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Solutions of the equation of Helmholtz in an angle. II

by

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1. Introduction

This paper is the second of the set of papers dealing with the problem mentioned in the introduction of the previous paper [1]. References to that paper will be indicated by I followed by the section number or formula number. We shall use here the same notation. Here the G-problem is solved in two different ways. In section 2 the problem is solved by representing the Green's function by a Fourier integral,the integrand of which satisfies a certain functional relation. The latter functional equation has been considered in connection with the F-problem treated in the preceding paper (cf I4). This method is a streamlined version of Van Dantzig's method [2]. In section 3 a few elementary cases are considered for which the method of the previous section can be successfully applied. In section 4 the G-problem is solved by a different method. The Green's function is now represented essentially as a Laplace transform, the integrand of which is a sectionally holomorphic function. The boundary conditions lead eventually to a certain generalized Wiener-Hopf problem. The latter problem may be formulated as follows. Let $g_1(w)$ be holomorphic in the strip $\varphi_1-\frac{1}{2}\pi \leq \mathrm{Im}\, w < \varphi_1+\frac{1}{2}\pi$ and symmetric with respect to $i\varphi_1$, let $g_2(w)$ be holomorphic in the strip $\varphi_2-\frac{1}{2}\pi \leq \mathrm{Im}\, w \leq \varphi_2+\frac{1}{2}\pi$ and symmetric with respect to $i\varphi_2$. It will be assumed that $0 < \varphi_2-\varphi_1 \leq \pi$ so that the two strips overlap or at least have a line in common. If in the common region $R(\varphi_2-\frac{1}{2}\pi \leq \mathrm{Im}\, w \leq \varphi_1+\frac{1}{2}\pi)$ $h_1(w)$, $h_2(w)$ and $k(w)$ are given functions, then the problem is to determine $g_1(w)$ and $g_2(w)$ from the following functional equation in R (cf. (4.14) with a slightly different notation).

$$h_2(w)g_2(w) + h_1(w)g_1(w) = k(w). \tag{1.1}$$

If $\varphi_2-\varphi_1=\pi$ R is a line and the problem is equivalent to a Hilbert problem in the z-plane, where z=sh w. The solution of (1.1) involves a factorisation problem of the quotient $h_1(w)/h_2(w)$ which appears to be equivalent to the abovementioned functional equation of I4.
The final form of the solution is given by (4.25) which may be transformed into the elegant form (4.28) or the series expansions (4.35). Contrary to the form of the solution obtained by the first method that obtained by the second method lends itself readily to the dis-

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1) Cf. H.A. Lauwerier (1959[b]).

2) Cf. D. van Dantzig (1958).

cussion of special cases. In section 5 we have considered the continuation of the Green's function across the sides of the angle in the Riemannian plane. The primary reflections of the logarithmic pole of the Green's function with respect to the sides are dipole tails upon which the continued function makes a jump. The secondary reflections are bundles of dipole tails. In the special case $\gamma_1=\gamma_2=0$ or $\gamma_1=\gamma_2=\frac{1}{2}\pi$ the tails are absent and only the repeated reflections of the logarithmic pole remain, just as in the well-known Sommerfeld problem.

The G-problem solved in this paper generalizes all sort of problems of diffraction of acoustic, electromagnetic and hydrodynamic waves arising from a finite or infinite source. The well-known problem of the sloping beach has been treated in a similar way by the same author [3].

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3) Cf. H.A. Lauwerier (1959[a]).

2. The G-problem, first method

Let $G(r,\varphi,r_0,\varphi_0)$ be a function of Green satisfying I(1.1) and I(1.2) which is continuous at r=0 and which vanishes sufficiently rapidly for $r \to \infty$, e.g. as exp-cr where c is a positive constant. Then we take its complex Fourier transform

$$(2.1) \qquad W(w,\varphi) = \pi^{-1}\text{chw} \int_0^\infty \exp(ir\,\text{shw})G(r,\varphi,r_0,\varphi_0)dr.$$

By partial integration it can easily be proved that

$$(2.2) \qquad \left(\frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial \varphi^2}\right) W = \pi^{-1}\text{chw} \int_0^\infty \exp(ir\text{shw})\left\{\left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial^2}{\partial \varphi^2} - r^2\right\} G\,dr.$$

Then it follows from (I 1.1) that

$$(2.3) \qquad \frac{\partial^2 W}{\partial w^2} + \frac{\partial^2 W}{\partial \varphi^2} = -\pi^{-1} r_0 \text{chw} \exp(ir_0\text{shw})\, \delta(\varphi-\varphi_0).$$

Hence in the (w,φ)-plane, where w is real, the function $W(w,\varphi)$ is a harmonic function with a line-source at $\varphi=\varphi_0$. Inversion of (2.1) gives the integral representation

$$(2.4) \qquad G(r,\varphi,r_0,\varphi_0) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp(-ir\text{shw})W(w,\varphi)dw.$$

The variable w in (2.1) may be complex with $0 \leq \text{Im } w \leq \pi$. In this region the function W is regular. We note that the right-hand side of (2.4) vanishes for $r < 0$.

If w is a real variable it will often be replaced by u. Then we may define functions $U(u,\varphi)$ and $V(u,\varphi)$ by means of

$$(2.5) \qquad U(u,\varphi) \overset{\text{def}}{=} \tfrac{1}{2}\{W(u,\varphi) + W(-u,\varphi)\}$$

and

$$(2.6) \qquad V(u,\varphi) \overset{\text{def}}{=} \frac{1}{2i}\{W(u,\varphi) - W(-u,\varphi)\}.$$

They represent respectively the cosine transform and the sine transform of G.

It follows from (2.4) that

$$(2.7) \qquad G(r,\varphi,r_0,\varphi_0) = \int_{-\infty}^{\infty} \cos(r\text{shu})\, U(u,\varphi)du$$

and

$$(2.8) \qquad G(r,\varphi,r_0,\varphi_0) = \int_{-\infty}^{\infty} \sin(r\text{shu})\, V(u,\varphi)du.$$

Since W is harmonic in w and φ we may put

(2.9) $$W(w,\varphi) = g_1(w+i\varphi)+g_2(-w+i\varphi),$$

where g_1 and g_2 are analytic functions of their arguments. Then it follows from (2.5) and (2.6) that we may put

(2.10) $$2U(u,\varphi) = g(u+i\varphi) + g(-u+i\varphi)$$

and

(2.11) $$2V(u,\varphi) = g(u+i\varphi) - g(-u+i\varphi),$$

where in both cases g(w) is an analytic function of the complex argument w. In view of the line-source at $\varphi = \varphi_0$ the function is sectionally holomorphic in $\varphi_1 < \operatorname{Im} w < \varphi_0$ and $\varphi_0 < \operatorname{Im} w < \varphi_2$ with a jump line at $\operatorname{Im} w = \varphi_0$. It follows from (2.3) that $W(w,\varphi)$ is continuous but that its partial derivative with respect to ψ makes the following jump

(2.12) $$\frac{\partial W}{\partial \varphi}\Big|_{\varphi_0-0}^{\varphi_0+0} = -\pi^{-1} r_0 \operatorname{ch} w \exp(ir_0 \operatorname{sh} w).$$

In view of (2.5) and (2.6) we have similarly for $U(u,\varphi)$ and $V(u,\varphi)$

(2.13) $$\frac{\partial U}{\partial \varphi}\Big|_{\varphi_0-0}^{\varphi_0+0} = -\pi^{-1} r_0 \operatorname{ch} u \cos(ir_0 \operatorname{sh} u),$$

and

(2.14) $$\frac{\partial V}{\partial \varphi}\Big|_{\varphi_0-0}^{\varphi_0+0} = -\pi^{-1} r_0 \operatorname{ch} u \sin(ir_0 \operatorname{sh} u).$$

Then it follows from (2.10) and (2.13) that

(2.15) $$g(u+i\varphi)\Big|_{\varphi_0-0}^{\varphi_0+0} = -(\pi i)^{-1} \sin(r_0 \operatorname{sh} u).$$

Similarly it follows from (2.11) and (2.14) that

(2.16) $$g(u+i\varphi)\Big|_{\varphi_0-0}^{\varphi_0+0} = (\pi i)^{-1} \cos(r_0 \operatorname{sh} u).$$

After these preliminaries we shall introduce the boundary conditions (I 1.2). Since the representation (2.4) implies (2.7) as well as (2.8) we consider (2.4) only. Then by using (2.9) we find for j=1 and j=2

$$(2.17)\qquad \int_{-\infty}^{\infty} \exp(-ir\,\mathrm{sh}\,w)\left\{ \mathrm{ch}(w-i\gamma_j)g_1(w+i\varphi_j)-\mathrm{ch}(w+i\gamma_j)g_2(-w+i\varphi_j)\right\}dw=0 .$$

These conditions are satisfied if

$$(2.18)\qquad \mathrm{ch}(w-i\gamma_j)g_1(w+i\varphi_j)=\mathrm{ch}(w+i\gamma_j)g_2(-w+i\varphi_j).$$

In the remaining part of this section we shall treat the cases $\mathrm{Re}\,\gamma_1 \leq \mathrm{Re}\,\gamma_2$, and $\mathrm{Re}\,\gamma_1 > \mathrm{Re}\,\gamma_2$ separately.

a Re $\gamma_1 \leq \mathrm{Re}\,\gamma_2$.

In this case the cosine representation (2.7) has certain advantages. In view of (2.9) and (2.10) we may put

$$(2.19)\qquad g_1(w) = g_2(w) = g(w).$$

We shall next introduce a new unknown sectionally holomorphic function $P(w)$ by means of

$$(2.20)\qquad g(w) = P(w)\,\emptyset(w)$$

where the auxiliary function $\emptyset(w)$ is given by (cf. I(5.10))

$$(2.21)\qquad \emptyset(w) = \frac{e(w-i\varphi_1,\gamma_2)}{e(w-i\varphi_2,\gamma_1)} .$$

Then substitution of (2.20) in (2.18) gives for $w=u$ with real u the symmetry relations

$$(2.22)\qquad P(i\varphi_j-u) = P(i\varphi_j+u).$$

From (2.13) it follows that $P(u+i\varphi)$ makes the following jump at $\varphi=\varphi_0$

$$(2.23)\qquad P(u+i\varphi)\Big|_{\varphi_0-0}^{\varphi_0+0} = \frac{-\sin(r_0\,\mathrm{sh}\,u)}{\pi i\;\emptyset(u+i\varphi_0)} .$$

In order to construct a solution of (2.22) and (2.23) we consider the following function

$$(2.24)\qquad F(w) \overset{\mathrm{def}}{=} \frac{\nu}{2\pi i}\int_{-\infty}^{\infty} f(u_0)\,\frac{\mathrm{sh}\,\nu(u_0+i\varphi_0)}{\mathrm{ch}\,\nu(u_0+i\varphi_0)-\mathrm{ch}\,\nu w}\,du_0,$$

where $f(u_0)$ is absolutely integrable in $(-\infty,\infty)$. This function is clearly holomorphic in the strips $\varphi_1 \leq \mathrm{Im}\,w < \varphi_0$ and $\varphi_0 < \mathrm{Im}\,w \leq \varphi_2$, satisfies the symmetry relations (2.22) and makes the jump $f(u)$ at the line $w=u+i\varphi_0$. Hence a solution of (2.22) and (2.23) is obtained by an appropriate choice of f(u) viz.

$$(2.25)\qquad P(w) = \frac{\nu}{2\pi^2} \int_{-\infty}^{\infty} \frac{\sin(r_0 \operatorname{sh} u_0)}{\emptyset(u_0+i\varphi_0)} \; \frac{\operatorname{sh}\nu(u_0+i\varphi_0)}{\operatorname{ch}\nu(u_0+i\varphi_0)-\operatorname{ch}\nu w}\, du_0 .$$

Substitution of this result in (2.20), (2.10) and (2.7) gives finally [4)]

$$(2.26)\qquad G(r,\varphi,r_0,\varphi_0) = \frac{1}{2\Theta\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(r \operatorname{sh} u)\sin(r_0 \operatorname{sh} u_0)\, \frac{\emptyset(u+i\varphi)}{\emptyset(u_0+i\varphi_0)} \cdot \frac{\operatorname{sh}\nu(u_0+i\varphi_0)}{\operatorname{ch}\nu(u_0+i\varphi_0)-\operatorname{ch}\nu(u+i\varphi)}\, du\, du_0 .$$

b Re γ_1 > Re γ_2

In this case the homogeneous Helmholtz equation has a solution which is continuous at r=0 (cf.I(5.13)). Therefore the solution of the problem of Green is not unique. However, a unique Green's function may be determined by requiring that

$$(2.27)\qquad G(0,\varphi,r_0,\varphi_0) = 0.$$

Now the sine representation (2.8) is appropriate. In view of (2.9) and (2.11) we put

$$(2.28)\qquad g_1(w) = -g_2(w) = ig(w)$$

and next as in (2.20)

$$(2.29)\qquad g(w) = P(w)\, \emptyset(w),$$

where again $\emptyset(w)$ is given by (2.21). For P(w) clearly the symmetry relations (2.22) are obtained but the jump condition at $\varphi=\varphi_0$ is here

$$(2.30)\qquad P(u+i\varphi)\Big|_{\varphi_0-0}^{\varphi_0+0} = \frac{\cos(r_0 \operatorname{sh} u)}{\pi i\, \emptyset(u+i\varphi_0)} .$$

By a similar argument as in the previous case we find the solution

$$(2.31)\qquad P(w) = -\frac{\nu}{2\pi^2} \int_{-\infty}^{\infty} \frac{\cos(r_0 \operatorname{sh} u_0)}{\emptyset(u_0+i\varphi_0)} \; \frac{\operatorname{sh}\nu w}{\operatorname{ch}\nu(u_0+i\varphi_0)-\operatorname{ch}\nu w}\, du_0 .$$

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4) Cf. D. van Dantzig (1958), formula (5.10).

Substitution of this result in (2.29), (2.11) and (2.8) finally gives [5]

$$(2.32) \quad G(r,\varphi,r_0,\varphi_0) = \frac{1}{2\theta\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sin(r \operatorname{sh} u)\cos(r_0 \operatorname{sh} u_0) \frac{\emptyset(u+i\varphi)}{\emptyset(u_0+i\varphi_0)} \frac{\operatorname{sh}\nu(u+i\varphi)}{\operatorname{ch}\nu(u+i\varphi)-\operatorname{ch}\nu(u_0+i\varphi_0)} \, du \, du_0 .$$

The results (2.26) and (2.32) can be obtained from each other by replacing γ_1 and γ_2 by $-\gamma_1$ and $-\gamma_2$, i.e.

$$(2.33) \quad G(r,\varphi,r_0,\varphi_0,\gamma_1,\gamma_2) = G(r_0,\varphi_0,r,\varphi,-\gamma_1,-\gamma_2).$$

The latter relation may also be derived from Green's theorem.

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5) Cf. D. van Dantzig (1958), formula (5.11).

3. Examples

a. The following Green's function

(3.1) $$G_o(r,\varphi,r_o,\varphi_o) \overset{def}{=} (2\pi)^{-1} K_o(\sqrt{r^2+r_o^2-2rr_o\cos(\varphi-\varphi_o)})$$

satisfies the Helmholtz equation I(1.1) in the full r,φ-plane, i.e. for $0 \leq r < \infty$, $0 \leq \varphi \leq 2\pi$. We note that G_o has also a meaning for negative values of r but that the reflection $r \to -r$ may be considered as equivalent to the translation $\varphi \to \varphi \pm \pi$. According to (2.1) the complex Fourier transform $W_o(w,\varphi)$ will be taken with only positive values of r. In order to carry out the transformation we need a few auxiliary formulae. We have the following integral representation of the modified Bessel function K_o

(3.2) $$K_o(\sqrt{x^2+y^2}) = \tfrac{1}{2}\int_{-\infty}^{\infty} \exp-\{y\,ch(u+ic)+ix\,sh(u+ic)\}du,$$

where c is an arbitrary real constant. This expression converges in the halfplane $x \sin c < y \cos c$. By changing c this halfplane may swing round the origin so that the full x,y-plane is covered. Substitution of polar coordinates $x=-r_o\cos\varphi_o+r\cos\varphi$, $y=-r_o\sin\varphi_o+r\sin\varphi$ gives with either $c=\varphi$ or $c=\varphi\pm\pi$ the result

(3.3) $$K_o(\sqrt{r^2+r_o^2-2rr_o\cos(\varphi-\varphi_o)})\tfrac{1}{2}\int_{-\infty}^{\infty} \exp-\{irshu-ir_osh(u+i|\varphi-\varphi_o|)\}du.$$

Using this we find for $W_o(w,\varphi)$ without difficulty

(3.4) $$W_o(w,\varphi) = \frac{1}{4\pi^2 i}\int_{-\infty}^{\infty} \frac{chw}{sht-shw}\exp\{ir_osh(t+i|\varphi-\varphi_o|)\}\,dt.$$

This function is clearly holomorphic in the strip $0 \leq \mathrm{Im}\, w \leq \pi$ and satisfies there the symmetry relation

(3.5) $$W_o(w,\varphi) + W_o(\pi i-w,\varphi) = 0.$$

Further we note that $W_o(w,\varphi)$ is continuous at $\varphi=\varphi_o$. However, $\partial W_o/\partial\varphi$ makes a jump at $\varphi=\varphi_o$ the amount of which can be determined in the following way

$$\frac{\partial W_o}{\partial\varphi}\Bigg|_{\varphi_o-0}^{\varphi_o+0} = -\frac{r_o}{2\pi^2 i}\int_{-\infty}^{\infty}\frac{chw\,cht}{sht-shw}\exp(ir_osht)dt =$$

$$= -\frac{r_ochw}{2\pi^2 i}\int_{-\infty}^{\infty}\frac{\exp ir_oz}{z-shw}dz = -\pi^{-1}r_ochw\exp(ir_oshw),$$

which confirms (2.12).

Any function of Green satisfying I(1.1) and some boundary conditions can be considered as the sum of G_0, the singular part, and a function with continuous derivatives, the regular part. Since the complex Fourier transform of the regular part is regular at $\varphi=\varphi_0$ the preceding argument may be considered as an independent proof of the jump condition (2.12).

In order to obtain from (3.4) an illustration of the property (2.9) we need the identity

$$\frac{\operatorname{ch} w}{\operatorname{sh} t-\operatorname{sh} w} = \frac{e^t}{e^t-e^w} - \frac{e^t}{e^t+e^{-w}} . \tag{3.6}$$

Then it can be easily derived that (2.9) holds with

$$g_j(w) = \frac{\varepsilon}{4\pi^2 i} \int_{-\infty}^{\infty} \frac{e^w}{e^{t+i\varphi_0}-e^w} \exp(\varepsilon i r_0 \operatorname{sh} t)dt, \tag{3.7}$$

where $\varepsilon=+$ for $j=1$ and $\varepsilon=-$ for $j=2$.
Evidently $g_1(w)$ and $g_2(w)$ are sectionally holomorphic with the boundary line at Im $w=\varphi_0$. Writing $w=u+i\varphi$ with real u we have at $\varphi=\varphi_0$

$$g_j(u+i\varphi)\Big|_{\varphi_0-0}^{\varphi_0+0} = \varepsilon(2\pi)^{-1}\exp(\varepsilon i r_0 \operatorname{sh} u). \tag{3.8}$$

b If the complex Fourier transform $W_1(w,\varphi)$ is taken with positive and negative values of r we obtain

$$\begin{aligned} W_1(w,\varphi) &= \pi^{-1}\operatorname{ch} w \int_{-\infty}^{\infty} \exp(ir\operatorname{sh} w)G_0 dr = \\ &= (2\pi)^{-1} \exp\{ir_0 \operatorname{sh}(w+i|\varphi-\varphi_0|)\} . \end{aligned} \tag{3.9}$$

This function is holomorphic in the strip $-\pi \leq \operatorname{Im} w \leq \pi$ but the relation (3.5) no longer holds. In this case we have (2.9) with

$$g_1(w) = \begin{cases} 0 & \text{for } \varphi<\varphi_0 \\ (2\pi)^{-1}\exp\{ir_0\operatorname{sh}(w-i\varphi)\} & \text{for } \varphi>\varphi_0, \end{cases} \tag{3.10}$$

and

$$g_2(w) = \begin{cases} (2\pi)^{-1}\exp-\{ir_0\operatorname{sh}(w-i\varphi)\} & \text{for } \varphi<\varphi_0 \\ 0 & \text{for } \varphi>\varphi_0 . \end{cases} \tag{3.11}$$

From this the jump condition (2.12) follows without difficulty.

4. The G-problem, second method

In this section the G-problem for the angle $\varphi_1<\varphi<\varphi_2$ will be solved by a different method. We shall start the discussion by assuming that (cf. I(5.17))

$$(4.1)\qquad \tfrac{1}{2}\pi<\theta<\pi\ ,\quad \mathrm{Re}\ \gamma_1<\theta-\tfrac{1}{2}\pi\ ,\quad \mathrm{Re}\ \gamma_2>\tfrac{1}{2}\pi-\theta.$$

In that case we know from I(5.19) and I(5.20) that the "regular" solution of the F-problem can be written in the form

$$(4.2)\qquad F(r,\varphi)=\int_{-\infty+ic}^{\infty+ic}\exp\{-r\,ch(w-i\varphi)\}\ H(w)dw,$$

where

$$(4.3)\qquad \varphi_1+\mathrm{Re}\ \gamma_1<c<\varphi_2+\mathrm{Re}\ \gamma_2.$$

The function H(w) is holomorphic in the strip given by (4.3) so that the representation (4.2) gives a regular solution in the larger angle (cf.I(5.18))

$$(4.4)\qquad \varphi_1-\tfrac{1}{2}\pi+\mathrm{Re}\ \gamma_1<\varphi<\varphi_2+\tfrac{1}{2}\pi+\mathrm{Re}\ \gamma_2.$$

A Green's function may be considered as the sum of a singular part for which the function G_0 of (3.1) can be taken and a regular part which is a solution of the homogeneous Helmholtz equation I(3.1). It will be tried to represent a function of Green satisfying I(1.1) and I(1.2) by

$$(4.5)\qquad G(r,\varphi,r_0,\varphi_0)=G_0(r,\varphi,r_0,\varphi_0)+\frac{1}{4\pi}\int_{-\infty+ic}^{\infty+ic}\exp\{-r\,ch(w-i\varphi)\}\ g(w)dw,$$

where g(w) is holomorphic in the strip (4.3) and of the order $O(\exp-\varepsilon|\mathrm{Re}\ w|)$ as $\mathrm{Re}\ w\to\pm\infty$.

In order to introduce the boundary conditions I(1.2) in (4.5) we note that from (3.1) and (3.3) it follows that for j=1,2 at $\varphi=\varphi_j$

$$(4.6)\qquad \left(\cos\gamma_j\,\frac{1}{r}\frac{\partial}{\partial\varphi}-\sin\gamma_j\,\frac{\partial}{\partial r}\right)G_0=$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{-i\varepsilon_j r\,sh w}\ ch(w+i\gamma_j)\exp\{i\varepsilon_j sh(w+i\varphi_0-i\varphi_j)\}\,dw,$$

where $\varepsilon_j=+1$ for $j=1$ and $\varepsilon_j=-1$ for $j=2$.

Then the boundary condition at $\varphi=\varphi_j$ gives

$$(4.8) \qquad \int_{-\infty}^{\infty} e^{-i\varepsilon_j r \operatorname{sh} w} g_j(w) d\operatorname{sh} w = 0,$$

where

$$(4.9) \qquad g_j(w)\operatorname{ch} w \stackrel{\text{def}}{=} \operatorname{ch}(w-i\gamma_j)g(w+i\varphi_j-\tfrac{1}{2}i\pi)$$
$$-\operatorname{ch}(w+i\gamma_j)\exp\{i\varepsilon_j r_0 \operatorname{sh}(w+i\varphi_0-i\varphi_j)\} \quad .$$

The conditions (4.8) should be fulfilled for $r>0$ only so that it is sufficient to assume that $g_1(w)$ is holomorphic in the upper strip $0<\operatorname{Im} w<\pi$ and $g_2(w)$ holomorphic in the lower strip $-\pi<\operatorname{Im} w<0$, that $g_j(w)$ is symmetric with respect to $\frac{1}{2}\varepsilon_j\pi i$, and that $g_j(w)$ vanishes at $\operatorname{Re} w\rightarrow\pm\infty$.

The relation (4.9) gives either

$$(4.10) \qquad g(w) = h_1(w)g_1(w-i\varphi_1-\tfrac{1}{2}i\pi)+k_1(w)$$

valid in the strip $\varphi_1-\frac{1}{2}\pi<\operatorname{Im} w<\varphi_1+\frac{1}{2}\pi$,

or

$$(4.11) \qquad g(w) = h_2(w)g_2(w-i\varphi_2+\tfrac{1}{2}i\pi)+k_2(w)$$

valid in the strip $\varphi_2-\frac{1}{2}\pi<\operatorname{Im} w<\varphi_2+\frac{1}{2}\pi$,

where the functions $h_j(w)$ and $k_j(w)$ are defined by

$$(4.12) \qquad h_j(w) \stackrel{\text{def}}{=} \operatorname{sh}(w-i\varphi_j)/\operatorname{sh}(w-i\varphi_j-i\gamma_j),$$

and

$$(4.13) \qquad k_j(w) \stackrel{\text{def}}{=} \exp\{r_0\operatorname{ch}(w-2i\varphi_j+i\varphi_0)\}\ \operatorname{sh}(w-i\varphi_j+i\gamma_j)/\operatorname{sh}(w-i\varphi_j-i\gamma_j).$$

From (4.10) and (4.11) there follows for the common strip $\varphi_2-\frac{1}{2}\pi<\operatorname{Im} w<\varphi_1+\frac{1}{2}\pi$ that

$$(4.14) \qquad h_2(w)g_2(w-i\varphi_2+\tfrac{1}{2}i\pi)-h_1(w)g_1(w-i\varphi_1-\tfrac{1}{2}i\pi) = k_1(w)-k_2(w).$$

The solution of the functional equation (4.14) is the crucial point of the method of this section. The problem (4.14) may be formulated as representing a generalization of a Hilbert problem or a Wiener-Hopf problem. It does represent a Hilbert problem on a line in the special cases $\varphi_1=-\frac{1}{2}\pi$ and $\varphi_2=\frac{1}{2}\pi$. By transforming (4.14) from the w-plane into the complex z-plane by means of z=shw we obtain with a similar notation

$$(4.15) \qquad h_2(z)g_2(z) - h_1(z)g_1(z) = k(z)$$

valid for real z, where $h_1(z)$,$h_2(z)$ and $k(z)$ are given functions and where it is required that $g_2(z)$ is holomorphic in the upper halfplane $\operatorname{Im} z > 0$ and $g_1(z)$ is holomorphic in the lower halfplane $\operatorname{Im} z < 0$. Hence (4.15) represents a Hilbert problem on the real axis of the z-plane, the solution of which involves the Wiener-Hopf factorisation of the quotient $h_1(z)/h_2(z)$.

The more general problem (4.14) might be also interpreted in the z-plane which is now a Riemannian plane with branch points at $z=\pm i$. Therefore (4.14) may be considered as a Wiener-Hopf problem in a Riemannian plane. Its solution involves the following factorisation. To find functions $H_j(w)$, $j=1,2$, which are free from poles and zeros in the strip $\varphi_j-\frac{1}{2}\pi < \operatorname{Im} w < \varphi_j+\frac{1}{2}\pi$, which are symmetric with respect to $i\varphi_j$ and which satisfy

$$(4.16) \qquad h_1(w)H_1(w) = h_2(w)H_2(w) \overset{\text{def}}{=} H_o(w).$$

From the symmetry relations of the $H_j(w)$ we may derive a set of functional relations for H(w). We have by using (4.12)

$$(4.17) \qquad \frac{H_o(i\varphi_j+w)}{H_o(i\varphi_j-w)} = \frac{h_j(i\varphi_j+w)}{h_j(i\varphi_j-w)} = \frac{\operatorname{sh}(w+i\gamma_j)}{\operatorname{sh}(w-i\gamma_j)} .$$

But these are exactly the functional relations of the function H(w) discussed in I(5.20) and I(5.22).
Therefore the function $H_o(w)$ can be identified with the latter function H(w).

Hence a factorisation (4.16) is obtained by taking

$$(4.18) \qquad H_j(w) = H(w)/h_j(w).$$

As a verification of the fact that e.g. $H_1(w)$ is free from poles and zeros in the strip $\varphi_1-\frac{1}{2}\pi < \operatorname{Im} w < \varphi_1+\frac{1}{2}\pi$ we note that the nearest zeros and poles of $H_1(w)$ are a zero $i(\varphi_1+\theta)$ and a pole $i(\varphi_1+\theta+\gamma_2)$ on the upper side and a zero $i(\varphi_1-\theta)$ and a pole $i(\varphi_1-\theta-\gamma_2)$ on the lower side. It may be remarked that at this point an essential use of the inequalities (4.1) is made.

By using the factorisation (4.16) the relation (4.14) can be brought in the following form

(4.19) $$\psi_2(w) - \psi_1(w) = \psi(w),$$

where for j=1 and j=2

(4.20) $$\psi_j(w) \overset{\text{def}}{=} g_j(w-i\varphi_j-\varepsilon_j\tfrac{1}{2}i\varphi)/H_j(w)$$

and

(4.21) $$\psi(w) \overset{\text{def}}{=} \{k_1(w)-k_2(w)\}/H(w).$$

The unknown function $\psi_j(w)$ is holomorphic in the strip $\varphi_j-\frac{1}{2}\pi < \text{Im } w < \varphi_j+\frac{1}{2}\pi$ and symmetric with respect to $i\varphi_j$.

The problem (4.19), a simpler version of (4.14) may be interpreted in the following way. Let $\psi(w)$ be a given analytic function which is holomorphic in a strip $\alpha_1 < \text{Im } w < \alpha_2$ with $\alpha_2-\alpha_1 < \pi$. Then $\psi(w)$ should be split in two parts $\psi_1(w)$ and $\psi_2(w)$ as indicated by (4.19) where $\psi_1(w)$ is holomorphic and symmetric in the lower strip $\alpha_2-\pi < \text{Im } w < \alpha_2$ and $\psi_2(w)$ is holomorphic and symmetric in the upper strip $\alpha_1 < \text{Im } w < \alpha_1+\pi$ (see figure 1).

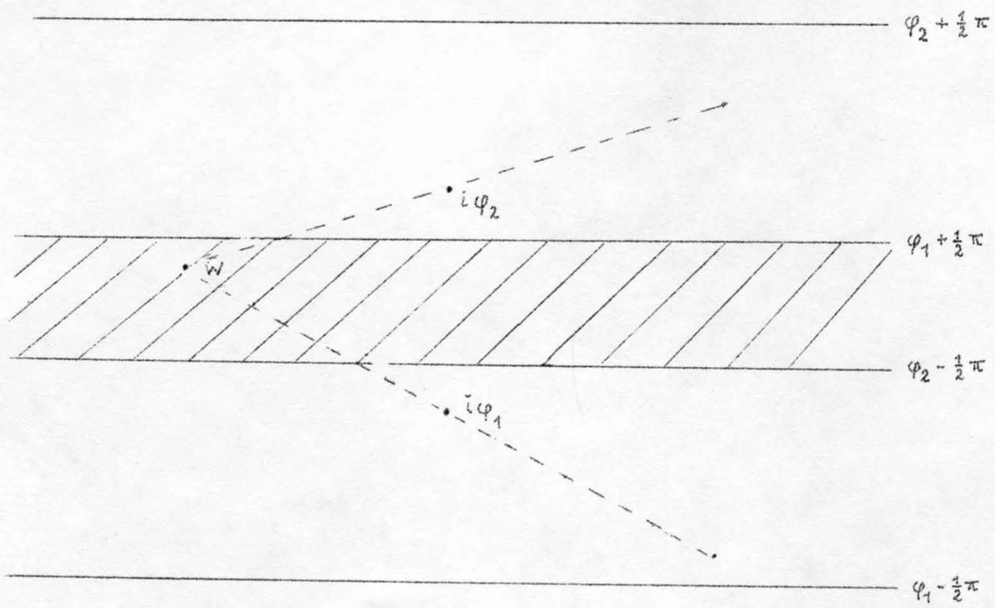

figure 1

The equation (4.19) holds in the strip $\varphi_2-\frac{1}{2}\pi < \text{Im } w < \varphi_1+\frac{1}{2}\pi$. The symmetry relations of the functions $\psi_j(w)$ are explicitly

(4.22) $$\psi_j(i\varphi_j+w) = \psi_j(i\varphi_j-w).$$

Then it is easily seen that (4.19) is solved by

$$(4.23)\qquad \psi_j(w) \doteq \frac{\nu}{2\pi i}\int_{L_j} \psi(w_o)\,\frac{\operatorname{sh}\nu(w_o-i\varphi_j)}{\operatorname{ch}\nu(w_o-i\varphi_j)-\operatorname{ch}\nu(w-i\varphi_j)}\,dw_o,$$

where L_1 and L_2 are contours in the region of regularity of $\psi(w_o)$ with Re w_o running from $-\infty$ to $+\infty$ and where w is enclosed between L_2 and L_1 as shown in figure 2.

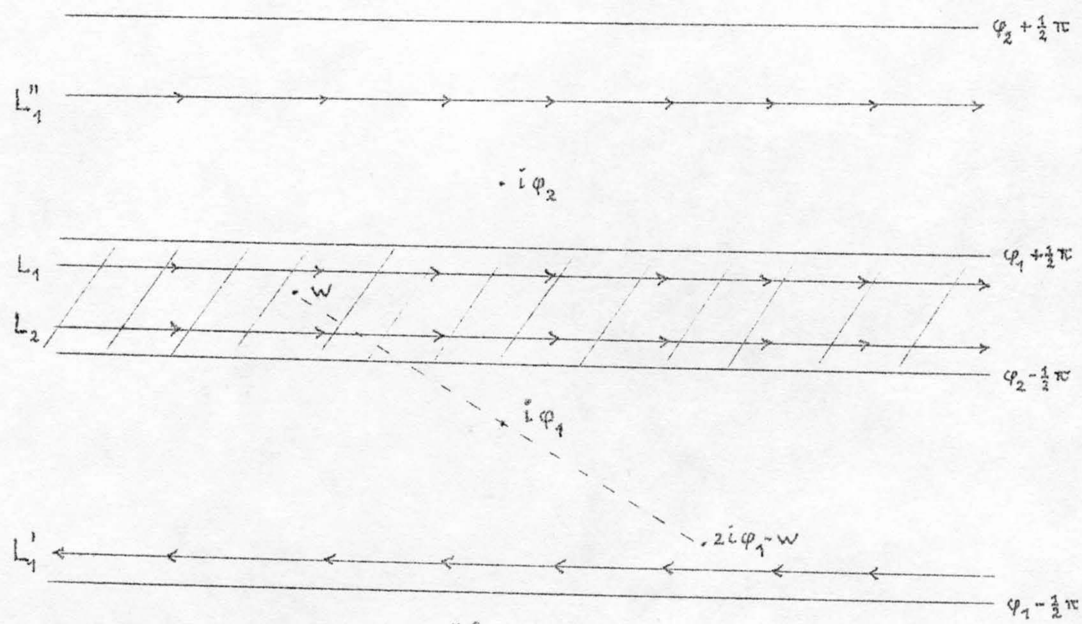

figure 2

We shall assume that contours L_j can be found such that the integrals defining $\psi_j(w)$ converge.

Substitution of the expressions (4.20) and (4.21) in (4.23) gives

$$(4.24)\qquad g_j(w-i\varphi_j-\varepsilon_j\tfrac{1}{2}i\pi) = H_j(w)\,\frac{\nu}{2\pi i}\int_{L_j}\frac{k_1(w_o)-k_2(w_o)}{H(w_o)}\cdot\frac{\operatorname{sh}\nu(w_o-i\varphi_j)}{\operatorname{ch}\nu(w_o-i\varphi_j)-\operatorname{ch}\nu(w-i\varphi_j)}\,dw_o.$$

From (4.5), (4.10) and (4.24) for $j=1$ the following solution is obtained

$$2\pi G(r,\varphi,r_o,\varphi_o) = K_o\left(\sqrt{r^2+r_o^2-2rr_o\cos(\varphi-\varphi_o)}\right)+\frac{1}{2}\int_{-\infty+ic}^{\infty+ic} e^{-r\operatorname{ch}(w-i\varphi)}k_1(w)dw+$$

$$(4.25)\qquad +\frac{\nu}{2\pi i}\int_{-\infty+ic}^{\infty+ic} e^{-r\operatorname{ch}(w-i\varphi)}H(w)dw\int_{L_1}\frac{k_1(w_o)-k_2(w_o)}{H(w_o)}\cdot\frac{\operatorname{sh}\nu(w_o-i\varphi_1)}{\operatorname{ch}\nu(w_o-i\varphi_1)-\operatorname{ch}\nu(w-i\varphi_1)}\,dw_o.$$

A similar expression may be obtained from (4.5), (4.11) and (4.24) for j=2.

In order to obtain an interpretation of (4.25) we consider the halfplane case $\varphi_1=0$, $\varphi_2=\pi$ and next $\gamma_1=\gamma_2=\gamma$. Then it follows from (4.13) that $k_1(w)=k_2(w)$ so that (4.25) reduces to

$$(4.26)\quad \begin{aligned} 2\pi G(r,\varphi,r_0,\varphi_0) &= K_0\left(\sqrt{r^2+r_0^2-2rr_0\cos(\varphi-\varphi_0)}\right)+ \\ &+\frac{1}{2}\int_{-\infty+ic}^{\infty+ic} \frac{sh(w+i\gamma)}{sh(w-i\gamma)} \exp\left\{-r\,ch(w-i\varphi)+r_0\,ch(w+i\varphi_0)\right\}dw. \end{aligned}$$

A more detailed interpretation of this important special case will be postponed to the following section but here we remark that with $\mathrm{Re}\,\gamma < c < \pi+\mathrm{Re}\,\gamma$ the integral on the right-hand side of (4.26) converges in the halfplane $r\cos(c-\varphi) > r_0\cos(c+\varphi_0)$ or $(x-x_0)\cos c+(y+y_0)\sin c > 0$ (see figure 3). Hence it represents a kind of "oblique reflection" of the singularity (x_0,y_0) with respect to the X axis

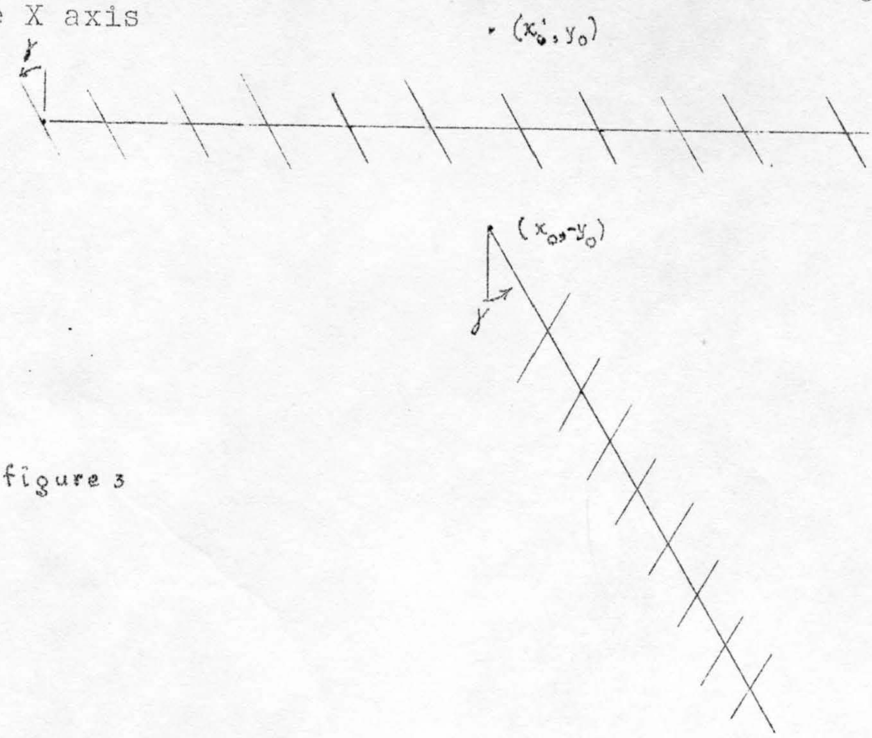

figure 3

Therefore the right-hand side of (4.25) represents the Green's function as the sum of the elementary Green's function G_0 of the full plane, its "oblique reflection" with respect to one side of the angle and a correction term.

The expression (4.25) may be transformed in the following way. By using (4.13) and the functional relations (4.17) we have

$$\int_{L_1} \frac{k_1(w_o)-k_2(w_o)}{H(w_o)} \frac{\operatorname{sh}\nu(w_o-i\varphi_1)}{\operatorname{ch}\nu(w_o-i\varphi_1)-\operatorname{ch}\,(w-i\varphi_1)}\,dw_o =$$

$$= \int_{L_1}\left\{\frac{\exp r_o\operatorname{ch}(w_o-2i\varphi_1+i\varphi_o)}{H(-w_o+2i\varphi_1)} - \frac{\exp r_o(w_o-2i\varphi_2+i\varphi_o)}{H(-w_o+2i\varphi_2)}\right\} \cdot$$

$$\cdot \frac{\operatorname{sh}\nu(w_o-i\varphi_1)}{\operatorname{ch}\nu(w_o-i\varphi_1)-\operatorname{ch}\nu(w-i\varphi_1)}\,dw_o.$$

If L_1' is the reflection of the path L_1 with respect to $w_o=i\varphi_1$ with Re w_o running from +oo to -oo and if L_1'' is the reflection with respect to $w_o=i\varphi_2$ with Re w_o running from -oo to +oo then the latter expression may be changed into

$$(4.27)\quad \int_{L_1'+L_1''} \frac{\exp r_o\operatorname{ch}(w_o-i\varphi_o)}{H(w_o)} \frac{\operatorname{sh}\nu(w_o-i\varphi_1)}{\operatorname{ch}\nu(w_o-i\varphi_1)-\operatorname{ch}\nu(w-i\varphi_1)}\,dw_o.$$

The denominator $\operatorname{ch}\nu(w_o-i\varphi_1)-\operatorname{ch}\nu(w-i\varphi_1)$ gives two poles $w_o=w$ and $w_o=2i\varphi_1-w$ lying between L_1' and L_1''. There are no other poles in this strip. If the expression (4.27) is substituted in the third term on the right-hand side of (4.25) the residues of the poles $w_o=w$ and $w_o=-2i\varphi_1-w$ cancel the first and the second term on the right-hand side of (4.25). Therefore we obtain the result

$$(4.28)\quad 2\pi G(r,\varphi,r_o,\varphi_o) = \frac{\nu}{4\pi i}\int_{-\infty+ic}^{\infty+ic} \exp\{-r\operatorname{ch}(w-i\varphi)\}\,H(w)\,dw$$

$$\int_L \exp\{r_o\operatorname{ch}(w_o-i\varphi_o)\}\,H^{-1}(w_o)\,\frac{\operatorname{sh}\nu(w_o-i\varphi_1)}{\operatorname{ch}\nu(w_o-i\varphi_1)-\operatorname{ch}\nu(w-i\varphi_1)}\,dw_o,$$

where the path L is of the form of I fig.3.

The expression (4.28) is no longer restricted to values of θ, γ_1 and γ_2 which satisfy the inequalities (4.1) but can easily be extended to all θ, γ_1 and γ_2 values.

By way of illustration we shall consider the special case $\gamma_1=\gamma_2=0$, $\varphi_1=0$ and $\varphi_2=\theta$ with θ arbitrary. Then $H(w)\equiv 1$. If moreover ν is an integer, say $\nu=m$, the contour L may be transformed into the lines L_1' and L_1'' for which we take Im $w=\pm\pi+\varphi_o$. Then the inner integral of (4.28) equals the sum of the contributions from L_1' and L_1'' and a number of poles viz.

$$(4.29)\qquad w_o = \pm w+2j\theta i\ , \qquad j=0,1,\ldots m-1.$$

The contributions from L_1' and L_1'' are equal and opposite in sign so that finally

$$2\pi G(r,\varphi,r_o,\varphi_o) = \sum_{j=0}^{m-1} \tfrac{1}{2} \int_{-\infty+ic}^{\infty+ic} \exp\{-r\,ch(w-i\varphi)+r_o ch(w_o-i\varphi_o)\}dw,$$

or

$$(4.30)\quad G(r,\varphi,r_o,\varphi_o) = \sum_{j=0}^{m-1}\{G_o(r,\varphi,r_o,\varphi_o+2j\pi/m)+G_o(r,\varphi,r_o,-\varphi_o+2j\pi/m)\}.$$

This is in agreement with the result derived by a direct method in I(3.5).

If $\gamma_1=\gamma_2=0$, $\varphi_1=0$ and $\varphi_2=\theta$ but ν not necessarily an integer from (4.28) also the previous result I(3.6) can be derived. For $\mathrm{Re}\, w_o > \mathrm{Re}\, w$ we have the expansion

$$(4.31)\quad sh\,\nu w_o/(ch\,\nu w_o - ch\,\nu w) = \sum_{m=0}^{\infty} \varepsilon_m chm\nu w\, e^{-m\nu w_o},$$

where $\varepsilon_o=1$ and $\varepsilon_m=2$ for $m \geq 1$.

A similar expansion holds of course for $\mathrm{Re}\, w_o < \mathrm{Re}\, w$. If (4.31) is substituted in (4.28) we obtain at first after some elementary reductions

$$(4.32)\quad \frac{\nu}{2\pi i}\int_L \exp\{r_o ch(w_o-i\varphi_o)\}\frac{sh\,\nu w_o}{ch\,\nu w_o - ch\,\nu w}dw_o =$$

$$= \sum_{m=0}^{\infty} \varepsilon_m chm\nu w \cos m\nu\varphi_o \cdot \frac{\nu}{\pi i}\int_{L^+}\exp\{r_o chw_o - m\nu w_o\}dw_o,$$

where L^+ is the right-hand part of L (cf. I (5.25)).

The right-hand side of (4.32) equals

$$(4.33)\quad 2\nu \sum_{m=0}^{\infty} \varepsilon_m chm\nu w \cos m\nu\varphi\; I_{m\nu}(r_o).$$

Substitution of this expression in (4.28) gives at once the required result

$$(4.34)\quad G(r,\varphi,r_o,\varphi_o) = \theta^{-1}\sum_{m=0}^{\infty} \varepsilon_m \cos m\nu\varphi \cos m\nu\varphi_o\; K_{m\nu}(r)\, I_{m\nu}(r_o).$$

This expansion converges for $r > r_o$.

The same process can also be applied upon the general expression (4.28). Then we obtain eventually

$$(4.35)\quad G(r,\varphi,r_o,\varphi_o) = \theta^{-1}\sum_{m=0}^{\infty}\varepsilon_m F_m(r,\varphi)F_m^*(r_o,\varphi_o),$$

where (cf. I(5.23)) $F_m(r,\varphi)$ is defined by

$$(4.36)\qquad F_m(r,\varphi) = \tfrac{1}{2}\int_{-\infty+ic}^{\infty+ic} \exp\{-r\,ch(w-i\varphi)\}\, ch\, m\nu(w-i\varphi_1)H(w)dw,$$

and (cf. I(5.24)) with σ =sgn Re w

$$(4.37)\qquad F_m^*(r,\varphi) = \frac{1}{4\pi i}\int_L \exp\{r\,ch(w-i\varphi)-\sigma m\nu(w-i\varphi_1)\}H^{-1}(w)dw.$$

We note that $F_m(r,\varphi)$ and $F_m^*(r,\varphi)$ both are solutions of the homogeneous Helmholtz equation $(\Delta-1)F=0$. However, $F_m(r,\varphi)$ satisfies the same boundary conditions as G but $F_m^*(r,\varphi)$ satisfies the adjoint boundary conditions with $\gamma_j \to -\gamma_j$. The expansion (4.35) obviously converges for $r > r_0$. In view of the symmetry relation (2.33) from (4.35) at once a similar expansion valid for $r < r_0$ follows. It would be of interest to prove (4.35) by some direct method.

5. Discussion of the results

In the case of a halfplane with $\varphi_1=0$, $\varphi_2=\theta$, $\gamma_1=\gamma_2=\gamma$ the solution of the G-problem is given by the expression (4.26). By this expression not only $G(r,\varphi,r_o,\varphi_o)$ in the relevant halfplane is given but also its continuation in the complementary halfplane. For simplicity we shall assume that γ is real and positive. It has been already noted in the previous section that (4.26) holds in the halfplane $(x-x_o)\cos c + (y+y_o)\sin c > 0$. Hence by varying the parameter c the required continuation can be obtained.

If we define

$$(5.1)\qquad R(x,y) \overset{def}{=} \tfrac{1}{2} \int_{-\infty+ic}^{\infty+ic} \frac{\mathrm{sh}(w+i\gamma)}{\mathrm{sh}(w-i\gamma)} \exp(-x\,\mathrm{ch}w+iy\,\mathrm{sh}w)dw,$$

the solution of the G-problem can be written as

$$(5.2)\qquad G(x,y,x_o,y_o) = G_o(x,y,x_o,y_o)+(2\pi)^{-1}R(x-x_o,y+y_o).$$

If differentiation in the direction $\frac{1}{2}\pi+\gamma$ with respect to the X-axis is symbolically written as D_γ the boundary condition at the X-axis viz.

$$(5.3)\qquad \cos\gamma\,\frac{\partial G}{\partial y} - \sin\gamma\,\frac{\partial G}{\partial x} = 0 \qquad \text{at } y=0,$$

may be written in the form

$$(5.4)\qquad D_\gamma G = 0 \qquad \text{at } y=0.$$

Now it follows at once from (5.1) that

$$(5.5)\qquad D_\gamma R(x,y) = D_{-\gamma} K_o(\sqrt{x^2+y^2}).$$

Hence $R(x,y)$ can be interpreted as the contribution from a line of dipoles which makes the angle $-\frac{1}{2}\pi+\gamma$ with the X-axis. The dipoles itself are making the reflected angle $\frac{1}{2}\pi-\gamma$ with the X-axis. Therefore (5.2) represents the sum of contributions from a logarithmic pole at (x_o,y_o) and a dipole tail radiating from $(x_o,-y_o)$ in the direction $-\frac{1}{2}\pi+\gamma$ (see figure 3). The dipole tail is equivalent to a line of normal dipoles plus a simple pole at its end. Geometrically this is obvious. Analytically this follows from the identity

$$(5.6)\qquad \mathrm{sh}(w+i\gamma)/\mathrm{sh}(w-i\gamma) = \cos 2\gamma + i\sin 2\gamma\,\mathrm{cth}(w-i\gamma).$$

Substitution of (5.6) in (5.1) gives

$$(5.7)\qquad R(x,y) = \cos 2\gamma\, K_o(\sqrt{x^2+y^2})+\tfrac{1}{2}i\sin 2\gamma \int_{-\infty+ic}^{\infty+ic} \mathrm{cth}(w-i\gamma)\cdot\exp(-x\,\mathrm{ch}w+iy\,\mathrm{sh}w)dw.$$

The first term on the right-hand side represents a simple pole of strength cos 2γ at the origin, the second term represents a tail of normal dipoles. By crossing this tail a jump is made, the amount of which obviously is determined by the pole $w=i\gamma$ of the integrand. Using polar coordinates (r,φ) we obtain

$$(5.8) \qquad R(r,-\tfrac{1}{2}\pi+\gamma+0)-R(r,-\tfrac{1}{2}\pi+\gamma-0) = \pi \sin 2\gamma \ .$$

If $\gamma=0$ the boundary condition at $y=0$ reduces to a simple Neumann condition. In that case we may say that according to (5.7) the dipoles annihilate each other and that only a single reflected pole at $(x_0,-y_0)$ remains.

If $\gamma=\frac{1}{4}\pi$ the dipole tail contains only normal dipoles.

If $\gamma=\frac{1}{2}\pi$ we have a Dirichlet condition at $y=0$. Again there are no dipoles and a single reflected pole of negative strength remains at $(x_0,-y_0)$, or as we may say, there is a source at (x_0,y_0) and a sink at $(x_0,-y_0)$.

A similar discussion applies to the general case the solution of which is given by (4.28). For simplicity we shall assume that γ_1 and γ_2 are real. The poles of

$$(5.9) \qquad \{\operatorname{ch} \nu(w_0-i\varphi_1)-\operatorname{ch}\nu(w-i\varphi_1)\}^{-1},$$

which are given by

$$(5.10) \qquad w_0=w-2m\theta i \quad \text{and} \quad w_0=-w+(2m\theta+2\varphi_1)i,$$

where m is an integer, determine the position of the logarithmic pole (r_0,φ_0) and its repeated reflections with respect to the sides $\varphi=\varphi_j$ of the angle. The geometric picture is of course a Riemannian plane in the form of a spiral staircase.
By taking the residues we have formally

$$(5.11) \qquad 2\pi G(r,\varphi,r_0,\varphi_0) = \sum \tfrac{1}{2} \int_{-\infty+ic}^{\infty+ic} \frac{H(w)}{H(w_0)} \exp\{-r\operatorname{ch}(w-i\varphi)+r_0\operatorname{ch}(w_0-i\varphi_0)\}dw.$$

We shall show that each term on the right-hand side of (5.11) may be interpreted as the contribution from a bundle of dipole tails radiating from a simple pole which is one of the repeated reflections of (r_0,φ_0). In order to show this we may as well take the simpler expressions

$$(5.12) \qquad R_m(r,\varphi) \overset{\text{def}}{=} \tfrac{1}{2} \int_{-\infty+ic}^{\infty+ic} \frac{H(w)}{H(w-2m\theta i)} \exp\{-r\operatorname{ch}(w-i\varphi)\}dw,$$

and

$$(5.13) \qquad R'_m(r,\varphi) \overset{\text{def}}{=} \tfrac{1}{2} \int_{-\infty+ic}^{\infty+ic} \frac{H(w)}{H(-w+2m\theta i+2\varphi_1 i)} \exp\{-r\,ch(w-i\varphi)\}dw.$$

From the functional relations (4.17) it follows that

$$(5.14) \qquad \frac{H(w)}{H(w+2\theta i)} = \frac{sh(w-i\varphi_1+i\gamma_1)sh(w-i\varphi_1-i\gamma_2+i\theta)}{sh(w-i\varphi_1-i\gamma_1)sh(w-i\varphi_1+i\gamma_2+i\theta)}$$

and

$$(5.15) \qquad \frac{H(w)}{H(-w+2i\varphi_1)} = \frac{sh(w-i\varphi_1+i\gamma_1)}{sh(w-i\varphi_1-i\gamma_1)} .$$

Therefore in both cases (5.12) and (5.13) we obtain something like

$$(5.16) \qquad \tfrac{1}{2} \int_{-\infty+ic}^{\infty+ic} \prod \frac{sh(w-i\alpha_j)}{sh(w-i\beta_j)} \exp\{-r\,ch(w-i\varphi)\}dw,$$

where the α_j and β_j easily follow from the relations (5.14) and (5.15). Instead of giving general formulae the β_j for the first few cases are given below.

	position of pole	β_j		
R'_2	$2\varphi_1-\varphi_0+4\theta$	$\varphi_1+\gamma_2+\theta$	$\varphi_1-\gamma_1+2\theta$	$\varphi_1+\gamma_2+3\theta$
R_1	$\varphi_0+2\theta$	$\varphi_1+\gamma_2+\theta$	$\varphi_1-\gamma_1+2\theta$	
R'_1	$2\varphi_1-\varphi_0+2\theta$	$\varphi_1+\gamma_2+\theta$		
R_0	φ_0	–		
R'_0	$2\varphi_1-\varphi_0$	$\varphi_1+\gamma_1$		
R_{-1}	$\varphi_0-2\theta$	$\varphi_1+\gamma_1$	$\varphi_1-\gamma_2-\theta$	
R'_{-1}	$2\varphi_1-\varphi_0-2\theta$	$\varphi_1+\gamma_1$	$\varphi_1-\gamma_2-\theta$	$\varphi_1+\gamma_1-2\theta$

The values of α_j are determined by the same scheme with γ_1 and γ_2 replaced by $-\gamma_1$ and $-\gamma_2$.

It can easily be verified that constants C_j $j=0,1,\ldots n$ exist such that

$$(5.17) \qquad \prod_{j=1}^{n} \frac{sh(w-i\alpha_j)}{sh(w-i\beta_j)} = C_0 + i \sum_{j=1}^{n} C_j \, cth(w-i\beta_j).$$

In fact we have

$$(5.18) \qquad C_0 = \cos\{\textstyle\sum(\alpha_j-\beta_j)\}$$

and

$$C_m = \prod_{j \neq m} \frac{\sin(\beta_m - \alpha_j)}{\sin(\beta_m - \beta_j)} \quad (5.19)$$

The identity (5.19) is obviously a generalization of (5.6) and its consequences are therefore similar to those of the latter relation.

Hence $R_m^{\pm}(r,\varphi)$ may be interpreted as the sum of contributions from a simple pole at the origin and a bundle of tails of normally directed dipoles. The number and the direction of these tails is determined by the scheme given above.

An illustration of the kind of reflections of the logarithmic pole (r_0,φ_0) with respect to the sides of the angle is given below in figure 4 [6]. We have chosen the following numerical values $\theta=45^\circ$ and $\gamma_1=\gamma_2=15^\circ$.

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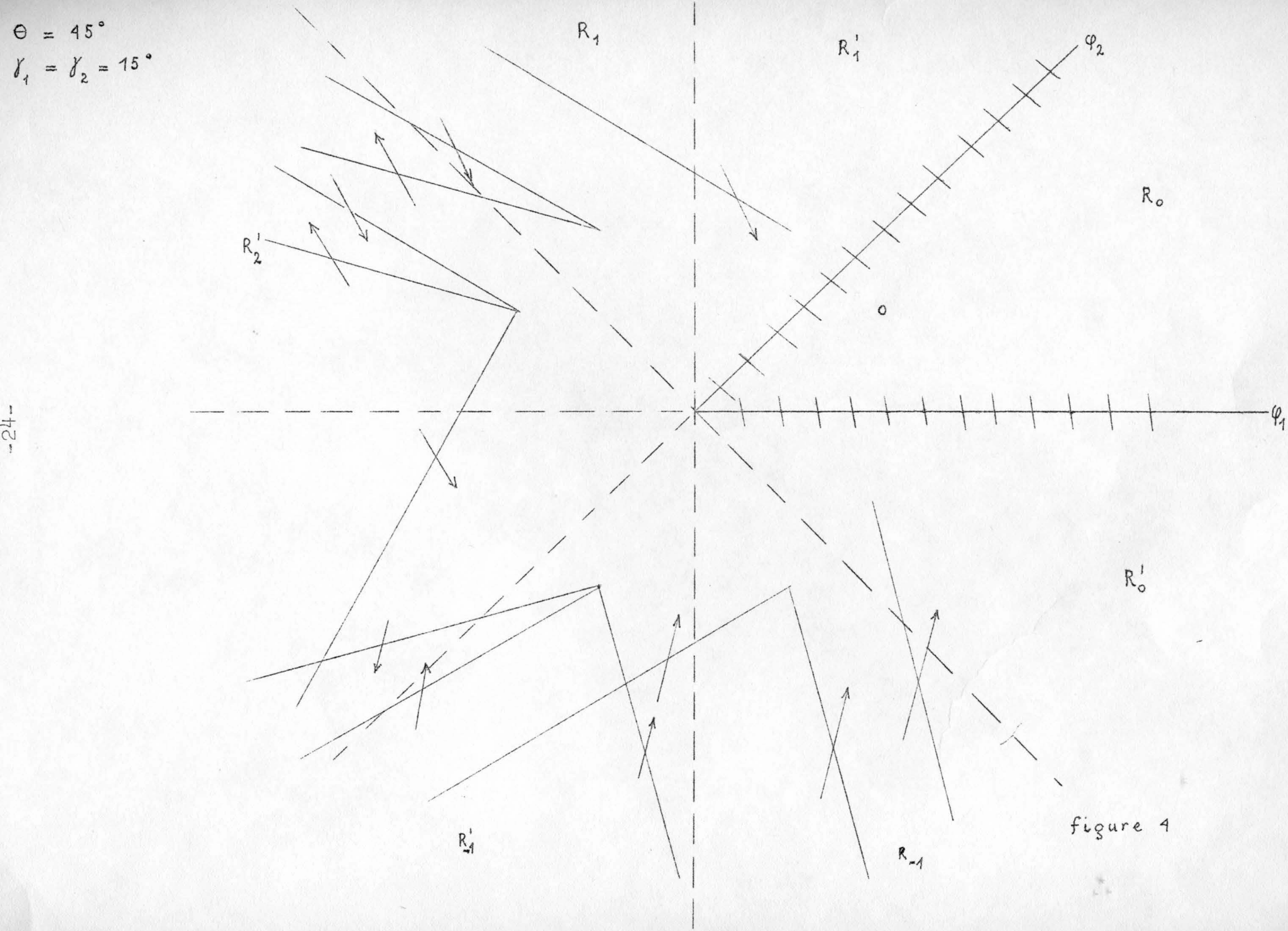
6) The author wishes to express his thanks to Dr D.J. Hofsommer who carefully checked the table of reflection angles and constructed the geometrical illustration of figure 4.



$\Theta = 45°$
$\gamma_1 = \gamma_2 = 15°$
R_1
R'_1
φ_2
R_0
R'_2
o
φ_1
R'_0
R'_{-1}
R_{-1}

figure 4

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