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Invariants of Functions On A Sphere

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MATHEMATICS

INVARIANTS OF FUNCTIONS ON A SPHERE [1)]

BY

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1. In this paper the following theorem will be proved. Let $f(\vartheta, \varphi)$ and $g(\vartheta, \varphi)$ be functions given on a unit-sphere which admit of expansions in terms of spherical harmonics, viz.

$$f(\vartheta, \varphi) = \frac{2k+1}{4\pi} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} F_k^m P_k^m(\cos \vartheta) e^{im\varphi}, \tag{1.1}$$

$$g(\vartheta, \varphi) = \frac{2k+1}{4\pi} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} G_k^m P_k^m(\cos \vartheta) e^{im\varphi}, \tag{1.2}$$

then the quantities

$$[f, g]_k \stackrel{\text{def}}{=} \oint \oint f(\vartheta, \varphi)\, g(\vartheta', \varphi')\, P_k(\cos \gamma)\, d\omega\, d\omega', \qquad k = 0, 1, 2, \ldots \tag{1.3}$$

with

$$\cos \gamma \stackrel{\text{def}}{=} \cos \vartheta \cos \vartheta' + \sin \vartheta \sin \vartheta' \cos (\varphi - \varphi') \tag{1.4}$$

are given by

$$[f, g]_k = \sum_{m=-k}^{k} F_k^m G_k^{-m}. \tag{1.5}$$

In (1.3) $d\omega = \sin \vartheta \, d\vartheta \, d\varphi$ is a surface element of the sphere and $\oint$ denotes integration over the entire surface of the sphere.

Since γ represents the angle between the radii to the points (ϑ, φ) and (ϑ', φ'), it follows immediately from (1.3) that the $[f, g]_k$ are invariants in the sense that their numerical values are unaffected by rotations of the coordinate system.

Instead of (1.1) we may write

$$f(\vartheta, \varphi) = \sum_{k=0}^{\infty} f_k(\vartheta, \varphi), \tag{1.6}$$

$$f_k(\vartheta, \varphi) = \frac{2k+1}{\pi} \sum_{m=-k}^{k} F_k^m P_k^m(\cos \vartheta) e^{im\varphi}, \tag{1.7}$$

and (1.2) similarly. This means, that the functions $f(\vartheta, \varphi)$ and $g(\vartheta, \varphi)$ are expanded in general spherical harmonics $f_k(\vartheta, \varphi)$ and $g_k(\vartheta, \varphi)$ respectively. These harmonics are orthogonal, i.e.

$$\oint f_m(\vartheta, \varphi)\, g_k(\vartheta, \varphi)\, d\omega = 0, \qquad m \neq k, \tag{1.8}$$

1) Report TW 60 of the Mathematical Centre, Amsterdam.

and they have the property [1])

$$\oint f_k(\vartheta', \varphi')\, P_k(\cos\gamma)\, d\omega' = \frac{4\pi}{2k+1} f_k(\vartheta, \varphi). \tag{1.9}$$

Because of the orthogonality of the f_m with respect to the special surface harmonic $P_k(\cos\gamma)$ we may replace f_k by f in (1.9), or

$$\oint f(\vartheta', \varphi')\, P_k(\cos\gamma)\, d\omega' = \frac{4\pi}{2k+1} f_k(\vartheta, \varphi). \tag{1.10}$$

Hence

$$\begin{aligned} [f, g]_k &= \oint\oint f(\vartheta', \varphi')\, g(\vartheta, \varphi)\, P_k(\cos\gamma)\, d\omega\, d\omega' \\ &= \frac{4\pi}{2k+1} \oint f_k(\vartheta, \varphi)\, g(\vartheta, \varphi)\, d\omega \\ &= \frac{4\pi}{2k+1} \oint f_k(\vartheta, \varphi)\, g_k(\vartheta, \varphi)\, d\omega, \end{aligned} \tag{1.11}$$

by aid of the orthogonality relation (1.8).

Since (1.7) is a Fourier expansion of f_k with respect to φ, Parseval's theorem is applicable. We find

$$\begin{aligned} \int_0^{2\pi} f_k(\vartheta, \varphi)\, g_k(\vartheta, \varphi)\, d\varphi &= \frac{(4k+1)^2}{8\pi} \sum_{m=-k}^{k} F_k^m G_k^{-m} P_k^m(\cos\vartheta)\, P_k^{-m}(\cos\vartheta) \\ &= \frac{(4k+1)^2}{8\pi} \sum_{m=-k}^{k} (-1)^m \frac{(k-m)!}{(k+m)!} F_k^m G_k^{-m} [P_k^m(\cos\vartheta)]^2, \end{aligned} \tag{1.12}$$

because of $(k-m)!\, P_k^m = (-1)^m (k+m)!\, P_k^{-m}$.

By means of the normalization integral for associated Legendre functions

$$\int_0^{\pi} [P_k^m(\cos\vartheta)]^2 \sin\vartheta\, d\vartheta = \frac{2}{2k+1} \frac{(k+m)!}{(k-m)!}, \tag{1.13}$$

it follows that

$$\begin{aligned} [f, g]_k &= \frac{4\pi}{2k+1} \int_0^{\pi} \int_0^{2\pi} f_k(\vartheta, \varphi)\, g_k(\vartheta, \varphi)\, d\varphi \sin\vartheta\, d\vartheta \\ &= \sum_{m=-k}^{k} F_k^m G_k^{-m}, \end{aligned}$$

which proves the theorem.

We note the following "addition formula"

$$[af+bg, cf+dg]_k = ac[f, f]_k + (ad+bc)[f, g]_k + bd[g, g]_k, \tag{1.14}$$

which follows immediately from (1.3). This formula will be used in another context [2].

2. The above theorem can be generalized to more dimensions in the following way.

[1]) cf. e.g. [1], art. 49.

Let $\mathbf{u}$ and $\mathbf{v}$ be two $(p+2)$-dimensional unitvectors and let $f(\mathbf{u})$ and $g(\mathbf{u})$ be two functions on the hypersurface of a $(p+2)$-dimensional unit hypersphere. Expansion of f and g in surface harmonics gives

$$(2.1) \qquad f(\mathbf{u}) = \sum_{k=0}^{\infty} f_k(\mathbf{u}), \qquad g(\mathbf{u}) = \sum_{k=0}^{\infty} g_k(\mathbf{u}).$$

Again the surface harmonics are orthogonal, i.e.

$$(2.2) \qquad \oint f_m(\mathbf{u})\, g_k(\mathbf{u})\, d\omega_u = 0, \qquad m \neq k,$$

where $d\omega_u$ is an element of the hypersurface.

Furthermore (1.9) is a special case of a theorem of FUNK and HECKE [1]), which states that

$$(1.3) \qquad \begin{cases} \oint F(\mathbf{u}\cdot\mathbf{v})\, f_k(\mathbf{u})\, d\omega_u = \lambda_k f_k(\mathbf{v}), \\ \lambda_k = \dfrac{(4\pi)^{\frac{1}{2}p}\, k!\, \Gamma(\frac{1}{2}p)}{(k+p-1)!} \displaystyle\int_{-1}^{1} (1-x^2)^{\frac{1}{2}(p-1)}\, C_k^{\frac{1}{2}p}(x)\, F(x)\, dx, \end{cases}$$

where $C_k^{\frac{1}{2}p}(x)$ is a Gegenbauer function.

If we take $F(\mathbf{u}\cdot\mathbf{v}) = C_k^{\frac{1}{2}p}(\mathbf{u}\cdot\mathbf{v})$ which is a special surface harmonic depending on one variable only, (2.3) becomes

$$(2.4) \qquad \oint C_k^{\frac{1}{2}p}(\mathbf{u}\cdot\mathbf{v})\, f_k(\mathbf{u})\, d\omega_u = \frac{4\pi^{\frac{1}{2}p+1}}{(p+2k)\Gamma(\frac{1}{2}p)}\, f_k(\mathbf{v}).$$

The λ_k is evaluated by means of the normalization integral for Gegenbauer polynomials

$$(2.5) \qquad \int_{-1}^{1} [C_k^{\frac{1}{2}p}(x)]^2 (1-x^2)^{\frac{1}{2}(p-1)}\, dx = \frac{2^{2-p}\pi(k+p-1)!}{(p+2k)\, k!\, [\Gamma(\frac{1}{2}p)]^2}$$

Because of the orthogonality of the surface harmonics it follows from (2.4) that

$$(2.6) \qquad \oint C_k^{\frac{1}{2}p}(\mathbf{u}\cdot\mathbf{v})\, f(\mathbf{u})\, d\omega_u = \frac{4\pi^{\frac{1}{2}p+1}}{(p+2k)\Gamma(\frac{1}{2}p)}\, f_k(\mathbf{v}).$$

Defining

$$(2.7) \qquad [f,g]_k \overset{\text{def}}{=} \oint \oint C_k^{\frac{1}{2}p}(\mathbf{u}\cdot\mathbf{v})\, f(\mathbf{u})\, g(\mathbf{v})\, d\omega_u\, d\omega_v,$$

we have, because of (2.6)

$$[f,g]_k = \frac{4\pi^{\frac{1}{2}p+1}}{(p+2k)\Gamma(\frac{1}{2}p)} \oint f_k(\mathbf{v})\, g(\mathbf{v})\, d\omega_v,$$

or, again using the orthogonality relation (2.2),

$$(2.8) \qquad [f,g]_k = \frac{4\pi^{\frac{1}{2}p+1}}{(p+2k)\Gamma(\frac{1}{2}p)} \oint f_k(\mathbf{u})\, g_k(\mathbf{u})\, d\omega_u,$$

which is a generalization of (1.11).

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1) cf. e.g. [3] art. 11.4.

In order to derive a generalization of the theorem (1.5) the surface harmonics $f_k(\boldsymbol{u})$ and $g_k(\boldsymbol{u})$ of degree k should be expanded in a series of standard surface harmonics. (It is known, that for a given degree k, a complete set of harmonics exists, containing $\frac{2k+p}{k+p}\binom{k+p}{p}$ members [1]).

Substitutions of these series in the r.h. member of (2.8) then gives an expression containing the expansion coefficients. With increasing number of dimensions, however, it becomes increasingly difficult to label the expansion coefficients satisfactorily. Also several sets of standard harmonics of degree k can be used, depending on the choice of the coordinate system. For these reasons a generalization of (1.5) is not well feasible.

The above remarks will be illustrated for the case $p=2$ in the next section.

3. To designate the points on a fourdimensional unit hypersphere, two sets of coordinates are well suited, defined by the following sets of equations

$$\text{(3.1)} \qquad \begin{cases} x_1 = \cos\vartheta \quad , & x_2 = \sin\vartheta\cos\varphi \quad , \\ x_3 = \sin\vartheta\sin\varphi\cos\psi, & x_4 = \sin\vartheta\sin\varphi\cos\psi, \end{cases}$$

and

$$\text{(3.2)} \qquad \begin{cases} x_1 = \cos\vartheta\cos\varphi, & x_2 = \cos\vartheta\sin\varphi, \\ x_3 = \sin\vartheta\cos\psi, & x_3 = \sin\vartheta\sin\psi. \end{cases}$$

Using the set (3.1) the surface harmonics f_k and g_k can be expanded according to

$$\text{(3.3)} \qquad \begin{cases} f_k = \sum\limits_{n=0}^{k} \sum\limits_{m=-n}^{n} F_k^{m,n} \sin^n\vartheta \, C_{k-n}^{n+1}(\cos\vartheta) \, P_n^m(\cos\varphi) \, e^{im\psi} \\ g_k = \sum\limits_{n=0}^{k} \sum\limits_{m=-n}^{n} G_k^{m,n} \sin^n\vartheta \, C_{k-n}^{n+1}(\cos\vartheta) \, P_n^m(\cos\varphi) \, e^{im\psi} \end{cases}$$

The surface element reads

$$\text{(3.4)} \qquad d\omega_u = \sin^2\vartheta \, d\vartheta \sin\varphi \, d\varphi \, d\psi.$$

Application of Parseval's theorem and substitution of the normalization integrals for the associated Legendre functions (1.13) and for the Gegenbauer polynomials (2.5) results in the expression

$$\text{(3.5)} \qquad [f, g]_k = \frac{4\pi^4}{k+1} \sum_{n=0}^{k} \frac{4^{-n}}{2n+1} \binom{k}{n} \binom{k+n+1}{n} \sum_{m=-n}^{n} F_k^{m,n} \, G_k^{-m,n}.$$

Using the latter set of coordinates (3.2), the surface harmonics f_k and g_k can be expanded according to

$$\text{(3.6)} \qquad \begin{cases} f_k = \sum\limits_{n=-k}^{k} \sum\limits_{m=-k+|n|}^{k-|n|} F_k^{m,n} \, P_{\frac{1}{2}k}^{\frac{1}{2}m, \frac{1}{2}n} (\cos 2\vartheta) \, e^{i(m\varphi+n\psi)}, \\ g_k = \sum\limits_{n=-k}^{k} \sum\limits_{m=-k+|n|}^{k-|n|} G_k^{m,n} \, P_{\frac{1}{2}k}^{\frac{1}{2}m, \frac{1}{2}n} (\cos 2\vartheta) \, e^{i(m\varphi+n\psi)}, \end{cases}$$

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[1]) cf. e.g. [3] art. 11.2.

where, in the summation over m, only those terms are retained for which $k-m-n$ is even.

The functions $P_s^{q,r}$ are generalized associated Legendre functions, defined by (comp [4])

$$(3.7)\quad \begin{cases} P_s^{q,r}(\cos 2\vartheta) \overset{\text{def}}{=} \dfrac{\cos^{2s}\vartheta \tan^{2q}\vartheta}{(2q)!\,(s-q+r)!\,(s-q-r)!} \cdot \\ \qquad \cdot F(-s+q+r,\ -s+q-r;\ 1+2q;\ -\tan^2\vartheta). \end{cases}$$

They are orthogonal, i.e.

$$(3.8)\qquad \int_{-1}^{1} P_s^{q,r}(x)\, P_t^{q,r}(x)\, dx = 0,$$

and their normalization is given by ($s \pm q \pm r$ integer).

$$(3.9)\qquad \int_{-1}^{1} [P_s^{q,r}(x)]^2\, dx = \frac{2}{(2s)!\,(2s+1)!} \binom{2s}{s+q+r} \binom{2s}{s+q-r}.$$

Furthermore

$$(3.10)\qquad P_s^{q,r} = P_s^{q,-r} = (-1)^{2q} P_s^{-q,r} \quad (2q \text{ integer}).$$

The surface element now equals

$$(3.11)\qquad d\omega_u = \tfrac{1}{2} \sin 2\vartheta\, d\vartheta\, d\varphi\, d\psi.$$

Twofold application of Parseval's theorem and substitution of the normalization integral (3.9) and the surface element (3.11) results in the expression

$$(3.12)\quad [f,g] = \left\{\frac{2\pi^2}{(k+1)!}\right\}^2 \sum_{n=-k}^{k} \sum_{m=-k+|n|}^{k-|n|} \binom{k}{\frac{1}{2}(k+m+n)} \binom{k}{\frac{1}{2}(k+m-n)} F_k^{m,n} G_k^{-m,-n},$$

where, in the summation over m, only those terms are retained for which $k-m-n$ is even.

4. The results of section 1 have been obtained in the course of an investigation concerning the expansion of the Earth's topography in a series of spherical harmonics [5]. A related result has been obtained by H. C. van der Hulst. In a private communication to the author, A. Erdélyi suggested that the theorem be generalized to more dimensions. This is done in the remaining sections.

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