqrt{1-iz}}\ \frac{z-i\ \cos\gamma}{\sqrt{1+iz}}$$

the relation 2.5 may be brought into the form

$$\frac{z+i\ \cos\gamma}{\sqrt{1-iz}}\ \phi^+(z) - \frac{\sqrt{1+iz}}{z-i\ \cos\gamma}\ \phi^-(z) = f(z)\ , \qquad 2.6$$

where

$$f(z) = \frac{-i\ \sqrt{1+iz}}{(z+\varepsilon i)(z-i\ \cos\gamma)}\ .$$

The simpler Hilbert problem 2.6 has the solution

$$\frac{z+i\ \cos\gamma}{\sqrt{1-iz}}\ \phi^+(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t-z}\,dt\ , \quad \text{Im } z>0. \qquad 2.7$$

Without difficulty we obtain

$$G(z) = \frac{\sqrt{1-iz}}{(z+\varepsilon i)(z+i\cos\gamma)} . \qquad 2.8$$

The solution 2.1 becomes

$$\bar{\mathcal{T}}(x,y,p) = \frac{1}{2\pi}\int_{-\infty+\varepsilon i}^{\infty+\varepsilon i} \exp -\left\{ixz + |y|\sqrt{1+z^2}\right\}\left\{\frac{y}{|y|} + \frac{iz\,\mathrm{tg}\,\gamma}{\sqrt{1+z^2}}\right\}\frac{\sqrt{1-iz}}{z(z+i\cos\gamma)}\,dz \qquad 2.9$$

In polar coordinates $x=r\cos\varphi$ , $y=r\sin\varphi$ the expression 2.9 becomes after the substitution $z=\mathrm{sh}\,w$

$$\bar{\mathcal{T}}(r,\varphi,p) = \frac{1}{2\pi}\int_L e^{-ir\,\mathrm{sh}(w-i\varphi)}\left\{\frac{1}{2\mathrm{ch}\frac{w}{2}} - \frac{i}{2\mathrm{sh}\frac{w}{2}} + \frac{\sqrt{1-\cos\gamma}}{\mathrm{ch}\frac{1}{2}(w-\gamma i-\frac{\pi i}{2})}\right\}dw, \qquad 2.10$$

where $0 < \varphi < 2\pi$ and where L is a horizontal path with $\mathrm{Min}(0,\varphi-\pi) < \mathrm{Im}\,w < \varphi$, $-\infty < \mathrm{Re}\,w < \infty$ .

From 2.10 we may derive

$$\bar{\mathcal{T}}(r,\varphi,p) = \tfrac{1}{2}\psi(r,\tfrac{\pi}{2}-\varphi) - \tfrac{1}{2}\psi(r,\tfrac{3\pi}{2}-\varphi) + \sqrt{1-\cos\gamma}\;\psi(r,\pi+\gamma-\varphi), \qquad 2.11$$

$$\psi(r,\alpha) = e^{r\cos\alpha}\,\mathrm{erfc}(\sqrt{2r}\cos\tfrac{\alpha}{2}). \qquad 2.12$$

In particular we obtain at the barrier

$$\bar{\mathcal{T}}(x,\pm 0,p) = \mp\,\mathrm{erf}\sqrt{x} + \sqrt{1-\cos\gamma}\;e^{-x\cos\gamma}\left\{1 \pm \mathrm{erf}\sqrt{x(1-\cos\gamma)}\right\} \qquad 2.13$$

## § 3. Applications.

We shall consider only the particular windfield

$$U = 0 \qquad V = \begin{cases} 0 & t < 0 \\ -1 & t > 0 \end{cases} , \qquad 3.1$$

then we have at the upper side of the barrier

$$\bar{\mathcal{T}}(x,+0,p) = \frac{1}{pq}\left\{\mathrm{erf}\sqrt{qx} - \sqrt{1-\cos\gamma}\;e^{-qx\cos\gamma}(1+\mathrm{erf}\sqrt{qx(1-\cos\gamma)})\right\}, \qquad 3.2$$

where $q \cos\gamma = \sqrt{p^2 + \lambda p}$.

a. $\lambda = 0$ , $\Omega = 0$ .

Formula 3.2 reduces to

$$\bar{\varphi} = \frac{\text{erf}\sqrt{px}}{p^2} \quad . \tag{3.3}$$

The inverse transform becomes

$$\varphi = \begin{cases} t & 0 < t < x \\ \frac{2t}{\pi}\arcsin\sqrt{\frac{x}{t}} + \frac{2}{\pi}\sqrt{x(t-x)} , & t > x \end{cases} \quad . \tag{3.4}$$

We note that for $t \to \infty$

$$\varphi = \frac{4}{\pi}\sqrt{xt} + O(t^{-\frac{1}{2}}) \quad . \tag{3.5}$$

b. $\lambda \neq 0$ , $\Omega = 0$.

We have
$$\bar{\varphi} = \frac{\text{erf}\sqrt{qx}}{pq} \tag{3.6}$$

where
$$q = \sqrt{p^2 + \lambda p}.$$

The inverse transform is elementary only for $t < x$. We have

$$\varphi = \int_0^t e^{-\frac{\lambda\tau}{2}} I_0\left(\frac{\lambda\tau}{2}\right)d\tau , \qquad 0 < t < x , \tag{3.7}$$

$$\varphi = \frac{2x^{1/2}t^{1/4}}{\pi^{1/2}\lambda^{1/4}\Gamma(\frac{5}{4})} + O(t^{-1/4}) , \qquad t \to \infty. \tag{3.8}$$

c. $\lambda = 0, \Omega \neq 0$.

We have

$$\bar{\varphi} = \frac{1}{pq}\left\{\text{erf}\sqrt{qx} - \frac{\sqrt{q-p}}{\sqrt{q}}e^{-px}\left(1 + \text{erf}\sqrt{(q-p)x}\right)\right\} \tag{3.9}$$

where $q = \sqrt{p^2 + \Omega^2}$.

The inverse transform is elementary only for $t < x$.

$$\varphi = \int_0^t J_0(\Omega\tau)d\tau , \qquad 0 < t < x. \tag{3.10}$$

Since for $p \longrightarrow 0$ $\quad \bar{\zeta} = -\frac{1}{pq}\left\{1+O(p)\right\}$

we have

$$\zeta \longrightarrow -\frac{1}{\Omega}, \quad t \rightarrow \infty \qquad 3.11$$

In figures 1 and 2 the elevation at the upper side of the barrier is sketched in the cases

a $x = 10$     b $x = 10$, $\lambda = 0.1$     c $x = 10$, $\Omega = 0.4$ .

Also the graphs of $\zeta$ for an infinite barrier $(-\infty < x < \infty)$ are given. For $0 < t < x$ they coincide with the graphs of $\zeta$ for a semi-infinite barrier.

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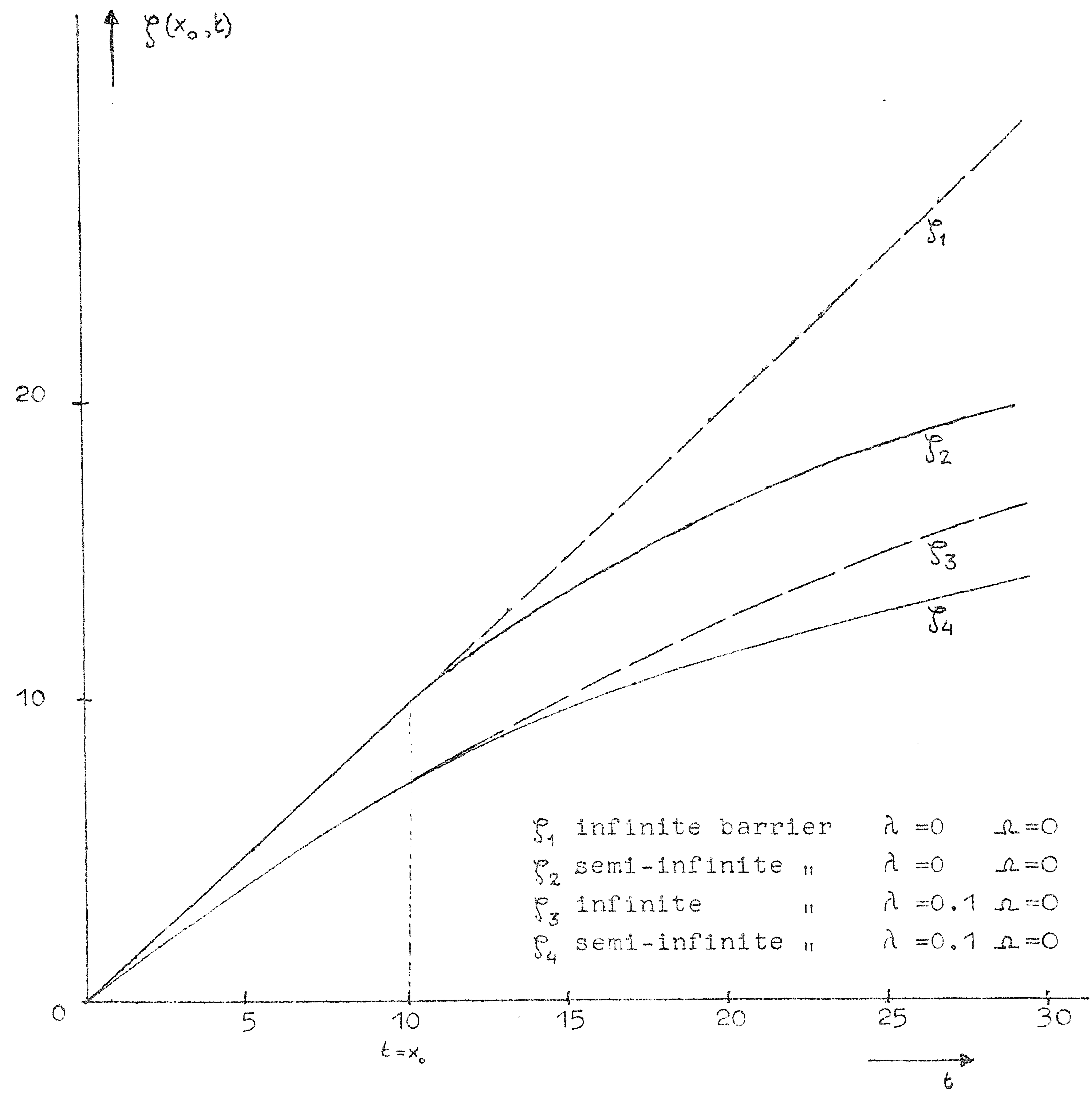


fig. 1

Elevation at the upper side of the barrier for $x_0=10$ ; $\lambda=0$ , 0.1 ; $\Omega=0$ .

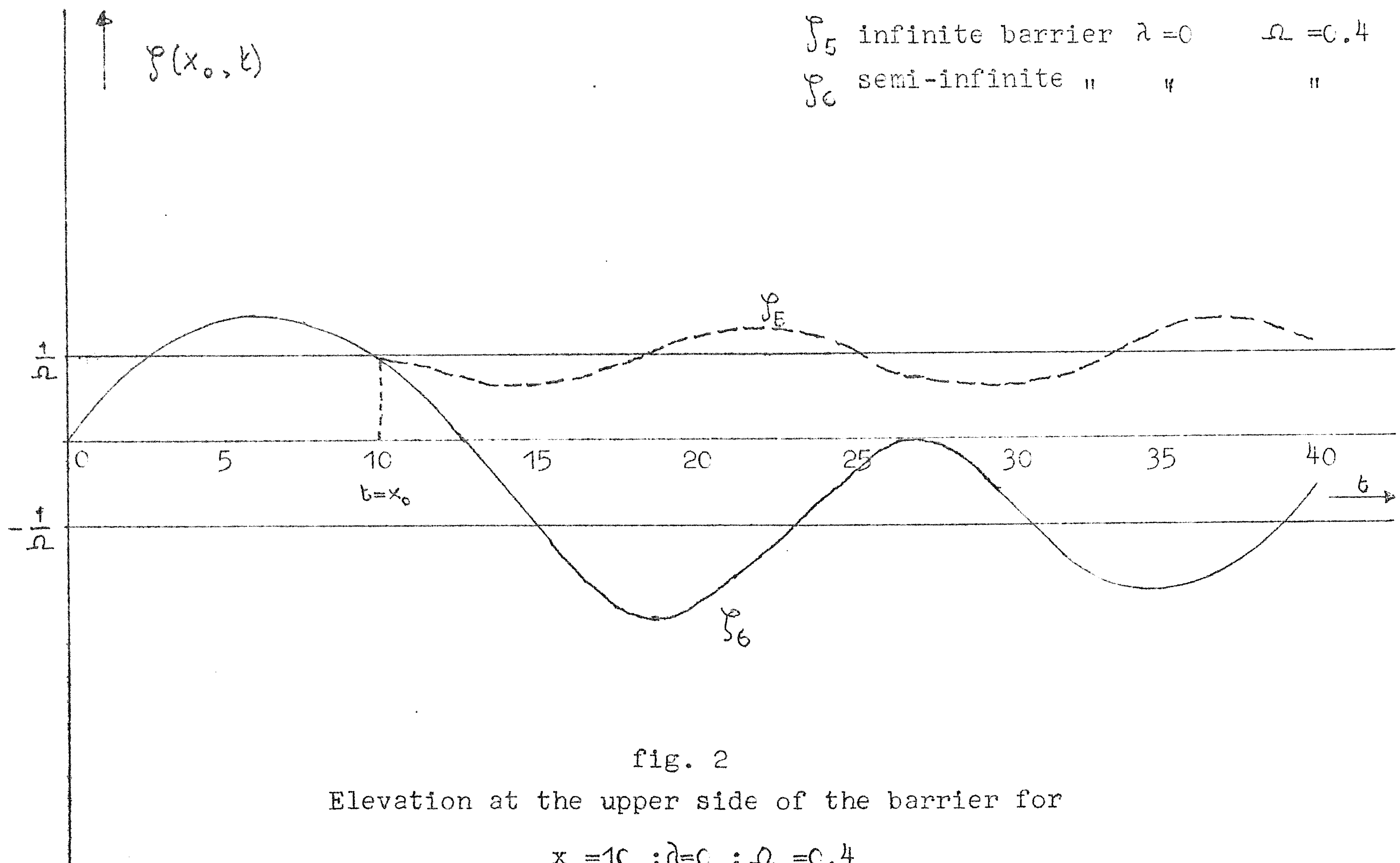


fig. 2

Elevation at the upper side of the barrier for

$x_0 = 10$ ; $\lambda = 0$ ; $\Omega = 0.4$