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The North Sea Problem. IV

Free Oscillations of a Rotating Rectangular Sea

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1960

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Reprinted from Proceedings, Series A, **63**, No. 4 and Indag. Math., **22**, No. 4, 1960

MATHEMATICS

THE NORTH SEA PROBLEM. IV

FREE OSCILLATIONS OF A ROTATING RECTANGULAR SEA

BY

D. VAN DANTZIG [1]) AND H. A. LAUWERIER

(Communicated by Prof. J. F. KOKSMA at the meeting of March 26, 1960)

List of symbols

x, y Cartesian coordinates. The undisturbed surface of the water is given by $z=0$ and the bottom by $z=h$ (h is a constant);
t the time;
u, v the components of the total stream;
ζ the elevation of the water-surface;
Ω the coefficient of Coriolis;
λ the coefficient of friction;
g the constant of gravity;
c $c=\sqrt{gh}$.

1. *Introduction*

The present paper is a revised and corrected version of a report the late VAN DANTZIG wrote in connection with the hydrodynamic problem of the North Sea. In this paper the free oscillatory motions in a rotating rectangular sea are studied with special reference to the North Sea. The free motions depend on two parameters: the coefficient of friction λ, and the coefficient of Coriolis Ω. Essentially the case $\lambda=0$ is studied. In that case the eigenvalues of the problem are purely imaginary and can be written as $\pm i\omega$, $\omega>0$. However, the influence of a small value of λ upon the eigenvalues can easily be determined.

This investigation may be considered as a continuation of the work of a number of British investigators. RAYLEIGH [1] [2] considered the free oscillations when Ω is small, but some of his results were in error. TAYLOR [1] gave the first complete solution for a closed rectangular sea. JEFFREYS [1] criticized Taylor's conclusions and pointed out that a double infinity of eigenvalues was implied in Taylor's solution and that there might exist modes moving round the basin in both directions. LAMB [1] derived by a different method the approximations to the lowest eigenvalues when Ω is small. For a square sea he obtained in particular

$$\omega=\omega_0 \pm \frac{4}{\pi^2}\,\Omega+\dots. \tag{1.1}$$

[1]) Deceased July 22nd 1959.

GOLDSBROUGH [1] gave an approximate solution for the free oscillations in a rotating rectangular sea. In particular the case of a square sea was investigated and Lamb's formula was confirmed. PROUDMAN [1] re-examined Rayleigh's investigation and showed that by a correct application of Rayleigh's principle Lamb's formula could be obtained. CORKAN and DOODSON [1] considered free oscillations in a rotating closed square sea. By the use of iteration methods a number of cases were treated numerically.

The storm-flood of the 1st February 1953 stimulated VAN DANTZIG to further research in this field. In VAN DANTZIG [1] a review is given of some results obtained at the Mathematical Centre at that time. The final verson of Van Dantzig's work appeared in 1958 as a report [2] of the Mathematical Centre. It contains some generalizations of the results of the above-mentioned authors and, more important, a discussion of the influence of a small value of Ω upon the eigenvalues of a rotating rectangular bay which on three sides is bounded by coasts and which on the remaining side borders on an infinitely deep ocean. In particular he obtained for the lowest eigenvalue $\omega_0 = \frac{1}{2}c\pi/b$ of the rectangular bay $0<x<a$, $0<y<b$, where $y=b$ represents the ocean boundary and where $b \approx 2a$, the following result

$$(1.2) \qquad \omega = \omega_0 + 0.504\,\frac{a\Omega^2}{c\pi} + \dots .$$

Hence, the rotation of the Earth tends to increase the eigenvalues or to lower the period. This is in contrast to the case of the closed rectangular lake $0<x<a$, $-b<y<b$. There assuming $b \approx a$ van Dantzig obtained the result

$$(1.3) \qquad \omega = \omega_0 - 0.151\,\frac{a^3\Omega^2}{b^2c\pi^2} + \dots$$

for the influence of Ω upon the lowest eigenvalue $\omega_0 = \frac{1}{2}c\pi/b$.

Section 2 follows closely Van Dantzig's text. In it it is shown how the case $\lambda \neq 0$ can be reduced to that of $\lambda = 0$. In section 3 where Van Dantzig's text has been slightly shortened the elementary case of the infinite channel is treated. In section 4 the rotating rectangular lake is considered. By introducing the operator formulation at an earlier point the treatment could be made more elegant and concise yet preserving the essentials of the original text. In section 5 the influence of a small value of the parameter Ω upon the lower eigenvalues is studied. This section has been completely revised and some minor mistakes have been corrected. Section 6 which deals with the rectangular bay has been treated in a similar way as section 4. The same applies to section 7 which is a counterpart to section 5.

Quoting Van Dantzig's own words: the treatment in this paper is not in all respects satisfactory since the answers obtained are valid for small Ω only. Therefore the results obtained here are applicable to a laboratory model of a rectangular basin rotating with not too large velocity, rather than to the North Sea where Ω is large.

However, a subsequent paper by the second author will contain a number of rather recent results which treat the influence of a *large* value of the parameter Ω. Moreover the existence of non-oscillatory free motions when $\lambda \neq 0$ will be proved. These free motions which have the character of a pure damping seem to have been overlooked in previous publications.

2. *The mathematical problem*

The free motions of a plane sheet of water are determined by the equations (I 2.6) and (I 2.7) without the wind terms,

$$\left\{\begin{aligned} &\left(\frac{\partial}{\partial t}+\lambda\right) u - \Omega v + gh\,\frac{\partial \zeta}{\partial x} = 0, \\ &\left(\frac{\partial}{\partial t}+\lambda\right) v + \Omega u + gh\,\frac{\partial \zeta}{\partial y} = 0, \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0. \end{aligned}\right. \tag{2.1}$$

We consider a rectangular basin $0<x<a$, $b_1<y<b_2$ which is bounded either on all sides by coasts or partly by coasts and partly by an infinitely deep ocean. The variables of (2.1) will be replaced by dimensionless variables according to

$$\left\{\begin{aligned} &u,\, v \rightarrow hcu,\, hcv; \\ &\zeta \rightarrow h\zeta; \\ &x,\, y \rightarrow a\pi^{-1}x,\, a\pi^{-1}y; \\ &t \rightarrow a\pi^{-1}c^{-1}t, \end{aligned}\right. \tag{2.2}$$

where $c = \sqrt{gh}$.

This has a.o. the effect that $gh=1$ in (2.1).

We want to study free motions where u, v and ζ are the real parts of quantities proportional to $\exp(i\omega t)$, where $\operatorname{Im}\omega \geqslant 0$. Denoting the proportionality factors by the same symbols, (2.1) passes into

$$\left\{\begin{aligned} &(i\omega+\lambda)u - \Omega v + \frac{\partial \zeta}{\partial x} = 0, \\ &(i\omega+\lambda)v + \Omega u + \frac{\partial \zeta}{\partial y} = 0, \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\omega\zeta = 0. \end{aligned}\right. \tag{2.3}$$

The sides $x=0$ and $x=\pi$ will always be coasts so that we have the boundary conditions

$$u=0 \qquad \text{for } x=0 \quad \text{and} \quad x=\pi. \tag{2.4}$$

The sides $y=b_j$ $(j=1, 2)$ may be a coast or an ocean boundary so that

$$\left\{\begin{aligned} &v=0 \qquad \text{for } y=b_j \qquad \text{if coast,} \\ &\zeta=0 \qquad \text{for } y=b_j \qquad \text{if ocean.} \end{aligned}\right. \qquad (j=1,\,2) \tag{2.5}$$

By means of a complex change of scale the equations (2.3) may be reduced to a system of the same type but without friction [1]). If we put

$$(2.6)\qquad \begin{cases} \omega' \stackrel{\text{def}}{=} \omega(1-i\lambda/\omega)^{\frac{1}{2}}, \\ \Omega' \stackrel{\text{def}}{=} \Omega(1-i\lambda/\omega)^{-\frac{1}{2}}, \\ \zeta' \stackrel{\text{def}}{=} \zeta(1-i\lambda/\omega)^{-\frac{1}{2}}, \end{cases}$$

a system of the form (2.3) is obtained with $u'=u$, $v'=v$ and $\lambda'=0$. If the eigenvalues of this system (i.e. the free frequencies) have been obtained in the form $\omega'=\varphi(\Omega')$ then ω is obtained by solving the equation

$$(2.7)\qquad \omega(1-i\lambda/\omega)^{\frac{1}{2}}=\varphi\{\Omega(1-i\lambda/\omega)^{-\frac{1}{2}}\}.$$

If $|\omega|>>\lambda$ we obtain the first order approximation

$$(2.8)\qquad \omega=\varphi(\Omega)+\tfrac{1}{2}i\lambda\{1+\Omega\varphi'(\Omega)/\varphi(\Omega)\}+O(\lambda^2/\omega^2).$$

Hence the periods remain unaltered to a first approximation, but the oscillations are damped with the damping factor

$$\exp\ -\tfrac{1}{2}\lambda\{1+\Omega\varphi'(\Omega)/\varphi(\Omega)\}t.$$

Similarly, if $\zeta'=\psi(\Omega', x, y)$ then

$$(2.9)\qquad \zeta=\psi(\Omega)-\tfrac{1}{2}i\lambda\omega^{-1}\{\psi(\Omega)-\Omega\psi'(\Omega)\}+O(\lambda^2/\omega^2).$$

Hence the amplitude of ζ is obtained to a first approximation from that of $\lambda=0$ by multiplication with the damping factor

$$\exp\ -\tfrac{1}{2}\lambda\omega^{-1}\{\psi(\Omega)-\Omega\psi'(\Omega)\}t.$$

If ω and λ are of comparable magnitude the discussion of the equation (2.7) becomes very difficult. This important case will be considered in the following paper by Lauwerier.

In view of (2.8) and (2.9) the remainder of this paper is entirely devoted to the discussion of the frictionless case.

From the equations (2.3) it follows easily that (cf. I 2.11, I 2.19, I 2.20)

$$(2.10)\qquad (\Delta-\varkappa^2)u=0 \qquad (\Delta-\varkappa^2)v=0 \qquad (\Delta-\varkappa^2)\zeta=0,$$

where (cf. I 2.13)

$$(2.11)\qquad \varkappa^2 \stackrel{\text{def}}{=} \Omega^2-\omega^2.$$

Moreover we notice the relations (cf. I 2.18)

$$(2.12)\qquad \begin{cases} \varkappa^2 u = -\, i\omega\,\dfrac{\partial\zeta}{\partial x}-\Omega\,\dfrac{\partial\zeta}{\partial y}, \\ \varkappa^2 v = \dfrac{\partial\zeta}{\partial x}-i\omega\,\dfrac{\partial\zeta}{\partial y}. \end{cases}$$

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[1]) Cf. Veltkamp [1].

3. *Infinite channel*

For convenience we reformulate the long known solutions for an infinite straight channel. This case is determined by the equations (2.3) with $\lambda=0$ and only the boundary conditions (2.4). We consider solutions (u, v, ζ) only which are (for almost every y) quadratically integrable (shortly $\in L^2$) over $0<x<\pi$ so that the theories of Fourier series and of Hilbert spaces can be applied. It can easily be verified that

$$u = \sum_{n=1}^{\infty} (n^2+\Omega^2) \sin nx(C_n^+ e^{-\nu_n y} + C_n^- e^{\nu_n y}), \tag{3.1}$$

where

$$\nu_n \stackrel{\text{def}}{=} (n^2+\varkappa^2)^{\frac{1}{2}}, \tag{3.2}$$

satisfies the first equation of (2.11) and the boundary conditions at $x=0$ and $x=\pi$. The expression (3.1) is also the most general solution. The at first sight arbitrary insertion of the factor $n^2+\Omega^2$ will prove to be convenient later on. The elementary solutions

$$u = \sin nx \, e^{\mp\nu_n y} \tag{3.3}$$

are often called "Poincaré waves". The corresponding expressions for v and ζ are

$$v = (\varepsilon n\nu_n \cos nx + i\omega\Omega \sin nx)e^{-\varepsilon\nu_n y}, \tag{3.4}$$

and

$$\zeta = (in\omega \cos nx + \varepsilon\Omega\nu_n \sin nx)e^{-\varepsilon\nu_n y}, \tag{3.5}$$

where ε denotes either the $+$ sign or the $-$ sign.

We note that for $u=0$ the first equation of (2.12) may have still a non-vanishing solution which is of the form $f(\Omega x - i\omega y)$. Substitution in the third equation of (2.10) shows that f is an exponential function. This leads to the following two elementary solutions

$$\begin{cases} u = 0, \\ v = \exp \varepsilon\{\Omega(x-\tfrac{1}{2}\pi) - i\omega y\}, \\ \zeta = \varepsilon \exp \varepsilon\{\Omega(x-\tfrac{1}{2}\pi) - i\omega y\}. \end{cases} \tag{3.6}$$

The solutions (3.6) are often called the "Kelvin waves".

Hence the expressions of v and ζ which correspond to (3.1) contain not only the components (3.4) and (3.5) of the Poincaré waves but also those of the Kelvin waves (3.6) viz.

$$\begin{cases} v = \sum_{n=1}^{\infty} C_n^+(n\nu_n \cos nx + i\omega\Omega \sin nx)e^{-\nu_n y} + \\ \qquad - \sum_{n=1}^{\infty} C_n^-(n\nu_n \cos nx - i\omega\Omega \sin nx)e^{\nu_n y} + \\ \qquad - C_0^+ \exp\{\Omega(x-\tfrac{1}{2}\pi) - i\omega y\} - C_0^- \exp\{-\Omega(x-\tfrac{1}{2}\pi) + i\omega y\}, \end{cases} \tag{3.7}$$

and

$$(3.8)\quad \begin{cases} \zeta = \sum_{n=1}^{\infty} C_n^+(in\omega \cos nx + \Omega\nu_n \sin nx)e^{-\nu_n y} + \\ \qquad\qquad + \sum_{n=1}^{\infty} C_n^-(in\omega \cos nx - \Omega\nu_n \sin nx)e^{\nu_n y} + \\ \quad - C_0^+ \exp\{\Omega(x-\frac{1}{2}\pi) - i\omega y\} + C_0^- \exp\{-\Omega(x-\frac{1}{2}\pi) + i\omega y\}. \end{cases}$$

The expressions (3.1) and (3.7) will only be considered as solutions of our problem if u and v belong to the class L^2. The same then is true for $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$, and a fortiori for ζ itself. The condition is equivalent with saying that the sequences $n^2 C_n^{\pm} \exp \mp \nu_n y$ belong for all y with $b_1 < y < b_2$ to the class l^2 of quadratically summable sequences, or also

$$(3.9)\qquad \sum n^4 |C_n^{\pm}|^2 \exp 2b|\mathrm{Re}\ \nu_n| < \infty,$$

where $b = \max(|b_1|, |b_2|)$. The series (3.1), (3.7) and (3.8) need not be convergent everywhere of course.

4. *Rectangular lake*

We consider the rectangular basin, $0 < x < \pi$, $-b < y < b$ in dimensionless coordinates, which is bounded on all sides by coasts. Hence the solution of the previous section must satisfy the further conditions

$$(4.1)\qquad v(x, -b) = v(x, b) = 0.$$

By taking sum and difference it follows from (3.7) that

$$(4.2)\quad \begin{cases} \sum_{n=1}^{\infty} A_n^+ \cos nx + i\omega\Omega \sum_{n=1}^{\infty} \frac{\mathrm{cth}\ \nu_n b}{n\nu_n} A_n^- \sin nx = a^+(x), \\ \sum_{n=1}^{\infty} A_n^- \cos nx + i\omega\Omega \sum_{n=1}^{\infty} \frac{\mathrm{th}\ \nu_n b}{n\nu_n} A_n^+ \sin nx = a^-(x), \end{cases}$$

where

$$(4.3)\quad \begin{cases} A_n^+ \overset{\mathrm{def}}{=} n\nu_n \,\mathrm{ch}\ \nu_n b\ (C_n^+ - C_n^-), \\ A_n^- \overset{\mathrm{def}}{=} n\nu_n \,\mathrm{sh}\ \nu_n b\ (C_n^+ + C_n^-), \end{cases}$$

and

$$(4.4)\quad \begin{cases} a^+(x) \overset{\mathrm{def}}{=} \cos \omega b (C_0^+ e^{\Omega(x-\frac{1}{2}\pi)} + C_0^- e^{-\Omega(x-\frac{1}{2}\pi)}), \\ a^-(x) \overset{\mathrm{def}}{=} i \sin \omega b (C_0^+ e^{\Omega(x-\frac{1}{2}\pi)} - C_0^- e^{-\Omega(x-\frac{1}{2}\pi)}). \end{cases}$$

In operator form the conditions (4.2) may be written in the form

$$(4.5)\quad \begin{cases} \varphi^+(x) + i\omega\Omega S^+ \varphi^-(x) = a^+(x), \\ \varphi^-(x) + i\omega\Omega S^- \varphi^+(x) = a^-(x), \end{cases}$$

where

$$(4.6)\qquad \varphi^{\varepsilon}(x) \overset{\mathrm{def}}{=} \sum_{n=1}^{\infty} A_n^{\varepsilon} \cos nx,$$

if ε takes the $+$ sign and the $-$ sign respectively, and where

$$(4.7)\qquad \begin{cases} S^+\,\varphi(x) \overset{\text{def}}{=} \dfrac{2}{\pi}\displaystyle\int_0^\pi \left\{\sum_{n=1}^\infty \frac{\operatorname{cth}\nu_n b}{n\nu_n}\sin nx\cos nt\right\}\varphi(t)dt, \\ S^-\,\varphi(x) \overset{\text{def}}{=} \dfrac{2}{\pi}\displaystyle\int_0^\pi \left\{\sum_{n=1}^\infty \frac{\operatorname{th}\nu_n b}{n\nu_n}\sin nx\cos nt\right\}\varphi(t)dt. \end{cases}$$

The operators S^+ and S^- may be considered as operators in Hilbert space. It is obvious that they have finite norms.

We have now to find the non-trivial solutions of (4.5), subject these to the conditions

$$(4.8)\qquad \int_0^\pi \varphi^\varepsilon(x)dx = 0,$$

and finally to solve these for the C_0^ε. This is only possible for special values of ω which are the eigenvalues of the problem.

For $\Omega = 0$ the solution is trivial.

In the first place we may have either $\cos\omega\, b = 0$ or $\sin\omega\, b = 0$. Next we may have from (4.3) that either $\operatorname{ch}\nu_n b = 0$ or $\operatorname{sh}\nu_n b = 0$. All zeros arising from these equations are of the form

$$(4.9)\qquad \omega = (m^2 + \tfrac{1}{2}n^2\pi^2 b^{-2})^{\frac{1}{2}},$$

where m and n are integers.

The solutions belonging to an ω of the form (4.9) need not be unique, as more than one pair (m, n) may give the same ω, which is then a multiple eigenvalue. We shall not write down the solutions explicitly.

We return now to the general case. Eliminating in (4.5) either φ^- or φ^+ we obtain

$$(4.10)\qquad \begin{cases} (I + \omega^2\Omega^2 S^+S^-)\varphi^+(x) = h^+(x), \\ (I + \omega^2\Omega^2 S^-S^+)\varphi^-(x) = h^-(x), \end{cases}$$

where

$$(4.11)\qquad h^\varepsilon(x) \overset{\text{def}}{=} a^\varepsilon(x) - i\omega\Omega S^\varepsilon a^{-\varepsilon}(x).$$

The operators $I + \omega^2\Omega^2 S^\varepsilon S^{-\varepsilon}$ have, at least for sufficiently small $\omega\Omega$, an inverse V^ε, obtainable for small $\omega\Omega$ by a Neumann development. Then from (4.10) the following solution is obtained

$$(4.12)\qquad \varphi^\varepsilon(x) = V^\varepsilon h^\varepsilon(x).$$

5. *Approximation for small* Ω

The transcendental equations obtained from (4.8) and (4.12) are extremely complicated. For ω occurs *a*) implicitly in the factors ν_n, *b*) implicitly in the operators S^ε and V^ε, *c*) explicitly through the factors $\cos\omega b$ and $\sin\omega b$ of the $a^\varepsilon(x)$. The dependence on Ω is even more and that on b almost equally complicated.

However, by a development in powers of Ω^2 approximations to ω for small Ω can easily be obtained. Here we shall study the influence of Ω upon the first eigenvalue, $\omega_0 = \frac{1}{2}\pi b^{-1}$ for $\Omega = 0$, only. We may always assume that $b \geqq \frac{1}{2}\pi$. If $b = \frac{1}{2}\pi$, the case of a square sea, ω_0 is a double eigenvalue. We shall consider the non-degenerate case $b > \frac{1}{2}\pi$ first.

It will be found convenient to replace the expressions (4.4) by

$$(5.1)\qquad \begin{cases} a^+(x) = \cos \omega b\{B^+ \text{ch } \Omega(x-\frac{1}{2}\pi) + B^- \text{sh } \Omega(x-\frac{1}{2}\pi)\}, \\ a^-(x) = i \sin \omega b\{B^+ \text{sh } \Omega(x-\frac{1}{2}\pi) + B^- \text{ch } \Omega(x-\frac{1}{2}\pi)\}. \end{cases}$$

Then it follows from (4.11) and (4.12) that

$$(5.2)\qquad \begin{cases} V^+h^+(x) = B^+\{\cos \omega b V^+ \text{ch } \Omega(x-\frac{1}{2}\pi) + \\ \qquad\qquad + \omega\Omega \sin \omega b V^+S^+ \text{sh } \Omega(x-\frac{1}{2}\pi)\} + \\ \qquad + B^-\{\cos \omega b V^+ \text{sh } \Omega(x-\frac{1}{2}\pi) + \\ \qquad\qquad + \omega\Omega \sin \omega b V^+S^+ \text{ch } \Omega(x-\frac{1}{2}\pi)\}, \\ V^-h^-(x) = iB^+\{\sin \omega b V^- \text{sh } \Omega(x-\frac{1}{2}\pi) + \\ \qquad\qquad - \omega\Omega \cos \omega b V^-S^- \text{ ch } \Omega(x-\frac{1}{2}\pi)\} + \\ \qquad + iB^-\{\sin \omega b V^- \text{ ch } \Omega(x-\frac{1}{2}\pi) + \\ \qquad\qquad - \omega\Omega \cos \omega b V^-S^- \text{ sh } \Omega(x-\frac{1}{2}\pi)\}. \end{cases}$$

If we put for $\varepsilon = +$ and $\varepsilon = -$

$$(5.3)\qquad \begin{cases} V_1^\varepsilon = \dfrac{1}{\pi}\displaystyle\int_0^\pi V^\varepsilon \text{ ch } \Omega(x-\tfrac{1}{2}\pi)dx, \\ V_2^\varepsilon = \dfrac{1}{\pi}\displaystyle\int_0^\pi V^\varepsilon \text{ sh } \Omega(x-\tfrac{1}{2}\pi)dx, \end{cases}$$

and

$$(5.4)\qquad \begin{cases} W_1^\varepsilon = \dfrac{1}{\pi}\displaystyle\int_0^\pi V^\varepsilon S^\varepsilon \text{ ch } \Omega(x-\tfrac{1}{2}\pi)dx, \\ W_2^\varepsilon = \dfrac{1}{\pi}\displaystyle\int_0^\pi V^\varepsilon S^\varepsilon \text{ sh } \Omega(x-\tfrac{1}{2}\pi)dx, \end{cases}$$

then from (4.8) and (5.2) the following eigenvalue equation can be obtained

$$(5.5)\qquad \begin{vmatrix} \cos \omega b V_1^+ + \omega\Omega \sin \omega b W_2^+ & \sin \omega b V_2^- - \omega\Omega \cos \omega b W_1^- \\ \cos \omega b V_2^+ + \omega\Omega \sin \omega b W_1^+ & \sin \omega b V_1^- - \omega\Omega \cos \omega b W_2^- \end{vmatrix} = 0.$$

It can easily be derived that in general for small Ω

$$(5.6)\qquad \begin{cases} V_1^\varepsilon = \dfrac{\text{sh } \frac{1}{2}\Omega\pi}{\frac{1}{2}\Omega\pi} + O(\Omega^4), \\ V_2^\varepsilon = O(\Omega^3), \end{cases}$$

$$(5.7)\qquad \begin{cases} W_1^\varepsilon = O(\Omega^2), \\ W_2^\varepsilon = \dfrac{1}{\pi}\displaystyle\int_0^\pi S^\varepsilon \text{ sh } \Omega(x-\tfrac{1}{2}\pi)dx - \omega^2\Omega^2 \dfrac{1}{\pi}\int_0^\pi S^\varepsilon S^{-\varepsilon} S^\varepsilon \text{ sh } \Omega(x-\tfrac{1}{2}\pi)dx. + O(\Omega^5) \end{cases}$$

If further we put

$$\omega = \omega_0 - \delta, \tag{5.8}$$

where $\delta = O(\Omega^2)$, then

$$\begin{cases} \cos \omega b = b\delta + O(\Omega^6), \\ \sin \omega b = 1 + O(\Omega^4). \end{cases} \tag{5.9}$$

Substitution of (5.6) and (5.7) in (5.5) gives

$$\operatorname{cotg} \omega b = \frac{\frac{1}{2}\Omega\pi}{\operatorname{sh} \frac{1}{2}\Omega\pi} \{\omega\Omega M_1 + \omega^3\Omega^3 M_2 + O(\Omega^6)\}, \tag{5.10}$$

where

$$\begin{cases} M_1 \overset{\text{def}}{=} -\frac{1}{\pi} \int\limits_0^\pi S^+ \operatorname{sh} \Omega(x - \frac{1}{2}\pi) dx, \\ M_2 \overset{\text{def}}{=} \frac{1}{\pi} \int\limits_0^\pi S^+ S^- S^+ \operatorname{sh} \Omega(x - \frac{1}{2}\pi) dx. \end{cases} \tag{5.11}$$

Hence by means of (5.9)

$$\delta = b^{-1} \frac{\frac{1}{2}\Omega\pi}{\operatorname{sh} \frac{1}{2}\Omega\pi} \{\omega\Omega M_1 + \omega^3\Omega^3 M_2 + O(\Omega^6)\}. \tag{5.12}$$

From (4.7) and (5.11) it follows that

$$\begin{cases} M_1 = -\frac{4}{\pi^2} \sum_1 \frac{\operatorname{cth} \nu_n b}{n^2 \nu_n} \int\limits_0^\pi \cos nx \operatorname{sh} \Omega(x - \frac{1}{2}\pi) dx \\ \quad = \frac{8\Omega}{\pi^2} \operatorname{ch} \frac{1}{2}\Omega\pi \sum_1 \frac{\operatorname{cth} \nu_n b}{n^2(n^2 + \Omega^2)\nu_n}, \end{cases} \tag{5.13}$$

where $\sum_1$ denotes a summation over odd indices $n = 1, 3, 5, \ldots$ only. Hence the following first approximation is easily obtained

$$\delta = \frac{8\omega_0\Omega^2}{b\pi^2} \sum_1 \frac{\operatorname{cth} \nu_n{}^0 b}{n^4 \nu_n{}^0} + O(\Omega^4), \tag{5.14}$$

with

$$\nu_n{}^0 \overset{\text{def}}{=} (n^2 - \omega_0{}^2)^{\frac{1}{2}}.$$

Hence we have obtained

$$\omega = \frac{\pi}{2b} \left\{1 - \frac{8\Omega^2}{b\pi^2} \sum_1 \frac{\operatorname{cth} \nu_n{}^0 b}{n^4 \nu_n{}^0} + O(\Omega^4)\right\}. \tag{5.15}$$

By way of a numerical illustration we consider the case $b = \pi$ so that $\omega_0 = \frac{1}{2}$. Then (5.15) gives

$$\omega = 0.5 - 0.151\ \Omega^2 + \ldots. \tag{5.16}$$

For the second approximation we obtain

$$(5.17)\quad \delta = \frac{\frac{1}{2}\Omega\pi}{\operatorname{th}\frac{1}{2}\Omega\pi}\frac{8\omega_1\Omega^2}{b\pi^2}\sum\nolimits_1 \frac{\operatorname{cth}\nu_n' b}{n^2(n^2+\Omega^2)\nu_n'} - \frac{\omega_0{}^3\Omega^4}{b\pi}\int_0^\pi S^+S^-S^+(\tfrac{1}{2}\pi - x)\,dx + O(\Omega^6),$$

where

$$\omega_1 \stackrel{\text{def}}{=} \omega_0 - \frac{8\omega_0\Omega^2}{b\pi^2}\sum\nolimits_1 \frac{\operatorname{cth}\nu_n{}^0 b}{n^4\nu_n{}^0}, \text{ and } \nu_n' \stackrel{\text{def}}{=} (n^2+\Omega^2-\omega_1{}^2)^{\frac{1}{2}}.$$

Next we consider the case of the square sea $b=\frac{1}{2}\pi$. We shall restrict the discussion to that of the influence of Ω upon the lowest eigenvalue, $\omega_0=1$ for $\Omega=0$, which is degenerate. Therefore an expansion of the following type

$$(5.18)\qquad \omega = \omega_0 \pm p_1\Omega + p_2\Omega^2 + O(\Omega^3)$$

may be expected.

The discussion of this case follows closely that of the general case $b>\frac{1}{2}\pi$ with some minor variations. The main difference is that ν_1 vanishes for $\Omega=0$. In the notation of (5.8) we have

$$(5.19)\qquad \frac{\operatorname{cth}\nu_1 b}{\nu_1} = \frac{1}{\pi\delta} + \left(\frac{1}{2\pi} + \frac{\pi}{6} - \frac{\Omega^2}{2\pi\delta^2}\right) + O(\Omega).$$

The expressions (5.6), (5.7) and (5.9) remain true with slightly different order terms. The eigenvalue equation (5.10) or (5.12) becomes here

$$(5.20)\qquad \tfrac{1}{2}\pi\delta = \omega\Omega M_1 + \omega^3\Omega^3 M_2 + O(\Omega^3),$$

where M_1 and M_2 are still given by (5.11). We note that here $M_1=O(1)$ and $M_2=O(\Omega^{-1})$.

From (5.13) we obtain at once

$$(5.21)\qquad M_1 = \frac{8\Omega}{\pi^3\delta} + \frac{8\Omega}{\pi^2}\left(\frac{1}{2\pi} + \frac{\pi}{6} - \frac{\Omega^2}{2\pi\delta^2}\right) + \frac{8\Omega}{\pi^2}\sum\nolimits_1^* \frac{\operatorname{cth}\nu_n{}^0 b}{n^4\nu_n{}^0},$$

where $\sum_1{}^*$ denotes summation over the odd indices $n=3, 5, 7, \ldots$.

From (5.20) and (5.21) a first order approximation can be derived at once. We have

$$\tfrac{1}{2}\pi\delta = \frac{8\Omega^2}{\pi^3\delta} + O(\Omega^2),$$

so that in accordance with Lamb's formula (1.1)

$$(5.22)\qquad \delta = \pm\frac{4\Omega}{\pi^2} + O(\Omega^2),$$

which gives $p_1 = 4/\pi^2$.

For the second order approximation we need the limit of ΩM_2 for $\Omega \to 0$.

If we define

$$(5.23)\qquad \begin{cases} A \stackrel{\text{def}}{=} \lim\limits_{\Omega\to 0} \pi\Omega^{-1}\left(M_1 - \dfrac{8\Omega}{\pi^3\delta}\right), \\ B \stackrel{\text{def}}{=} \lim\limits_{\Omega\to 0} \pi\Omega M_2, \end{cases}$$

there follows from (5.18) and (5.20) that

$$p_1\Omega+p_2\Omega^2 = \frac{1}{\pi^2}\Omega(1+p_1\Omega)\left(\frac{16}{\pi^2(p_1+p_2\Omega)} - 2\Omega A\right) - \frac{2}{\pi^2}\Omega^2 B + O(\Omega^3).$$

Comparing the coefficients of Ω^2 on both sides we obtain without difficulty

$$p_2 = \frac{8}{\pi^4} - \frac{A}{\pi^2} - \frac{B}{\pi^2}. \tag{5.24}$$

There remains the determination of the constants A and B. The value of A follows from (5.21) viz.

$$A = \frac{4}{\pi^2} + \frac{4}{3} - \frac{\pi^2}{4} + \frac{8}{\pi}\sum\nolimits_1^* \frac{\operatorname{cth}\nu_n{}^0 b}{n^4\nu_n{}^0}. \tag{5.25}$$

In order to determine the value of B we note that (5.11) gives

$$M_2 = \frac{\Omega}{\pi}\int_0^\pi S^+S^-S^+(x-\tfrac{1}{2}\pi)\,dx + O(\Omega). \tag{5.26}$$

From (4.7) and (5.19) it follows that

$$S^+(x-\tfrac{1}{2}\pi) = -\frac{4}{\pi^2\delta}\sin x + O(1),$$

and next

$$S^-S^+(x-\tfrac{1}{2}\pi) = \frac{16}{\pi^3\delta}\sum\nolimits_2 \frac{\operatorname{th}\nu_n{}^0 b}{n(n^2-1)\nu_n{}^0}\sin nx + O(1),$$

where $\sum_2$ denotes summation over the even indices $n=2, 4, 6, \ldots$. Finally

$$S^+S^-S^+(x-\tfrac{1}{2}\pi) = \frac{64}{\pi^5\delta^2}\left(\sum\nolimits_2 \frac{\operatorname{th}\nu_n{}^0 b}{(n^2-1)^2\nu_n{}^0}\right)\sin x + O(\Omega^{-1}),$$

so that

$$B = \frac{8}{\pi}\sum\nolimits_2 \frac{\operatorname{th}\nu_n{}^0 b}{(n^2-1)^2\nu_n{}^0}. \tag{5.27}$$

Hence we have obtained the following result [1])

$$\left\{\begin{aligned}\omega = 1 \pm \frac{4}{\pi^2}\Omega + \frac{8}{\pi^3}\Big\{\frac{\pi^3}{32} - \frac{\pi}{6} + \frac{1}{2\pi} + \\ - \sum\nolimits_1^* \frac{\operatorname{cth}\tfrac{1}{2}\nu_n{}^0\pi}{n^4\nu_n{}^0} - \sum\nolimits_2 \frac{\operatorname{th}\tfrac{1}{2}\nu_n{}^0\pi}{(n^2-1)^2\nu_n{}^0}\Big\}\,\Omega^2 + O(\Omega^3),\end{aligned}\right. \tag{5.28}$$

or numerically

$$\omega = 1 \pm 0.405\,\Omega + 0.138\,\Omega^2 + \ldots \tag{5.29}$$

In the original dimensional variables we have of course

$$\omega = \frac{c\pi}{a}\left\{1 \pm 0.405\frac{a\Omega}{c\pi} + 0.138\left(\frac{a\Omega}{c\pi}\right)^2 + \ldots\right\}. \tag{5.30}$$

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[1]) In VAN DANTZIG's result the second series on the right-hand side was missing.

6. *Rectangular bay*

Under the same assumptions as in section 2 we consider a rectangular bay $0<x<\pi$, $0<y<b$ bounded by coasts $x=0$, $x=\pi$, $y=0$ and by an infinitely deep ocean at $y=b$. The solution of section 3 must now satisfy the conditions

$$v(x,\,0)=0 \quad \text{and} \quad \zeta(x,\,b)=0. \tag{6.1}$$

It follows from (3.7) and (3.8) that

$$\left\{\begin{aligned} &\sum_{n=1}^{\infty} C_n^+(n\nu_n \cos nx + i\omega\Omega \sin nx) + \\ &\qquad - \sum_{n=1}^{\infty} C_n^-(n\nu_n \cos nx - i\omega\Omega \sin nx) = a(x), \\ &\sum_{n=1}^{\infty} \mathrm{e}^{-\nu_n b} C_n^+(in\omega \cos nx + \Omega\nu_n \sin nx) + \\ &\qquad + \sum_{n=1}^{\infty} \mathrm{e}^{\nu_n b} C_n^-(in\omega \cos nx - \Omega\nu_n \sin nx) = b(x), \end{aligned}\right. \tag{6.2}$$

where

$$\left\{\begin{aligned} a(x) &\overset{\text{def}}{=} C_0^+ \exp \Omega(x-\tfrac{1}{2}\pi) + C_0^- \exp -\Omega(x-\tfrac{1}{2}\pi), \\ b(x) &\overset{\text{def}}{=} C_0^+ \exp\{\Omega(x-\tfrac{1}{2}\pi) - i\omega b\} - C_0^- \exp\{-\Omega(x-\tfrac{1}{2}\pi) + i\omega b\}. \end{aligned}\right. \tag{6.3}$$

If we define for $n \geqq 1$

$$\left\{\begin{aligned} A_n &\overset{\text{def}}{=} n\nu_n(C_n^+ - C_n^-), \\ B_n &\overset{\text{def}}{=} i\omega n(\mathrm{e}^{-\nu_n b} C_n^+ + \mathrm{e}^{\nu_n b} C_n^-), \end{aligned}\right. \tag{6.4}$$

and

$$\left\{\begin{aligned} A \cos \omega b &\overset{\text{def}}{=} C_0^+ + C_0^-, \\ B \cos \omega b &\overset{\text{def}}{=} \mathrm{e}^{-i\omega b} C_0^+ - \mathrm{e}^{i\omega b} C_0^-, \end{aligned}\right. \tag{6.5}$$

with the inverse relations

$$\left\{\begin{aligned} 2i\omega\, n\nu_n \operatorname{ch} \nu_n b\, C_n^+ &= \nu_n B_n + i\omega\, \mathrm{e}^{\nu_n b} A_n, \\ 2i\omega\, n\nu_n \operatorname{ch} \nu_n b\, C_n^- &= \nu_n B_n - i\omega\, \mathrm{e}^{-\nu_n b} A_n, \end{aligned}\right. \tag{6.6}$$

and

$$\left\{\begin{aligned} 2C_0^+ &= \mathrm{e}^{i\omega b} A + B, \\ 2C_0^- &= \mathrm{e}^{-i\omega b} A - B, \end{aligned}\right. \tag{6.7}$$

we obtain by substitution in the relations (6.2) the following conditions

$$\left\{\begin{aligned} &\sum_{n=1}^{\infty} A_n \cos nx + i\omega\Omega \sum_{n=1}^{\infty} \frac{\operatorname{th} \nu_n b}{n\nu_n} A_n \sin nx + \\ &\qquad + \Omega \sum_{n=1}^{\infty} \frac{1}{n \operatorname{ch} \nu_n b} B_n \sin nx = a(x), \\ &\sum_{n=1}^{\infty} B_n \cos nx + i\frac{\Omega}{\omega} \sum_{n=1}^{\infty} \frac{\nu_n \operatorname{th} \nu_n b}{n} B_n \sin nx + \\ &\qquad + \Omega \sum_{n=1}^{\infty} \frac{1}{n \operatorname{ch} \nu_n b} A_n \sin nx = b(x), \end{aligned}\right. \tag{6.8}$$

where

$$(6.9)\qquad \begin{cases} a(x) = A\cos\omega b \operatorname{ch} \Omega(x-\tfrac{1}{2}\pi) + (iA\sin\omega b + B)\operatorname{sh}\Omega(x-\tfrac{1}{2}\pi),\\ b(x) = (A - iB\sin\omega b)\operatorname{sh}\Omega(x-\tfrac{1}{2}\pi) + B\cos\omega b\operatorname{ch}\Omega(x-\tfrac{1}{2}\pi). \end{cases}$$

In operator form the conditions (6.8) may be written in the form (cf. (4.5))

$$(6.10)\qquad \begin{cases} \varphi(x) + i\omega\Omega S\varphi(x) + \Omega U\psi(x) = a(x),\\ \psi(x) + i\omega^{-1}\Omega T\psi(x) + \Omega U\varphi(x) = b(x), \end{cases}$$

where

$$(6.11)\qquad \begin{cases} \varphi(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} A_n \cos nx,\\ \psi(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} B_n \cos nx, \end{cases}$$

$$(6.12)\qquad \begin{cases} S f(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\operatorname{th} \nu_n b}{n\nu_n} \sin nx \, \frac{2}{\pi}\int_0^\pi \cos nt\, f(t)\, dt,\\ T f(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\nu_n \operatorname{th} \nu_n b}{n} \sin nx \, \frac{2}{\pi}\int_0^\pi \cos nt\, f(t)\, dt, \end{cases}$$

and

$$(6.13)\qquad U f(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n \operatorname{ch} \nu_n b} \sin nx \, \frac{2}{\pi}\int_0^\pi \cos nt\, f(t)\, dt.$$

The main difference in comparison with the case of a lake is the following. The operators S^ε of (4.7) have finite norms. The same is true for the operators S and U of (6.12) and (6.13) but not for T since

$$\sum_{n=1}^{\infty} |n^{-1}\nu_n \operatorname{th} \nu_n b|^2 = \infty.$$

Nevertheless T is bounded since from (6.12) it follows that

$$\|Tf(x)\|^2 \leqq C\|f(x)\|^2,$$

where C is a constant. We have tacitly assumed that $\operatorname{ch} \nu_n b \neq 0$ for all n. It follows that for all sufficiently small Ω/ω the operator $I + i\omega^{-1}\Omega T$ has an inverse. Defining

$$(6.14)\qquad \begin{cases} Q \stackrel{\text{def}}{=} (I + i\omega\Omega S)^{-1},\\ R \stackrel{\text{def}}{=} (I + i\omega^{-1}\Omega T)^{-1}, \end{cases}$$

it follows from (6.10) that

$$(6.15)\qquad \begin{cases} (I - \Omega^2 QURU)\,\varphi(x) = Q\,a(x) - \Omega QUR\,b(x),\\ (I - \Omega^2 RUQU)\,\psi(x) = R\,b(x) - \Omega RUQ\,a(x). \end{cases}$$

These equations are of the form (4.10) and their formal solution is obtained as in (4.12). The eigenvalue equation follows as in section 4 by elimination of A and B from the supplementary conditions

$$\frac{1}{\pi}\int_0^\pi \varphi(x)\,dx = 0 \quad \text{and} \quad \frac{1}{\pi}\int_0^\pi \psi(x)\,dx = 0. \tag{6.16}$$

If $\Omega=0$ the solution of the eigenvalue problem is almost trivial. It follows from (6.10) and (6.9) that for $\Omega=0$

$$\begin{cases} \varphi(x) = A\cos\omega b, \\ \psi(x) = B\cos\omega b, \end{cases} \tag{6.17}$$

so that eigenvalues are obtained from $\cos\omega b=0$.

In view of (6.6) eigenvalues are also obtained from $\mathrm{ch}\,\nu_n b=0$. Therefore the eigenvalues are all given by (compare (4.9))

$$\omega=\{m^2+(n+\tfrac{1}{2})^2\pi^2/b^2\}^{\frac{1}{2}}, \qquad m, n=0, 1, 2, \ldots. \tag{6.18}$$

7. *Approximation for small Ω and large b*

Because of the complicated nature of the eigenvalue problem of the preceding section we shall discuss in some more detail only approximate solutions under the conditions

1. that the length b of the bay is large,
2. that Ω is small.

We shall study in particular the influence of Ω upon the first eigenvalue, $\omega_0=\frac{1}{2}\pi b^{-1}$ for $\Omega=0$.

We suppose that b is so large that $\omega_0<1$ so that ν_1 is real. If we define

$$\beta \overset{\text{def}}{=} \mathrm{e}^{-\nu_1 b}, \tag{7.1}$$

then all $\exp -\nu_n b$ are $O(\beta^2)$ for $n\geqq 2$.

If the operators S^*, T^* and U^* are defined by

$$\begin{cases} S^* f(x) \overset{\text{def}}{=} \sum\limits_{n=1}^{\infty} \dfrac{\sin nx}{n\nu_n{}^0}\,\dfrac{2}{\pi}\displaystyle\int_0^\pi \cos nt\, f(t)\,dt, \\ T^* f(x) \overset{\text{def}}{=} \sum\limits_{n=1}^{\infty} \dfrac{\nu_n{}^0 \sin nx}{n}\,\dfrac{2}{\pi}\displaystyle\int_0^\pi \cos nt\, f(t)\,dt, \\ U^* f(x) \overset{\text{def}}{=} \dfrac{4\beta\sin x}{\pi}\displaystyle\int_0^\pi \cos t\, f(t)\,dt, \end{cases} \tag{7.2}$$

where

$$\nu_n{}^0 \overset{\text{def}}{=} (n^2-\omega_0{}^2)^{\frac{1}{2}},$$

it follows from (6.12) and (6.13) that, symbolically,

$$\begin{cases} S = S^* + O(\Omega^2)+O(\beta^2), \\ T = T^* + O(\Omega^2)+O(\beta^2), \\ U = U^* + O(\Omega^2)+O(\beta^2). \end{cases} \tag{7.3}$$

Then the relations (6.15) reduce to

$$(7.4)\qquad \begin{cases} \varphi(x) = (I - i\omega\Omega S^*)a(x) - \Omega U^* b(x) + O(\Omega^3) + O(\beta^2), \\ \psi(x) = (I - i\omega^{-1}\Omega T^*)b(x) - \Omega U^* a(x) + O(\Omega^3) + O(\beta^2). \end{cases}$$

Substitution of (6.9) gives with neglect of the order terms the approximate relations

$$(7.5)\qquad \begin{cases} \varphi(x) = A \cos \omega b + i\Omega(A \sin \omega b - iB)(x - \tfrac{1}{2}\pi) + \\ \qquad + \omega\Omega^2(\mathrm{A} \sin \omega b - iB)S^*(x - \tfrac{1}{2}\pi) - \Omega^2(A - iB \sin \omega b)U^*(x - \tfrac{1}{2}\pi), \\ \psi(x) = B \cos \omega b - i\Omega(B \sin \omega b + iA)(x - \tfrac{1}{2}\pi) + \\ \qquad - \omega^{-1}\Omega^2(B \sin \omega b + iA)T^*(x - \tfrac{1}{2}\pi) - \Omega^2(B + iA \sin \omega b)U^*(x - \tfrac{1}{2}\pi). \end{cases}$$

If we put

$$(7.6)\qquad \begin{cases} S_0 \overset{\text{def}}{=} \dfrac{1}{\pi} \displaystyle\int_0^\pi S^*(x - \tfrac{1}{2}\pi)\, dx, \\ T_0 \overset{\text{def}}{=} \dfrac{1}{\pi} \displaystyle\int_0^\pi T^*(x - \tfrac{1}{2}\pi)\, dx, \\ U_0 \overset{\text{def}}{=} \dfrac{1}{\pi} \displaystyle\int_0^\pi U^*(x - \tfrac{1}{2}\pi)\, dx, \end{cases}$$

the conditions (6.16) give, again without the usual order terms,

$$(7.7)\qquad \begin{cases} A(\cos \omega b + \omega\Omega^2 S_0 \sin \omega b - \Omega^2 U_0) - iB(\omega\Omega^2 S_0 - \Omega^2 U_0) \sin \omega b = 0, \\ -iA(\omega^{-1}\Omega^2 T_0 + \Omega^2 U_0 \sin \omega b) + \\ \qquad + B(\cos \omega b - \omega^{-1}\Omega^2 T_0 \sin \omega b - \Omega^2 U_0) = 0. \end{cases}$$

From this the approximate eigenvalue equation easily follows. After some elementary simplifications we obtain

$$(7.8)\qquad \Omega^{-2} \cos \omega b = \sin \omega b(-\omega S_0 + \omega^{-1} T_0) + 2U_0 + O(\Omega^2) + O(\beta^2).$$

A simple calculation shows that

$$(7.9)\qquad \begin{cases} S_0 = -\dfrac{8}{\pi^2} \sum_1 \dfrac{1}{n^4 \nu_n^0}, \\ T_0 = -\dfrac{8}{\pi^2} \sum_1 \dfrac{\nu_n^0}{n^4}, \\ U_0 = -\dfrac{16\beta}{\pi^2}. \end{cases}$$

Hence it follows from (7.8) that

$$(7.10)\qquad \omega = \omega_0 + \Omega^2 b^{-1}(\omega_0 S_0 - \omega_0^{-1} T_0 - 2U_0) + O(\Omega^4) + O(\beta^2)$$

or finally

$$(7.11)\qquad \omega = \omega_0 + \Omega^2 \left\{ \frac{16}{\pi^3} \sum_1 \frac{n^2 - 2\omega_0^2}{n^4 \nu_n^0} + \frac{32\beta}{\pi^2 b} \right\} + O(\Omega^4) + O(\beta^2).$$

As a numerical illustration we consider the model of the North Sea $0<x<\pi$, $0<y<2\pi$. Then $\omega_0=1/4$ and according to (7.11)

$$\omega = 0.25+0.504\,\Omega^2+\dots \tag{7.12}$$

However, the value $\Omega=0.6$ of the North Sea is too large to permit its substitution in this expression.

In the original dimensional coordinates of (2.1) where b is approximately twice the value of a we have in view of (2.2)

$$\omega = \omega_0+0.504\,\frac{a\Omega^2}{c\pi}+\dots, \tag{7.13}$$

where

$$\omega_0 = \frac{c\pi}{2b}. \tag{7.14}$$

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