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The North Sea Problem. V

Free Motions of a Rotating Rectangular Bay

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MATHEMATICS

THE NORTH SEA PROBLEM. V
FREE MOTIONS OF A ROTATING RECTANGULAR BAY [1])

BY

H. A. LAUWERIER

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1. *Introduction*

This paper is a sequel to N.S.P. IV. The free motions of a rotating rectangular bay are studied with special reference to the North Sea. In particular it will be studied what happens if there is a small coefficient of friction λ and if the coefficient of Coriolis Ω is not small. The notation is that of the previous paper. The bay is determined by $0<x<\pi$ and $0<y<b$ in dimensionless Cartesian coordinates. The side $y=b$ represents the open end at the ocean.

In view of the considerable theoretical difficulties of the general case we have restricted ourselves to the discussion of the lowest eigenvalues with the assumption that b is large. Luckily enough the value of $b=2\pi$ for the North Sea case may be considered as very large. Further we have considered those free motions which are integrable along the ocean boundary and to which consequently a finite "energy integral" may be associated. It appears that for $\lambda \neq 0$ only the corner (π, b) is a singularity.

The pilot model of an infinitely wide bay which is discussed in section 3 suggests for the North Sea model the existence of a double infinity of triples of eigenvalues. Each triple consists of a negative real eigenvalue (pure damping) and two conjugate complex eigenvalues with a negative real part (damped oscillation).

In section 8 it is shown that for large b the lowest eigenvalues can be obtained from an equation which involves two unknown functions $A(x)$ and $B(x)$. These functions can be determined from a set of orthogonality relations and they are discussed in the sections 6 and 7.

It is shown in section 9 that there exists a small negative real eigenvalue which for the North Sea model is about -0.074.

If, however, $\lambda=0$ there are only purely imaginary eigenvalues $i\omega$. Then there are no eigenvalues in the interval $0<|\omega|<\Omega$. The lowest eigenvalues of the North Sea model are very near to $\pm i\Omega$.

2. *The mathematical problem*

The free motions

$$u=\bar{u}(x,y)e^{pt}, \quad v=\bar{v}(x,y)e^{pt}, \quad \zeta=\bar{\zeta}(x,y)e^{pt} \tag{2.1}$$

[1]) Report TW 65 of the Mathematical Centre, Amsterdam.

of the rectangular bay $0<x<\pi$, $0<y<b$ are determined by the equations

$$(2.2)\qquad \begin{cases} (p+\lambda)\bar{u}-\Omega\bar{v}+\zeta_x=0 \\ (p+\lambda)\bar{v}+\Omega\bar{u}+\zeta_y=0 \\ \bar{u}_x+\bar{v}_y+p\zeta=0\,. \end{cases}$$

and the boundary conditions

$$(2.3)\qquad \begin{cases} \bar{u}=0 \quad \text{for} \quad x=0 \quad \text{and} \quad x=\pi, \\ \bar{v}=0 \quad \text{for} \quad y=0\,, \\ \zeta=0 \quad \text{for} \quad y=b. \end{cases}$$

We note that the variables and constants are dimensionless and that they are identical with those of the preceding paper (cf. IV 2.2). In the numerical application to the North Sea case we shall take

$$(2.4)\qquad b=2\pi \qquad \Omega=0.6 \qquad \lambda=0.12.$$

Then the scale of the dimensionless time variable is approximately 1.4 hour.

Only those free motions will be considered for which the "energy integral" is finite, i.e.

$$(2.5)\qquad \int_0^\pi\int_0^b (\bar{u}\bar{u}^*+\bar{v}\bar{v}^*+\zeta\zeta^*)\,dx\,dy<\infty.$$

From the equations (2.2) there follows

$$(2.6)\qquad \mathrm{Re}\{(p+\lambda)(\bar{u}\bar{u}^*+\bar{v}\bar{v}^*)+\bar{u}^*\zeta_x+\bar{v}^*\zeta_y\}=0.$$

Integrating this result over the rectangle $0<x<\pi$, $0<y<b$ we obtain by also using the boundary conditions (2.3)

$$(2.7)\qquad \mathrm{Re}\{(p+\lambda)\int_0^\pi\int_0^b(\bar{u}\bar{u}^*+\bar{v}\bar{v}^*)\,dx\,dy-\int_0^\pi\int_0^b(\bar{u}_x^*+\bar{v}_y^*)\zeta\,dx\,dy\}=0.$$

In view of the continuity equation we obtain

$$(2.8)\qquad \{\mathrm{Re}(p+\lambda)\}\int_0^\pi\int_0^b(\bar{u}\bar{u}^*+\bar{v}\bar{v}^*)\,dx\,dy+\{\mathrm{Re}\,p\}\int_0^\pi\int_0^b\zeta\zeta^*\,dx\,dy=0.$$

Hence the admissible eigenvalues are situated in the strip

$$(2.9)\qquad -\lambda\leqq\mathrm{Re}\,p\leqq 0.$$

From the equations (2.2) we may derive the following results (cf. I 2.11 sqq.)

$$(2.10)\qquad (\Delta-\varkappa^2)\bar{u}=0, \qquad (\Delta-\varkappa^2)\bar{v}=0, \qquad (\Delta-\varkappa^2)\zeta=0,$$

where

$$(2.11)\qquad \varkappa^2\overset{\text{def}}{=}p(p+\lambda)+\Omega^2p/(p+\lambda),$$

and (cf. I 2.18)

$$\begin{cases} p^{-1}\varkappa^2\bar{u} = -(\bar{\zeta}_x + \operatorname{tg}\gamma\bar{\zeta}_y) \\ p^{-1}\varkappa^2\bar{v} = -(\bar{\zeta}_y - \operatorname{tg}\gamma\bar{\zeta}_x) \end{cases}, \tag{2.12}$$

where

$$\operatorname{tg}\gamma \overset{\text{def}}{=} \Omega(p+\lambda)^{-1}, \qquad 0 \leqq \operatorname{Re}\gamma \leqq \tfrac{1}{2}\pi. \tag{2.13}$$

3. *The free motions of an infinitely wide bay*

As a pilot model of the rectangular bay $0<x<\pi$, $0<y<b$ we shall consider the strip $-\infty<x<\infty$, $0<y<b$ where $y=0$ is a coast and $y=b$ an ocean boundary. We shall consider only those free motions which do not depend on the variable x. Then the Helmholtz equations (2.10) reduce to simple ordinary differential equations e.g.

$$\left(\frac{d^2}{dy^2} - \varkappa^2\right)\bar{\zeta} = 0, \tag{3.1}$$

with the boundary conditions

$$\frac{\partial\bar{\zeta}}{\partial y} = 0 \qquad \text{for } y = 0, \tag{3.2}$$

and

$$\bar{\zeta} = 0 \qquad \text{for } y = b. \tag{3.3}$$

The free motions have the elementary form

$$\bar{\zeta} = \cos\,(2n+1)\frac{\pi y}{2b}, \tag{3.4}$$

where $n = 0, 1, 2, \ldots$.

The eigenvalues follow easily from (3.1) viz.

$$p(p+\lambda) + \Omega^2\frac{p}{p+\lambda} = -(2n+1)^2\frac{\pi^2}{4b^2}. \tag{3.5}$$

For each value of n there are consequently three eigenvalues, one real and negative and two conjugate complex with a negative real part.

For $\Omega = 0.6$ and $\lambda = 0.12$ the first few eigenvalues are

$n=0$	-0.0173	-0.111	$\pm\,0.649\,\mathrm{i}$
$n=1$	-0.0730	-0.084	$\pm\,0.958\,\mathrm{i}$
$n=2$	-0.0975	-0.071	$\pm\,1.385\,\mathrm{i}$.

We note that the lowest real eigenvalue is very small, which corresponds to a slow aperiodic damping.

For $\Omega = 0$ the equation (3.5) reduces to one of the second degree

$$p(p+\lambda) = -(n+\tfrac{1}{2})^2\frac{\pi^2}{b^2} \tag{3.6}$$

with for each integer n a conjugate pair of complex eigenvalues. Also for $\lambda = 0$ the degree is lowered. Then we have

$$p^2 = -\left\{\Omega^2 + (n+\tfrac{1}{2})^2\frac{\pi^2}{b^2}\right\} \tag{3.7}$$

with two purely imaginary eigenvalues for each n.

If we write $p = i\omega$ we may write (3.7) in the form

$$\omega = \left\{\Omega^2 + (n+\tfrac{1}{2})^2 \frac{\pi^2}{b^2}\right\}^{\frac{1}{2}}. \tag{3.8}$$

This means that the presence of the Coriolis force Ω tends to increase the eigenvalues ω when considered as a function of Ω. Moreover there are no eigenvalues in the range $(-\Omega, \Omega)$.

It is conjectured that the same features hold for a rectangular bay. The following tentative statements will be made

1. The eigenvalues of a rectangular bay appear in groups of three, one real and negative, two conjugate complex with a negative real part.
2. For $\lambda = 0$ there are no eigenvalues in the interval $(-\Omega, \Omega)$ other than the trivial one at the origin.
3. The "lowest" eigenvalue of the North Sea is real and negative.

4. *The free motions for* $\Omega = 0$

a. If $\lambda = \Omega = 0$ the problem reduces to

$$(\Delta - p^2)\,\zeta = 0 \tag{4.1}$$

with

$$\frac{\partial \zeta}{\partial x} = 0 \qquad \text{for } x = 0 \text{ and } x = \pi, \tag{4.2}$$

$$\frac{\partial \zeta}{\partial y} = 0 \qquad \text{for } y = 0, \tag{4.3}$$

$$\zeta = 0 \qquad \text{for } y = b. \tag{4.4}$$

The free motions have the elementary form (cf. 3.4)

$$\zeta = \cos mx \cos\,(2n+1)\,\frac{\pi y}{2b}, \tag{4.5}$$

where m and n are arbitrary integers. The eigenvalues are (cf. IV 6.18).

$$p = \pm\, i\{m^2 + (n+\tfrac{1}{2})^2 \pi^2/b^2\}^{\frac{1}{2}}. \tag{4.6}$$

The lowest eigenvalue, for which $m = n = 0$, is

$$p = \pm \frac{i\pi}{2b}. \tag{4.7}$$

The corresponding mode is

$$\zeta = \cos \frac{\pi y}{2b} \exp \pm \frac{i\pi t}{2b}. \tag{4.8}$$

With the North Sea data of $b = 2\pi$ and a time scale of 1.4 hour the main free period is accordingly about 35 hours.

b. If $\Omega = 0$ and $\lambda \neq 0$ the equation (4.1) must be replaced by

$$(\Delta - q^2)\,\zeta = 0, \tag{4.9}$$

where

$$q^2 \overset{\text{def}}{=} p^2 + \lambda p, \tag{4.10}$$

but the boundary conditions (4.2), (4.3) and (4.4) remain the same. The free motions (4.5) also remain the same but the eigenvalues are now

$$p = -\tfrac{1}{2}\lambda \pm \mathrm{i}\left\{m^2 + \frac{(2n+1)^2\pi^2}{4b^2} - \tfrac{1}{4}\lambda^2\right\}^{\frac{1}{2}}. \tag{4.11}$$

It follows from this expression that for $\lambda < \pi/b$ all modes are oscillatory with a small damping term. For the North Sea this inequality certainly holds. In the theoretical case $\lambda \geqq \pi/b$ the first few modes would be aperiodically damped motions.

The lowest eigenvalue is here

$$p = -\tfrac{1}{2}\lambda \pm \tfrac{1}{2}\mathrm{i}\left(\frac{\pi^2}{b^2} - \lambda^2\right)^{\frac{1}{2}}. \tag{4.12}$$

5. *The free motions in the general case*

The free motions in an infinite channel bounded by $x=0$ and $x=\pi$ are (cf. IV section 3).

a. two Kelvin waves

$$\begin{cases} \bar{u} = 0 \\ \bar{v} = p^{\frac{1}{2}} \exp \pm \{s(x-\tfrac{1}{2}\pi) - qy\} \\ \bar{\zeta} = \pm(p+\lambda)^{\frac{1}{2}} \exp \pm \{s(x-\tfrac{1}{2}\pi) - qy\}, \end{cases} \tag{5.1}$$

b. an infinity of Poincaré waves

$$\begin{cases} \bar{u} = (n^2+s^2) \sin nx \exp \pm \nu_n y \\ \bar{v} = n\nu_n (\cos nx \pm \alpha_n \sin nx) \exp \pm \nu_n y \\ \bar{\zeta} = (p+\lambda)n (\cos nx \pm \beta_n \sin nx) \exp \pm \nu_n y \end{cases} \tag{5.2}$$

for $n = 1, 2, 3, \ldots,$
and where

$$s^2 \overset{\text{def}}{=} \Omega^2 p(p+\lambda)^{-1}, \tag{5.3}$$

$$\nu_n{}^2 \overset{\text{def}}{=} n^2 + \varkappa^2, \tag{5.4}$$

$$\alpha_n \overset{\text{def}}{=} \frac{p\Omega}{n\nu_n}, \tag{5.5}$$

$$\beta_n \overset{\text{def}}{=} \frac{\Omega\nu_n}{(p+\lambda)n}. \tag{5.6}$$

The free motions in the rectangular bay $0<x<\pi$, $0<y<b$ can be represented as a linear combination of the Kelvin and Poincaré waves in the following way

$$\begin{cases} \bar{u} = \sum_{n=1}^{\infty} n^{-1}\nu_n{}^{-1}(n^2+s^2)C_n \sin nx\, \mathrm{e}^{-\nu_n y} + \\ \qquad - \sum_{n=1}^{\infty} n^{-1}\nu_n{}^{-1}(n^2+s^2)D_n \sin nx\, \mathrm{e}^{-\nu_n(b-y)}, \end{cases} \tag{5.7}$$

$$\begin{cases} \bar{v} = p^{\frac{1}{2}}(p+\lambda)^{-\frac{1}{2}} C\, \mathrm{e}^{s(x-\frac{1}{2}\pi)-qy} - p^{\frac{1}{2}}(p+\lambda)^{-\frac{1}{2}} D\, \mathrm{e}^{-s(x-\frac{1}{2}\pi)-q(b-y)} + \\ + \sum_{n=1}^{\infty} C_n(\cos nx + \alpha_n \sin nx)\, \mathrm{e}^{-\nu_n y} + \sum_{n=1}^{\infty} D_n(\cos nx - \alpha_n \sin nx)\, \mathrm{e}^{-\nu_n(b-y)}, \end{cases} \tag{5.8}$$

$$(5.9)\quad \begin{cases} \zeta = C\, e^{s(x-\frac{1}{2}\pi)-qy} + D\, e^{-s(x-\frac{1}{2}\pi)-q(b-y)} + \\ + \sum_{n=1}^{\infty} (p+\lambda)\, \nu_n^{-1}\, C_n(\cos nx + \beta_n \sin nx)\, e^{-\nu_n y} + \\ - \sum_{n=1}^{\infty} (p+\lambda)\, \nu_n^{-1}\, D_n(\cos nx - \beta_n \sin nx)\, e^{-\nu_n(b-y)}. \end{cases}$$

We have still to satisfy the conditions at $y=0$ and $y=b$. The coast condition at $y=0$ gives

$$(5.10)\quad \sum_{n=1}^{\infty} C_n(\cos nx + \alpha_n \sin nx) = \varphi(x),$$

where

$$(5.11)\quad \begin{cases} \varphi(x) = -\, p^{\frac{1}{2}}(p+\lambda)^{-\frac{1}{2}}\, C\, e^{s(x-\frac{1}{2}\pi)} + p^{\frac{1}{2}}(p+\lambda)^{-\frac{1}{2}}\, D\, e^{-s(x-\frac{1}{2}\pi)-qb} + \\ \qquad - \sum_{n=1}^{\infty} D_n\, e^{-\nu_n b}\, (\cos nx - \alpha_n \sin nx). \end{cases}$$

The ocean condition at $y=b$ gives

$$(5.12)\quad \sum_{n=1}^{\infty} (p+\lambda)\, \nu_n^{-1}\, D_n(\cos nx - \beta_n \sin nx) = \psi(x),$$

where

$$(5.13)\quad \begin{cases} \psi(x) = C\, e^{s(x-\frac{1}{2}\pi)-qb} + D\, e^{-s(x-\frac{1}{2}\pi)} + \\ \qquad + \sum_{n=1}^{\infty} (p+\lambda)\, \nu_n^{-1}\, C_n\, e^{-\nu_n b}\, (\cos nx + \beta_n \sin nx). \end{cases}$$

The coast condition (5.10) will be discussed in section 6. It turns out that there exists an orthogonal function $A(x)$ satisfying

$$(5.14)\quad \frac{1}{\pi}\int_0^{\pi} A(x)\, (\cos nx + \alpha_n \sin nx)\, dx = 0$$

for $n=1, 2, 3, \ldots$ and

$$(5.15)\quad \frac{1}{\pi}\int_0^{\pi} A(x)\, dx = 1.$$

Then it follows from (5.10) that

$$(5.16)\quad \int_0^{\pi} A(x)\, \varphi(x)\, dx = 0.$$

Further for $n \to \infty$ the coefficients C_n are of the following order

$$(5.17)\quad C_n = O(n^{-2}).$$

The ocean condition will be discussed in section 7. It will appear that there exists an integrable orthogonal function $B(x)$ with

$$(5.18)\quad \frac{1}{\pi}\int_0^{\pi} B(x)\, (\cos nx - \beta_n \sin nx)\, dx = 0$$

for $n=1, 2, 3, \ldots$ and

(5.19) $$\frac{1}{\pi}\int_0^\pi B(x)\,dx = 1,$$

except in the case $\lambda = 0$ and $|\omega| < \Omega$.

Further it follows that

(5.20) $$\int_0^\pi B(x)\psi(x)dx = 0,$$

and that for $n \to \infty$

(5.21) $$D_n = \frac{(-1)^{n-1}D'}{n^{1-2\gamma/\pi}} + O(n^{-1-2\gamma/\pi}),$$

where D' is a constant.

This result is also valid for $\lambda = 0$ and $|\omega| > \Omega$ when $\mathrm{Re}\,\gamma = 0$. However, for $\lambda = 0$ and $|\omega| < \Omega$ with $\mathrm{Re}\,\gamma = \frac{1}{2}\pi$ we have by putting

(5.22) $$p = \mathrm{i}\Omega \,\mathrm{th}\, \theta,$$

where θ is real

(5.23) $$D_n = (-1)^{n-1}D'\,n^{-2\mathrm{i}\theta/\pi} + D''\,n^{2\mathrm{i}\theta/\pi} + O(n^{-2}),$$

where D' and D'' are constants.

It follows from (5.7), (5.8), (5.9), (5.17) and (5.21) that for $0 < \mathrm{Re}\,\gamma < \frac{1}{2}\pi$ the elevation ζ is uniformly bounded in the rectangle $0 < x < \pi$, $0 < y < b$ but that the stream $\bar{u}, \bar{v}$ has a singularity at the ocean corner (π, b) of the following kind

(5.24) $$(\bar{u}\bar{u}^* + \bar{v}\bar{v}^*)^{\frac{1}{2}} = O(r^{-2\gamma/\pi}),$$

where r denotes the distance $\{(\pi - x)^2 + (b - y)^2\}^{\frac{1}{2}}$.

If $\mathrm{Re}\,\gamma = 0$ the right-hand side of (5.24) must be replaced by $O(\ln r)$, if $\mathrm{Re}\,\gamma = \frac{1}{2}\pi$ by $O(r^{-1})$.

It follows that the energy integral (2.5) is finite except when $\mathrm{Re}\,\gamma = \frac{1}{2}\pi$, i.e. when $\lambda = 0$ and $-\Omega < \omega < \Omega$.

6. *The coast condition*

In this section an orthogonal function $A(x)$ will be determined which satisfies (5.14) and (5.15). Let us assume the existence of the following expansion

(6.1) $$1 = \sum_{n=1}^{\infty} a_n(\sin nx + \alpha_n \cos nx), \qquad 0 < x < \pi,$$

then it can easily be shown that

(6.2) $$A(x) = \sum_{n=1}^{\infty} a_n \sin nx = 1 - \sum_{n=1}^{\infty} \alpha_n a_n \cos nx.$$

The proof follows from observing that for a_n the following two expressions hold

$$(6.3)\qquad a_n = \frac{2}{\pi}\int_0^\pi A(x)\sin nx\,dx$$

and

$$(6.4)\qquad -\alpha_n a_n = \frac{2}{\pi}\int_0^\pi A(x)\cos nx\,dx.$$

Elimination of a_n gives (5.14). The relation (5.15) follows at once from (6.2).

From (6.1) and (6.2) an integral equation for $A(x)$ can be derived. By using (6.3) we have

$$(6.5)\qquad 1 = A(x) + \sum_{n=1}^{\infty} \alpha_n \frac{2}{\pi}\int_0^\pi \cos nx \sin n\xi\, A(\xi)\,d\xi,$$

which may be written in the form

$$(6.6)\qquad A(x) + \frac{1}{\pi}\int_0^\pi K(x,\xi)\,A(\xi)\,d\xi = 1,$$

where

$$(6.7)\qquad K(x,\xi) = \sum_{n=1}^{\infty} \alpha_n \{\sin n(x+\xi) - \sin n(x-\xi)\},$$

The kernel function is apparently continuous and uniformly bounded in the square $0 \leqq x,\ \xi \leqq \pi$. Hence (6.6) represents an ordinary Fredholm equation. The existence of $A(x)$ is eventually secured by (6.6).

It follows from (6.6), (5.4) and (5.5) that $A(x)$ can be considered as a function of the two parameters $p\Omega$ and $\varkappa^2$. We shall now derive an expansion of $A(x)$ for small values of $p\Omega$.

We put

$$(6.8)\qquad A(x) = 1 + p\Omega A^{(1)}(x) + p^2\Omega^2 A^{(2)}(x) + \ldots$$

Then it follows from (6.5) that

$$A^{(1)}(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n\nu_n}\cos nx \int_0^\pi \sin n\xi\,d\xi,$$

or

$$(6.9)\qquad A^{(1)}(x) = -\frac{4}{\pi}\sum\nolimits_1 \frac{\cos nx}{n^2\nu_n},$$

where $\sum_1$ denotes a summation over the odd indices 1, 3, 5, … .

For $A^{(2)}(x)$ it can be derived that

$$(6.10)\qquad A^{(2)}(x) = \frac{4}{\pi}\sum\nolimits_2 \left\{\sum\nolimits_1 \frac{\Gamma_{mn}}{m^2\nu_m}\right\}\frac{\cos nx}{n\nu_n},$$

where

$$(6.11)\qquad \Gamma_{mn} \stackrel{\text{def}}{=} \frac{2}{\pi}\int_0^\pi \cos mx \sin nx\,dx,$$

and where $\sum_2$ denotes a summation over the even indices 2, 4, 6, … .
The explicit expression of Γ_{mn} is

$$\Gamma_{mn} = \frac{4}{\pi} \frac{n}{n^2 - m^2}, \tag{6.12}$$

if m and n have the same parity, otherwise $\Gamma_{mn} = 0$.

7. *The ocean condition* [1])

We repeat the ocean condition (5.12)

$$\sum_{n=1}^{\infty} (p+\lambda)\, \nu_n^{-1} D_n (\cos nx - \beta_n \sin nx) = \psi(x), \qquad 0 < x < \pi, \tag{7.1}$$

where $\psi(x)$ may be considered as a given function.

By changing x to $\pi - x$ this can be written in the standard form (see LAUWERIER [1] section 5).

$$\sum_{n=1}^{\infty} b_n (\sin nx + \theta_n \cos nx) = f(x), \tag{7.2}$$

where

$$\theta_n = \operatorname{cotg} \gamma \left(1 - \frac{\varkappa^2}{n^2}\right)^{-\frac{1}{2}} \tag{7.3}$$

with $0 \leqq \operatorname{Re} \gamma \leqq \frac{1}{2}\pi$.

The properties of the expansion (7.2) are as follows.

Unless $\operatorname{Re} \gamma = \frac{1}{2}\pi$ there exists an orthogonal function $l_0(x)$ such that

$$\int_0^{\pi} l_0(x) (\sin nx + \theta_n \cos nx)\, dx = 0 \tag{7.4}$$

for all n.

If

$$\int_0^{\pi} l_0(x) f(x)\, dx = 0 \tag{7.5}$$

the expansion (7.2) is possible and the asymptotic behaviour of the coefficients b_n is as follows

$$b_n = \frac{\beta}{n^{2-2\gamma/\pi}} + O(n^{-2-2\gamma/\pi}), \tag{7.6}$$

provided $\operatorname{Re} \gamma > 0$.

For the condition (7.1) this means that $B(x)$ exists for $\operatorname{Re} \gamma \neq \frac{1}{2}\pi$ and that for $n \to \infty$.

$$D_n = \frac{(-1)^{n-1} D'}{n^{1-2\gamma/\pi}} + O(n^{-1-2\gamma/\pi}), \tag{7.7}$$

where D' is a constant.

If $\operatorname{Re} \gamma = \frac{1}{2}\pi$ we may put

$$\gamma = \tfrac{1}{2}\pi - i\theta, \tag{7.8}$$

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[1]) Cf. also the corresponding section in N.S.P. II.

where θ is real. In that case there is no integrable orthogonal function and the expansion (7.1) would give

(7.9) $$D_n = (-1)^{n-1} D' n^{-2i\theta/\pi} + D'' n^{2i\theta/\pi} + O(n^{-2}),$$

where D' and D'' are constants.

We note that the relation (7.8) is equivalent to

(7.10) $$p = -\lambda + i\Omega \text{ th } \theta.$$

It follows from the discussion at the end of section 5 that free motions with a characteristic value determined by (7.10) have non-integrable singularities in both ocean corners. Therefore they do not belong to the class of free motions which are considered here.

The remaining part of this section will be devoted to the determination of the orthogonal function $B(x)$ in some special cases. As in the previous section $B(x)$ is also determined by the expansion

(7.11) $$1 = \sum_{n=1}^{\infty} b_n (\sin nx - \beta_n \cos nx), \qquad 0 < x < \pi.$$

We have

(7.12) $$B(x) = \sum_{n=1}^{\infty} b_n \sin nx = 1 + \sum_{n=1}^{\infty} \beta_n b_n \cos nx.$$

In view of (5.6) $B(x)$ can be considered as a function of the independent parameters $\text{tg}\,\gamma$ and $\varkappa^2$. We shall first assume that $\text{tg}\,\gamma$ is small so that $B(x)$ can be developed in the following power series

(7.13) $$B(x) = 1 + \text{tg}\,\gamma B^{(1)}(x) + \text{tg}^2\,\gamma B^{(2)}(x) + \dots$$

In a completely analogous way as in the case of $A(x)$ it can be derived that

(7.14) $$B^{(1)}(x) = \frac{4}{\pi} \sum\nolimits_1 \frac{\nu_n}{n^2} \cos nx$$

and

(7.15) $$B^{(2)}(x) = -\frac{4}{\pi} \sum\nolimits_2 \left\{ \sum\nolimits_1 \frac{\nu_m}{m^2} \Gamma_{mn} \right\} \frac{\nu_n}{n} \cos nx.$$

Next it will be assumed that $\varkappa^2$ is small and that $B(x)$ can be developed as follows

(7.16) $$B(x) = B_0(x) + \varkappa^2 B_1(x) + 0(\varkappa^4).$$

It follows from (5.18) and (5.19) that

(7.17) $$\frac{1}{\pi} \int_0^{\pi} B_0(x) (\cos nx - \text{tg}\,\gamma \sin nx)\, dx = \delta_{on}.$$

Hence we have in view of II (4.11)

(7.18) $$B_0(x) = \cos\gamma\, (\text{tg}\,\tfrac{1}{2}x)^{-2\gamma/\pi}.$$

The following term can be derived from (7.11) and (7.12).

By putting

(7.19) $$\beta_n = \operatorname{tg} \gamma \left\{ 1 + \frac{\varkappa^2}{2n^2} + O(\varkappa^4) \right\}$$

and

(7.20) $$b_n = \operatorname{cotg} \gamma \, e_n \left(\frac{2\gamma}{\pi} \right) + \varkappa^2 b_n^{(1)} + O(\varkappa^4)$$

it follows that

(7.21) $$\tfrac{1}{2} \cos \gamma \sum_{n=1}^{\infty} n^{-2} e_n \left(\frac{2\gamma}{\pi} \right) \cos nx = \sum_{n=1}^{\infty} b_n^{(1)} \sin (nx - \gamma)$$

and

(7.22) $$B_1(x) = \sum_{n=1}^{\infty} b_n^{(1)} \sin nx.$$

This problem may be solved in the following way. The set of biorthogonal functions $k_m(x)$ with respect to $\sin (nx - \gamma)$, $m, n = 1, 2, 3, \ldots$, is determined by (see LAUWERIER [1] section 3).

(7.23) $$\frac{2}{\pi} \int_0^{\pi} k_m(x) \cos (nx + \gamma) \, dx = \delta_{mn}.$$

The functions $k_m(x)$ can be derived from the expansions

(7.24) $$\cos mx = \sum_{n=1}^{\infty} b_n \sin (nx - \gamma)$$

by means of

(7.25) $$k_m(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

The explicit form of $k_m(x)$ is

(7.26) $$k_m(x) = - (\operatorname{tg} \tfrac{1}{2} x)^{1 - 2\gamma/\pi} \sum_{j=1}^{m} e_{m-j} \left(1 - \frac{2\gamma}{\pi} \right) \sin jx.$$

These functions are continuous and zero at $x = 0$ and $x = \pi$.

Hence it follows easily that

(7.27) $$B_1(x) = - \tfrac{1}{2} \cos \gamma \sum_{n=1}^{\infty} n^{-2} e_n \left(\frac{2\gamma}{\pi} \right) k_n(x).$$

Hence we have obtained the remarkable fact that $B_1(x)$ is continuous and zero at $x = 0$ and $x = \pi$.

8. *The eigenvalue equation*

In view of the considerable complications of the general case we shall restrict ourselves to the discussion of the lowest eigenvalues for which

$$\operatorname{Re} \varkappa^2 >> -1.$$

Moreover it will be assumed that b is so large that the terms with the exponential factors $\exp - \nu_n b$ occurring in (5.11) and (5.13) can be neglected. Then the coast condition (5.10) and the ocean condition (5.12) may be approximated by

$$(8.1)\quad \begin{cases} \sum_{n=1}^{\infty} C_n (\cos nx + \alpha_n \sin nx) = -p^{\frac{1}{2}}(p+\lambda)^{-\frac{1}{2}}\, C\, e^{s(x-\frac{1}{2}\pi)} + \\ \qquad + p^{\frac{1}{2}}(p+\lambda)^{-\frac{1}{2}}\, D\, e^{-s(x-\frac{1}{2}\pi)-qb}, \end{cases}$$

and

$$(8.2)\quad \sum_{n-1}^{\infty} (p+\lambda)\, \nu_n^{-1} D_n (\cos nx - \beta_n \sin nx) = C\, e^{s(x-\frac{1}{2}\pi)-qb} + D\, e^{-s(x-\frac{1}{2}\pi)}.$$

From (5.16) and (5.20) it follows that

$$(8.3)\quad C \int_0^{\pi} e^{s(x-\frac{1}{2}\pi)} A(x)\, dx - e^{-qb} D \int_0^{\pi} e^{-s(x-\frac{1}{2}\pi)} A(x)\, dx = 0,$$

and

$$(8.4)\quad e^{-qb} C \int_0^{\pi} e^{s(x-\frac{1}{2}\pi)} B(x)\, dx + D \int_0^{\pi} e^{-s(x-\frac{1}{2}\pi)} B(x)\, dx = 0.$$

From (8.3) and (8.4) the following approximate eigenvalue equation follows at once for the lowest eigenvalues

$$(8.5)\quad 2b\sqrt{p^2+\lambda p} = \pi i - \ln \frac{A_1}{A_2} + \ln \frac{B_1}{B_2},$$

where

$$(8.6)\quad A_1 \overset{\text{def}}{=} \frac{1}{\pi} \int_0^{\pi} e^{s(x-\frac{1}{2}\pi)} A(x)\, dx,$$

$$(8.7)\quad A_2 \overset{\text{def}}{=} \frac{1}{\pi} \int_0^{\pi} e^{-s(x-\frac{1}{2}\pi)} A(x)\, dx,$$

$$(8.8)\quad B_1 \overset{\text{def}}{=} \frac{1}{\pi} \int_0^{\pi} e^{s(x-\frac{1}{2}\pi)} B(x)\, dx,$$

$$(8.9)\quad B_2 \overset{\text{def}}{=} \frac{1}{\pi} \int_0^{\pi} e^{-s(x-\frac{1}{2}\pi)} B(x)\, dx.$$

It is interesting to see what becomes of (8.5) for $\Omega = 0$. Then it follows from (6.8) and (7.13) that $A(x) \equiv B(x) \equiv 1$ so that (8.5) is equivalent to (3.6) with $n = 0$ or (4.11) with $m = n = 0$.

We shall now suppose that $\lambda = 0$. Then we may put $p = i\omega$ with $\omega > \Omega$. It follows from (5.14) and (5.15) that

$$(8.10)\quad A(\pi - x) = A^*(x)$$

and from (5.18) and (5.19) that

$$(8.11)\quad B(\pi - x) = B^*(x).$$

Hence we must have $A_2 = A_1^*$ and $B_2 = B_1^*$. In that case the eigenvalue equation (8.5) becomes

$$(8.12)\quad b\omega = \tfrac{1}{2}\pi - \arg A_1 + \arg B_1.$$

This relation must be compatible with $\omega > \Omega$ whatever (large) value of b is chosen. In view of the discussions of the sections 6 and 7 this implies that

$$\text{(8.13)} \qquad \arg B_1 \to \infty \qquad \text{for } \omega \to \Omega + 0.$$

In section 10 this case will be considered in detail. Here the equation (8.12) will be studied for small values of Ω.

It follows from (8.6) with (6.8) that

$$\text{(8.14)} \qquad A_1 = 1 + \frac{8i\omega\Omega^2}{\pi^2} \sum_1 \frac{1}{n^4\sqrt{n^2-\omega^2}} + O(\Omega^4)$$

since A_1 is an even function of Ω. Hence

$$\text{(8.15)} \qquad \arg A_1 = \frac{8\omega\Omega^2}{\pi^2} \sum_1 \frac{1}{n^4\sqrt{n^2-\omega^2}} + O(\Omega^4).$$

In a similar way it follows from (8.8) and (7.13) that

$$\text{(8.16)} \qquad B_1 = 1 + \frac{8i\Omega^2}{\pi^2\omega} \sum_1 \frac{\sqrt{n^2-\omega^2}}{n^4} + O(\Omega^4),$$

and

$$\text{(8.17)} \qquad \arg B_1 = \frac{8\Omega^2}{\pi^2\omega} \sum_1 \frac{\sqrt{n^2-\omega^2}}{n^4} + O(\Omega^4).$$

Substitution of (8.15) and (8.17) in (8.12) gives

$$\text{(8.18)} \qquad b\omega = \tfrac{1}{2}\pi + \frac{8\Omega^2}{\pi^2\omega} \sum_1 \frac{n^2-2\omega^2}{n^4\sqrt{n^2-\omega^2}} + O(\Omega^4).$$

If this is applied to the rectangular model of the North Sea where $b = 2\pi$ there follows that (cf. IV 7.12).

$$\text{(8.19)} \qquad \omega = \tfrac{1}{4}\{1 + 2.02\,\Omega^2 + O(\Omega^4)\}.$$

Hence the force of Coriolis has the tendency to increase the lowest eigenvalue. However, the North Sea value of $\Omega = 0.6$ is certainly too large to justify the use of (8.19) since the outcome must exceed the value of Ω.

9. *The eigenvalue equation for real negative p*

We shall show that the eigenvalue equation (8.5) has a solution for a real negative value of p. If p is real we may take $\arg q = \arg s = \tfrac{1}{2}\pi$. Then $A(x)$ and $B(x)$ are real and $A_2 = A_1^*$ and $B_2 = B_1^*$. From (6.8), (6.9) and (8.6) it follows that

$$\text{(9.1)} \qquad A_1 = \frac{\operatorname{sh}\tfrac{1}{2}s\pi}{\tfrac{1}{2}s\pi}\left\{1 + \frac{4p\Omega}{\pi}\sum_1 \frac{s^2 \operatorname{cth}\tfrac{1}{2}s\pi}{n^2(n^2+s^2)\nu_n}\right\} + O(p^2\Omega^2).$$

From (7.16), (7.18) and (8.8) it follows that

$$\text{(9.2)} \qquad B_1 = \frac{\cos\gamma}{\pi}\int_0^\pi e^{s(x-\frac{1}{2}\pi)}\,(\operatorname{tg}\tfrac{1}{2}x)^{-2\gamma/\pi}\,dx + O(\varkappa^2).$$

If we write $q = i\varrho$ and $s = i\sigma$ so that ϱ and σ are positive and real the eigenvalue equation (8.5) can be approximated by

$$(9.3)\qquad b\varrho = \tfrac{1}{2}\pi - \frac{4p\Omega}{\pi} \sum\nolimits_1 \frac{\sigma^2 \operatorname{cotg} \frac{1}{2}\sigma\pi}{n^2(n^2-\sigma^2)\nu_n} + \arg \int_0^\pi e^{i\sigma(x-\frac{1}{2}\pi)} (\operatorname{tg} \tfrac{1}{2}x)^{-2\gamma/\pi}\, dx.$$

This equation may be reduced to a little rougher approximation by noting that for large γ

$$(9.4)\qquad \arg \int_0^\pi e^{i\sigma(x-\frac{1}{2}\pi)} (\operatorname{tg} \tfrac{1}{2}x)^{-2\gamma/\pi}\, dx \approx \arg e^{-\frac{1}{2}\sigma\pi i} = -\tfrac{1}{2}\sigma\pi.$$

Further we note that the contribution of the second term on the right hand side of (9.3) is rather small. Therefore we have approximately

$$(9.5)\qquad b\varrho = \tfrac{1}{2}\pi(1-\sigma).$$

For the North Sea case with $b = 2\pi$, $\lambda = 0.12$ and $\Omega = 0.6$ the equation (9.5) gives the real and negative root $p = -0.0745$.

We note that with this value of p

$$\varrho = 0.058 \qquad \sigma = 0.767$$
$$\operatorname{tg} \gamma = 13.2 \qquad 2\gamma/\pi = 0.95.$$

The more accurate equation (9.3) gives $p = -0.0744$.

10. *The case $\lambda = 0$ and $\omega \to \Omega + 0$*

For $\lambda = 0$ the eigenvalue equation (8.5) takes the form (8.12) or

$$(10.1)\qquad b\omega = \tfrac{1}{2}\pi - \arg A_1 + \arg B_1.$$

For $\omega \to \Omega + 0$ the orthogonal function $A(x)$ has a limit $A'(x)$ which is determined by

$$(10.2)\qquad \frac{1}{\pi} \int_0^\pi A'(x) \left(\cos nx + \frac{i\Omega^2}{n^2} \sin nx\right) dx = 0$$

and

$$(10.3)\qquad \frac{1}{\pi} \int_0^\pi A'(x)\, dx = 0.$$

Then also $\arg A_1$ has the limit

$$(10.4)\qquad \arg \frac{1}{\pi} \int_0^\pi e^{\Omega(x-\frac{1}{2}\pi)} A'(x)\, dx.$$

It follows from (6.8), (6.9) and (6.10) that

$$(10.5)\qquad \lim_{\omega\to\Omega} A_1 = \frac{\operatorname{sh} \frac{1}{2}\Omega\pi}{\frac{1}{2}\Omega\pi} \left\{1 + \frac{4i\Omega^2}{\pi} \sum\nolimits_1 \frac{\Omega^2 \operatorname{cth} \frac{1}{2}\Omega\pi}{n^3(n^2+\Omega^2)}\right\} + O(\Omega^6)$$

or

$$(10.6)\qquad \lim_{\omega\to\Omega} \arg A_1 = \frac{8\Omega^3}{\pi^2} \sum\nolimits_1 \frac{1}{n^5} + O(\Omega^6).$$

However, for $\omega \to \Omega + 0$ the orthogonal function $B(x)$ has no limit.

We shall write

$$\omega = \Omega \coth \tfrac{1}{2}\pi\theta \tag{10.7}$$

so that

$$\gamma = -\tfrac{1}{2}\pi\theta \mathrm{i} \tag{10.8}$$

and

$$\varkappa^2 = -\Omega^2/\mathrm{sh}^2\tfrac{1}{2}\pi\theta. \tag{10.9}$$

It follows from (7.16) and (7.18) that for $\theta \to \infty$

$$B(x) \approx \mathrm{ch}\tfrac{1}{2}\pi\theta\,(\mathrm{tg}\,\tfrac{1}{2}x)^{\theta\mathrm{i}}, \tag{10.10}$$

so that B_1 can be approximated by the expression

$$\beta(\theta) \stackrel{\mathrm{def}}{=} \pi^{-1}\,\mathrm{ch}\,\tfrac{1}{2}\pi\theta \int_0^{\pi} \mathrm{e}^{\Omega(x-\frac{1}{2}\pi)}\,(\mathrm{tg}\,\tfrac{1}{2}x)^{\mathrm{i}\theta}\,dx. \tag{10.11}$$

We shall then derive an expression for arg $\beta(\theta)$. By making the substitutions

$$\mathrm{tg}\,\tfrac{1}{2}x = t \qquad x = -\mathrm{i}\ln\frac{1+\mathrm{i}t}{1-\mathrm{i}t}, \tag{10.12}$$

we obtain

$$\beta(\theta) = 2\pi^{-1}\,\mathrm{e}^{-\frac{1}{2}\pi\Omega}\,\mathrm{ch}\,\tfrac{1}{2}\pi\theta \int_0^{\infty} \left(\frac{1+\mathrm{i}t}{1-\mathrm{i}t}\right)^{-\mathrm{i}\Omega} t^{\mathrm{i}\theta}\,\frac{dt}{1+t^2}. \tag{10.13}$$

If next a complex variable z is introduced by means of

$$z = \tfrac{1}{2}\pi + \mathrm{i}\ln t \qquad t = \mathrm{i}\mathrm{e}^{-\mathrm{i}w}, \tag{10.14}$$

we obtain

$$\beta(\theta) = \frac{1+\mathrm{e}^{-\pi\theta}}{2\pi\mathrm{i}} \int \mathrm{e}^{\theta w}\,(\mathrm{cotg}\,\tfrac{1}{2}w)^{\mathrm{i}\Omega}\,\frac{dw}{\sin w}, \tag{10.15}$$

where the path of integration is the vertical $\mathrm{Re}\, z = \frac{1}{2}\pi$.

For the expression (10.15) we may write for $\theta \to \infty$

$$\beta(\theta) = \frac{1+\mathrm{e}^{-\pi\theta}}{4\pi\mathrm{i}} \int^{0+} \mathrm{e}^{\theta w}\,(\tfrac{1}{2}w)^{-1-\mathrm{i}\Omega}\,\{1+\mathrm{O}(w^2)\}\,dw,$$

so that

$$\beta(\theta) = \frac{(2\theta)^{\mathrm{i}\Omega}}{\Gamma(1+\mathrm{i}\Omega)}\,\{1+\mathrm{O}(\theta^{-2})\}, \tag{10.16}$$

from which it follows that

$$\beta(\theta) = \left[\left(\frac{\mathrm{sh}\,\Omega\pi}{\Omega\pi}\right)^{\frac{1}{2}} \exp \mathrm{i}\,\{\Omega\ln 2\theta - \arg\Gamma(1+\mathrm{i}\,\Omega)\}\right]\{1+\mathrm{O}(\theta^{-2})\}. \tag{10.17}$$

Hence we have approximately

$$\arg B_1 \approx \Omega\ln 2\theta - \arg\Gamma(1+\mathrm{i}\Omega). \tag{10.18}$$

If Ω is very small we have

$$\Gamma(1+i\Omega) = -C\Omega + 0(\Omega^3), \tag{10.11}$$

where C is Euler's constant 0.5772 … .

Then (10.1) is approximately

$$b\omega = \tfrac{1}{2}\pi + \Omega(\ln 2\theta + C). \tag{10.12}$$

In the numerical case of the North Sea with $b=2\pi$ and $\Omega=0.6$ the relation (10.12) leads to roughly $\theta=35$ so that ω is unbelievably near to Ω.

It may be safely conjectured that an exact treatment of the eigenvalue equation must lead to a similar conclusion.

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