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MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

TOEGEPASTE WISKUNDE

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On the diffraction by a half-plane screen

by

D.J. Hofsommer

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1. Introduction

The two-dimensional diffraction problem for a half-plane screen has been solved by various authors, both for incident plane waves, c.f. e.g. [1],[4],[7] and for incident cylindrical waves, c.f. e.g. [2],[3],[6],[8]. The diffraction problem for cylindrical waves amounts to finding a Green's function. From this Green's function the solution of the plane wave problem can be obtained by letting the source travel to infinity. The solution of the plane wave diffraction problem, hence, is a limiting case of a Green's function.

In the present paper these well-known solutions will be rederived by a method which uses a Riemannian plane of two sheets with a branch-point at the edge of the screen. The solution for the half-plane diffraction problem with either Dirichlet or Neumann boundary conditions at the screen will be shown to depend on a Green's function for the full Riemannian plane, this function being uniquely determined by requiring that, on both sheets of the Riemannian plane, it vanish at infinity in a suitable way.

To fix ideas introduce polar coordinates r,ϑ with $0<r<\infty$ and $-2\pi<\vartheta<2\pi$ in the Riemannian plane in such a way that the screen coincides with the half-ray $\vartheta=0$ and the edge of the screen with the origin. A cut along the screen separates the Riemannian plane in a sheet $0<r<\infty$, $0<\vartheta<2\pi$ and a sheet $0<r<\infty$, $-2\pi<\vartheta<0$. In optical language these two sheets may be called "real space" and "virtual space" respectively.

Let $G(r,\vartheta,r_o,\vartheta_o)$, $0<r,r_o<\infty$, $-2\pi<\vartheta$, $\vartheta_o<2\pi$, be a Green's function, defined in the Riemannian plane without reference to the cut, which vanishes in a suitable way at infinity and which has a unit point-source at (r_o,ϑ_o). This Green's function satisfies $L(r,\vartheta)G(r,\vartheta,r_o,\vartheta_o)=$ $=-r^{-1}\delta(r-r_o)\delta(\vartheta-\vartheta_o)$, $L(r,\vartheta)$ being a linear differential operator. Now consider a "real" source, i.e. a source lying in real space, or $0<\vartheta_o<2\pi$, and its reflection with respect to the half-ray $\vartheta=0$. This is a "virtual" source, i.e. it lies in virtual space, and with it a Green's function $G(r,\vartheta,r_o,-\vartheta_o)$ is associated. Next construct the function

$$G_D(r,\vartheta,r_o,\vartheta_o) = G(r,\vartheta,r_o,\vartheta_o) - G(r,\vartheta,r_o,-\vartheta_o) \qquad (1.1)$$

This function satisfies the same conditions at infinity as $G(r,\vartheta,r_o,\vartheta_o)$. In general $G_D(r,0,r_o,\vartheta_o)\neq 0$. However, we shall prove that $G_D(r,0,r_o,\vartheta_o)=0$ if the differential operator $L(r,\vartheta)$ is symmetric in ϑ, or $L(r,\vartheta)=L(r,-\vartheta)$. The function $G(r,\vartheta,r_o,\vartheta_o)$ satisfies the differential equation

$L(r,\vartheta)G(r,\vartheta,r_o,\vartheta_o) = -r^{-1}\delta(r-r_o)\delta(\vartheta-\vartheta_o)$. In this equation replace ϑ by $-\vartheta$. Using the symmetry relation for $L(r,\vartheta)$ we find $L(r,\vartheta)G(r,-\vartheta,r_o,\vartheta_o) = -r^{-1}\delta(r-r_o)\delta(-\vartheta-\vartheta_o) = -r^{-1}\delta(r-r_o)\delta(\vartheta+\vartheta_o)$. This, however, is the differential equation for $G(r,\vartheta,r_o,-\vartheta_o)$. Assuming the conditions at infinity strong enough to secure a unique solution of the differential equation, we hence have the identity

$$G(r,\vartheta,r_o,-\vartheta_o) = G(r,-\vartheta,r_o,\vartheta_o) \qquad (1.2)$$

which holds for differential operators which are symmetrical in ϑ.

It now follows from (1.1) and (1.2) that G_D vanishes for $\vartheta=0$. The function G_D moreover has one real and one virtual source and hence is a Green's function for real space, which vanishes at the screen.

In a similar way we find that

$$\begin{aligned} G_N = (r,\vartheta,r_o,\vartheta_o) &= G(r,\vartheta,r_o,\vartheta_o) + G(r,\vartheta,r_o,-\vartheta_o) \\ &= G(r,\vartheta,r_o,\vartheta_o) + G(r,-\vartheta,r_o,\vartheta_o) \end{aligned}$$

represents a Green's function for real space whose normal derivative vanishes at the screen $\vartheta=0$.

It is seen that the applicability of the method of images, hence, depends on the symmetry properties of the differential operator.

2. Statement of the problem

Scalar diffraction in two dimensions is governed by the wave equation which, in Cartesian coordinates reads

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0. \qquad (2.1)$$

By means of a Laplace transformation with respect to ct we find

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - p^2\varphi = 0, \qquad \mathrm{Re}\ p > 0, \qquad (2.2)$$

where φ denotes the Laplace transform of ψ. Since we are interested in the quasi-stationary solution only, the right-hand member of (2.2) is put equal to zero. Another way to find the quasi-stationary solution, is to put $\psi = \varphi \exp i kct$. The resulting differential equation then is

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + k^2\varphi = 0. \qquad (2.2')$$

The former method is preferred in this paper, mainly because then φ must simply vanish at infinity, whilst in the latter method φ must satisfy there the more intricate Sommerfeld radiation condition. Our starting point, hence, will be the modified Helmholtz equation (2.2).

In accordance with the argument in section 1 we will derive the Green's function associated with equation (2.2) for a Riemannian plane of two sheets. This function satisfies the differential equation

$$\nabla^2 G - p^2 G = -\delta(\vec{r}-\vec{r}_0). \qquad (2.3)$$

The δ-function here represents a unit-source in one of the sheets of the Riemannian plane.
At infinity the Green's function will be required to satisfy the condition

$$\lim_{r\to\infty} G(\vec{r},\vec{r}_0) = 0. \qquad (2.4)$$

The case of an incident plane wave, which by a limiting process can be obtained from the case of a Green's function, will be treated independently. Since no source occurs in any finite region, the governing equation, again is given by (2.2). At infinity the condition (2.4) holds again, except for the direction of the incident wave. In that direction the solution φ should tend to the function φ_i that represents the incident wave. The function φ obtained in this way describes an incident plane wave in the Riemannian plane. Since it can be regarded to be a special case of a Green's function, the solution of the half-plane diffraction problem, again, can be found by adding or subtracting the reflected function.

3. Plane waves

As stated in the preceding section the governing differential equation in the case of incident plane waves reads

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}-p^2\varphi=0, \qquad (3.1)$$

An incident plane wave, which makes an angle ϑ_0 $(0<\vartheta_0<2\pi)$ with the screen is represented by

$$\varphi_i=\exp p(x\cos\vartheta_0+y\sin\vartheta_0), \qquad (3.2)$$

where the coordinate system is chosen in such a way, that the screen coincides with the positive X-axis. Now introduce a Riemannian plane with one branch-point, which coincides with the edge of the screen. By means of a parabolic transformation this Riemannian plane is mapped on an ordinary one. In the sequel we will try to solve our problem by aid of the method of separation of variables. The application of this method is simplified considerably if the incident wave itself assumes a separated form in parabolic coordinates. Fortunately this can be achieved by judiciously choosing the axes of the parabolic coordinate system. In effect by putting (see figure 1)

$$\left.\begin{aligned} x &= u\,v\sin\vartheta_0+\tfrac{1}{2}(u^2-v^2)\cos\vartheta_0,\\ y &= -u\,v\cos\vartheta_0+\tfrac{1}{2}(u^2-v^2)\sin\vartheta_0,\end{aligned}\right\} \qquad (3.3)$$

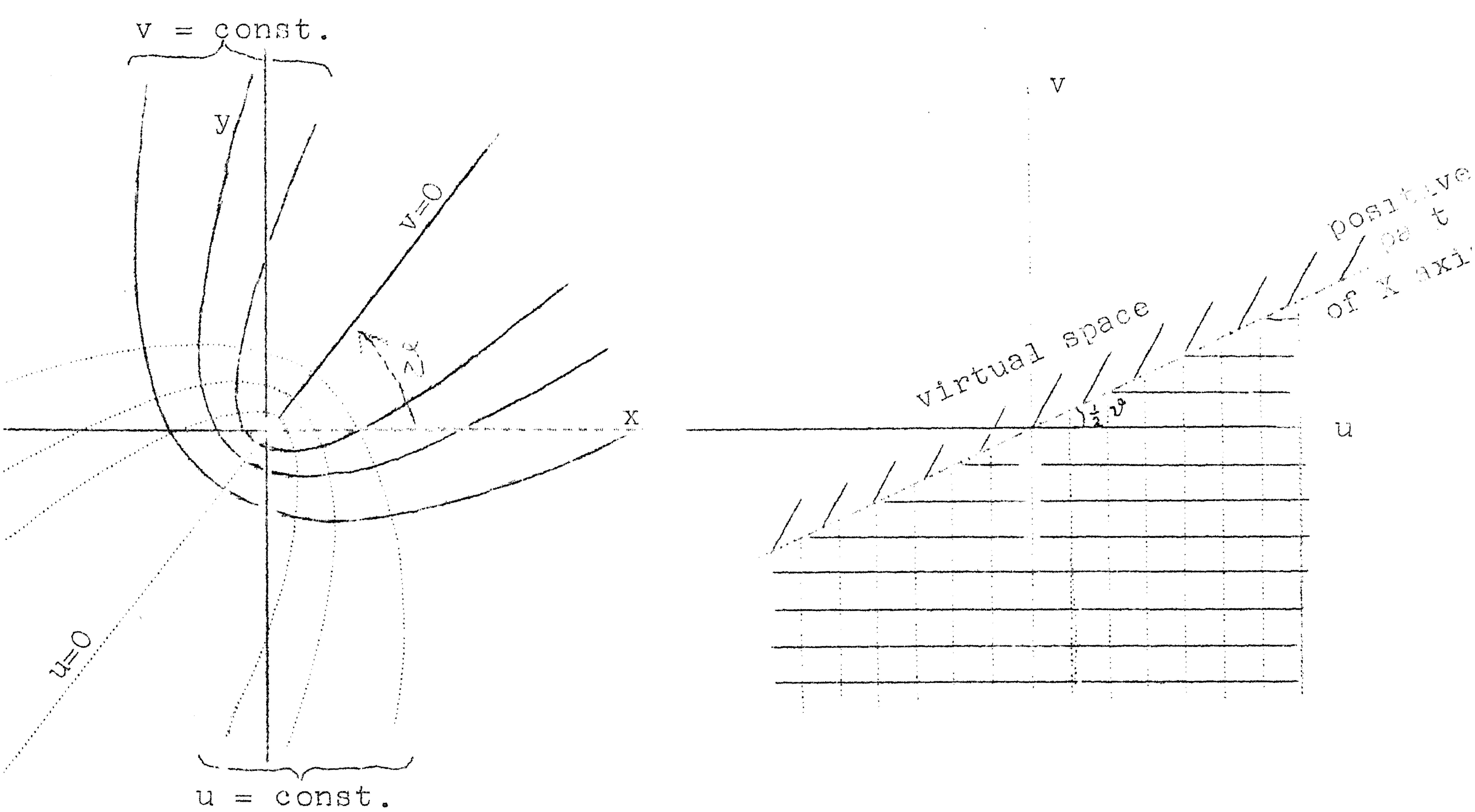

fig.1 The parabolic transformation 3.3

the incident plane waves assume the form

$$\varphi_i = \exp \tfrac{1}{2}p\,(u^2-v^2) = \exp \tfrac{1}{2}p\, u^2 \cdot \exp -\tfrac{1}{2}p\, v^2. \qquad (3.4)$$

LAMB [4] also introduces parabolic coordinates, but his u-axis coincides with the screen.

The ambiguity in (3.3) that both points (u,v) and (-u,-v) are representation of (x,y) will be removed by the requirement that the positive u-axis lies in real space, or, in other words, that the incident plane wave travels along the positive u-axis.

By means of (3.3) the modified Helmholtz equation (3.1) is transformed in the separable equation

$$\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2} - p^2(u^2+v^2)\varphi = 0, \qquad (3.5)$$

which, according to the requirements of section 2 should be solved under the conditions

$$\varphi \to 0 \quad , \quad \text{for } v \to \pm\infty , \qquad (3.6)$$

$$\varphi \to 0 \quad , \quad \text{for } u \to -\infty , \qquad (3.7)$$

$$\varphi \to \exp \tfrac{1}{2}p(u^2-v^2) \quad , \quad \text{for } u \to +\infty . \qquad (3.8)$$

Putting $\varphi(u,v) = U(u)V(v)$ we find

$$U'' - p(pu^2+n)U = 0, \qquad V'' - p(pv^2-n)V = 0, \qquad (3.9)$$

which differential equations determine parabolic cylinder functions.

It is found that (3.4) satisfies (3.9) for n=1. In general a solution of (3.5) will have the form of an infinite series of products of parabolic cylinder functions. In the present case, however, it seems plausible to try in the first instance a solution containing the terms for n=1 only. In that case independent solutions of (3.9) are

$$e^{\frac{1}{2}pu^2} \quad (3.10a) \quad , \quad e^{\frac{1}{2}pu^2}\int_{-\infty}^{u} e^{-pt^2}\,dt, \quad (3.10b)$$

$$e^{-\frac{1}{2}pv^2} \quad (3.10c) \quad , \quad e^{-\frac{1}{2}pv^2}\int_{0}^{v} e^{pt^2}\,dt. \quad (3.10d)$$

In view of the conditions (3.6) and (3.7) the functions (3.10a) and (3.10d) cannot occur. We thus are led to suppose a solution of the form

$$\varphi = Ae^{\frac{1}{2}p(u^2-v^2)} \int_{-\infty}^{u} e^{-pt^2}\,dt. \qquad (3.11)$$

For $u \to \infty$ it follows from (3.11) that

$$\varphi \to A\sqrt{\pi/p}\; e^{\frac{1}{2}p(u^2-v^2)},$$

which is identical with condition (3.8) provided we take $A = \sqrt{p/\pi}$.

Substitution of this value for A in (3.11) finally yields

$$\varphi = \sqrt{p/\pi}\; e^{\frac{1}{2}p(u^2-v^2)} \int_{-\infty}^{u} e^{-pt^2}dt$$
$$= \tfrac{1}{2}e^{\frac{1}{2}p(u^2-v^2)}\operatorname{erfc}(-u\sqrt{p}). \qquad (3.12)$$

This function satisfies all conditions and thus validates our initial guess, that only the solutions of (3.9) with n=1 play a part in the present problem. We also could have started with a trial solution in the form of a series of products of parabolic cylinder functions. We would then have found, of course, that all expansion coefficients, except one, became zero.

Equation (3.12) describes the behaviour of an incident plane wave in a Riemannian plane of two sheets with one branch-point. Before constructing a solution of the half-plane diffraction problem with Dirichlet or Neumann conditions at the screen by means of the method of images, we must ascertain, that this method is applicable. The screen is given by $x > 0$, $y=0$, or $v=u \tan \frac{1}{2}\vartheta_0$, and reflection should be carried out with respect to this line. It is easily seen that the operator $L(u,v) = \partial^2/\partial u^2 + \partial^2/\partial v^2 - (u^2+v^2)$ is symmetric with respect to every line through the origin. An analysis, analogous to that given in section 1 for polar coordinates, then shows that the method of images indeed is applicable if reflection is carried out with respect to a line through the origin. The image of a point P(u,v) with respect to the line $v=u \tan\frac{1}{2}\vartheta_0$ is the point $P'(u\cos\vartheta_0 + v\sin\vartheta_0,\; u\sin\vartheta_0 - v\cos\vartheta_0)$. The reflection φ_r of the field (3.12) then is

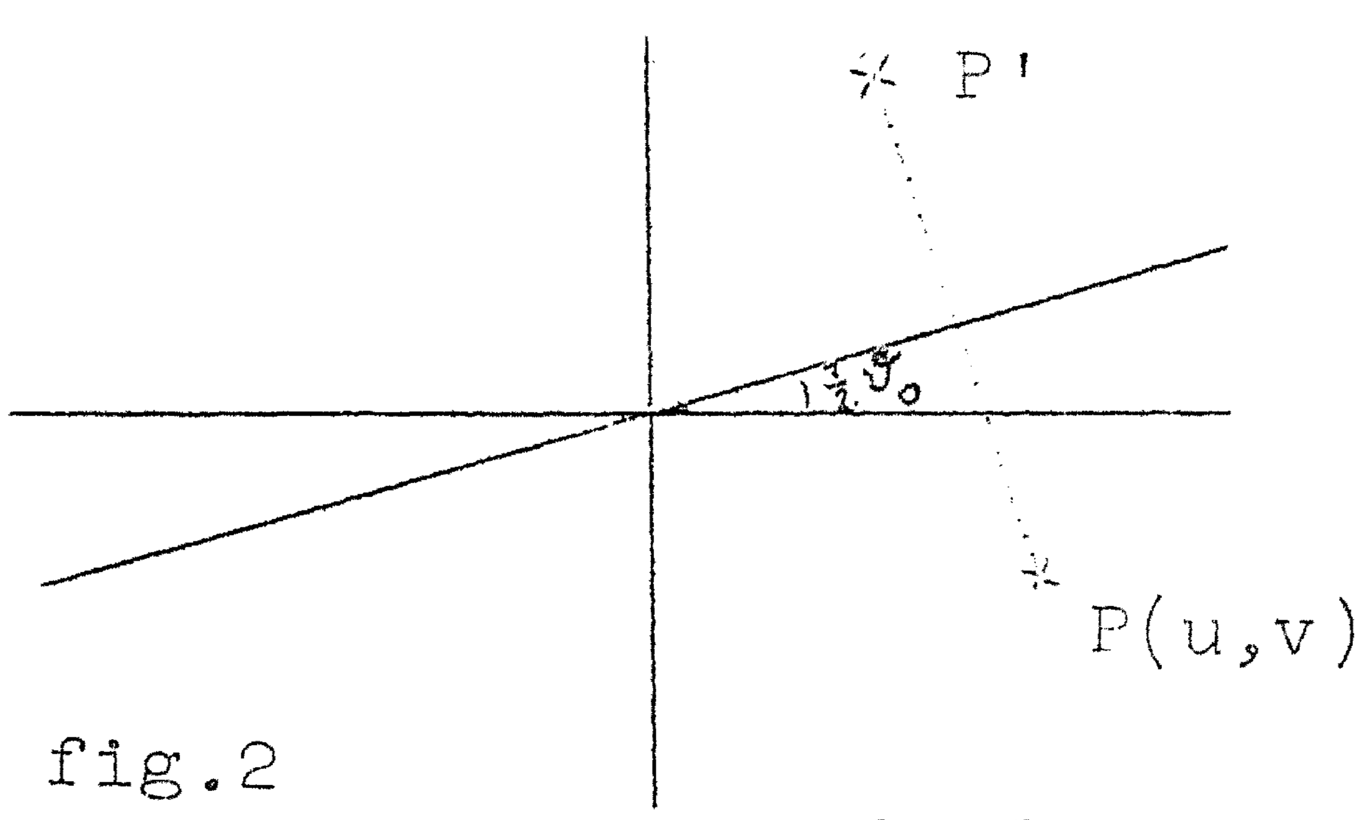

fig.2

$$\varphi_r = \tfrac{1}{2}e^{\frac{1}{2}p(u^2-v^2)\cos 2\vartheta_0 + puv\sin 2\vartheta_0}\operatorname{erfc}(-u\sqrt{p}\cos\vartheta_0 - v\sqrt{p}\sin\vartheta_0). \qquad (3.13)$$

We can reduce (3.12) and (3.13) to a more familiar form by introducing polar coordinates r, ϑ by means of

$$u = \sqrt{2r}\cos\tfrac{1}{2}(\vartheta_0-\vartheta),$$
$$v = \sqrt{2r}\sin\tfrac{1}{2}(\vartheta_0-\vartheta). \qquad (3.14)$$

Substitution of these values in (3.3) yields

$$x = r\cos\vartheta, \quad y = r\sin\vartheta,$$

as it should be. The screen is given by $\vartheta=0$, and the solutions (3.12) and (3.13) become

$$\varphi = \tfrac{1}{2}e^{pr\ \cos(\vartheta-\vartheta_0)}\operatorname{erfc}\left[-\sqrt{2pr}\ \cos\tfrac{1}{2}(\vartheta-\vartheta_0)\right], \qquad (3.15)$$

$$\varphi_r = \tfrac{1}{2}e^{pr\ \cos(\vartheta+\vartheta_0)}\operatorname{erfc}\left[-\sqrt{2pr}\ \cos\tfrac{1}{2}(\vartheta+\vartheta_0)\right], \qquad (3.16)$$

4. Green's function. First method

In general a cylindrical wave can be decomposed in plane waves. For the full one-valued x-y plane this decomposition appears in the following reduction.

Green's function for the full one-valued x-y plane, belonging to the modified Helmholtz equation (2.2) is that solution of

$$\frac{\partial^2 G_o}{\partial x^2} + \frac{\partial^2 G_o}{\partial y^2} - p^2 G_o = -\delta(x-x_o)\delta(y-y_o), \qquad (4.1)$$

which vanishes at infinity. In fact

$$G_o = \frac{1}{2\pi} K_o(pR) , \qquad (4.2)$$

where

$$R = \sqrt{(x-x_o)^2 + (y-y_o)^2},$$

or, in polar coordinates $x = r\cos\vartheta$, $y = r\sin\vartheta$, $x_o = r_o\cos\vartheta_o$, $y_o = r_o\sin\vartheta_o$,

$$R = \sqrt{r^2+r_o{}^2-2rr_o\cos(\vartheta-\vartheta_o)}, \qquad (4.3)$$

is the distance between the points (r,ϑ) and (r_o,ϑ_o).

Using a well-known integral representation for the modified Bessel function K_o, (4.2) can be written as

$$G_o = \frac{1}{4\pi}\int_{-\infty}^{\infty} \exp(-pR\operatorname{ch}t)dt. \qquad (4.4)$$

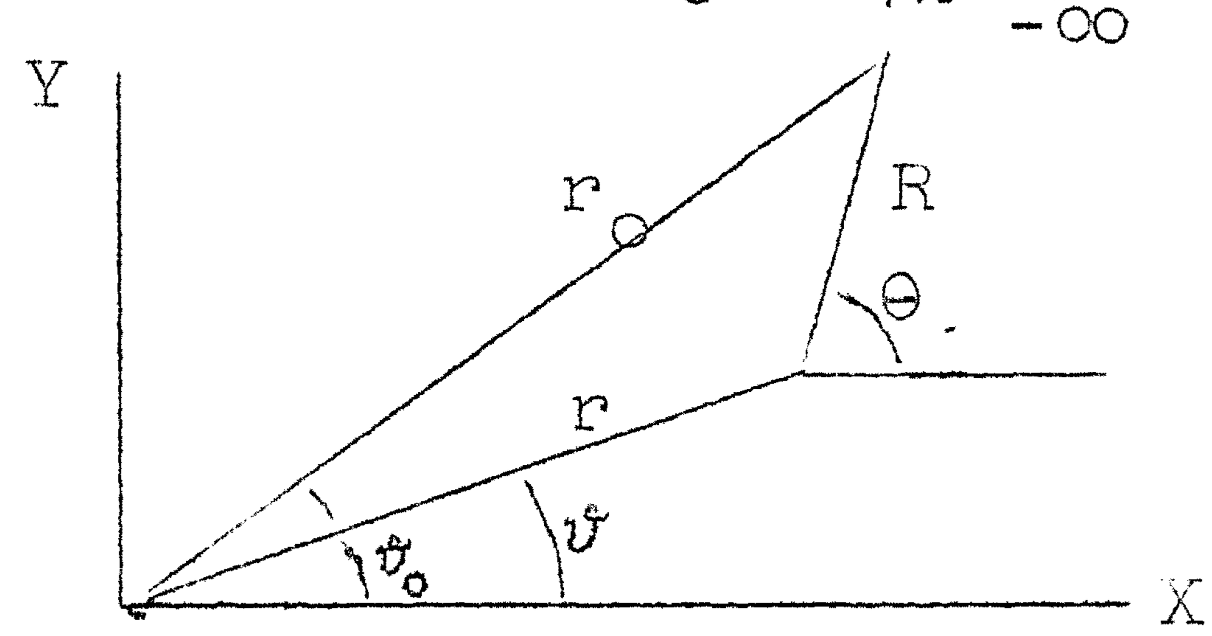

fig.3

Now define (comp. fig.3)

$$Re^{i\theta} = r_o e^{i\vartheta_o} - re^{i\vartheta} . \qquad (4.5)$$

Hence

$$Re^t = r_o e^{t+i(\vartheta_o-\theta)} - re^{t+i(\vartheta-\theta)} . \qquad (4.6)$$

Substitution of (4.6) in (4.4) gives

$$G_o = \frac{1}{4\pi}\int_{-\infty}^{\infty} \exp\left[-pr_o\cos(\vartheta_o-\theta-it)\right] \exp\left[pr\cos(\vartheta-\theta-it\right] dt. \qquad (4.7)$$

The factor $\exp[pr\cos(\vartheta-\theta-it)]$ in the integrand has the form of a plane wave which is "inclined" over an "angle" $\theta+it$ to the x-axis. The factor $\exp[-pr_o\cos(\vartheta_o-\theta-it)]$ may be regarded as a kind of amplitude. In (4.7), hence, the decomposition of a cylindrical wave in plane waves is achieved.

The representation (4.7) of the Green's function for real space can be used for the determination of the Green's function for the

Riemannian plane in the following way. In the preceding section, the incident plane wave for Riemannian plane was required to tend at infinity to the incident plane wave for real space. We shall prescribe the same requirement for the respective Green's functions. Since in (4.7) the Green's function for real space is expressed in terms of incident plane waves this simply means that in (4.7) these plane waves have to be replaced by the corresponding plane waves in the Riemannian plane, i.e. by

$$\tfrac{1}{2}\exp\left[\, pr\cos(\vartheta-\theta-it)\right] \operatorname{erfc}\left[-\sqrt{2pr}\,\cos\tfrac{1}{2}(\vartheta-\theta-it)\right] .$$

The Green's function for the Riemannian plane hence becomes

$$G(r,\vartheta,r_o,\vartheta_o) = \frac{1}{8\pi}\int_{-\infty}^{\infty} \exp(-pR\operatorname{ch}t)\operatorname{erfc}\left[-\sqrt{2pr}\,\cos\tfrac{1}{2}(\vartheta-\theta-it)\right]dt. \tag{4.8}$$

The remaining part of this section will be devoted to reducing the representation (4.8) to

$$G(r,\vartheta,r_o,\vartheta_o) = \frac{1}{4\pi}\int_{-\infty}^{\beta} \exp(-pR\operatorname{ch}y)\,dy, \qquad \beta = \operatorname{arsh} 2(rr_o)^{\frac{1}{2}}R^{-1}\cos\tfrac{1}{2}(\vartheta-\vartheta_o). \tag{4.9}$$

If this Green's function is used for constructing Green's functions for real space with Dirichlet or Neumann conditions at a half-plane the results are MacDonald's formula [6] . The following reduction of (4.8) to (4.9) is due to Lauwerier [5] .

In (4.8) we introduce the substitution $t=u_o-i(\vartheta_o-\theta)$. We then find

$$G = \frac{1}{8\pi}\int_{-\infty}^{\infty} \exp(-pr_o\cosh u_o)\exp\left[pr\cos(\vartheta-\vartheta_o-iu_o)\right] \operatorname{erfc}\left[-\sqrt{2pr}\,\cos\tfrac{1}{2}(\vartheta-\vartheta_o-iu_o)\right]du_o . \tag{4.10}$$

For the complementary error function the following integral representation is known

$$\exp z^2 \operatorname{erfc} z = \frac{1}{\pi}\, z\int_0^{\infty} \frac{\exp(-t)\,dt}{t^{\frac{1}{2}}(t+z^2)} , \quad \operatorname{Re} z>0. \tag{4.11}$$

By means of the substitutions $z=\sqrt{2pr}\,\cos\frac{1}{2}\gamma$, $t=2pr\sinh^2\frac{1}{2}u$ (4.11) becomes

$$\begin{aligned}
&\exp(pr\cos\gamma)\operatorname{erfc}(\sqrt{2pr}\,\cos\tfrac{1}{2}\gamma) = \\
&\frac{2}{\pi}\cos\tfrac{1}{2}\gamma\int_0^{\infty} \frac{\exp(-pr\cosh u)\cosh\frac{1}{2}u}{\cosh u+\cos\gamma}\,du = \\
&\frac{1}{2\pi}\int_0^{\infty}\exp(-pr\cosh u)\left[\frac{1}{\cosh\frac{1}{2}(u+i\gamma)}+\frac{1}{\cosh\frac{1}{2}(u-i\gamma)}\right]du = \\
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp(-pr\cosh u)}{\cosh\frac{1}{2}(u+i\gamma)}\,du, \qquad -\pi<\operatorname{Re}\gamma<\pi .
\end{aligned} \tag{4.12}$$

A simple direct derivation of (4.12) was given by Lauwerier [5]. If $|\vartheta-\vartheta_o|>\pi$ (4.12) is not immediately applicable to (4.10). However, by means of the property erfc(-z)=2-erfz we also have

$$G = \frac{1}{4\pi}\int_{-\infty}^{\infty} \exp(-p\,R\cosh t)dt + g, \tag{4.13}$$

$$g = -\frac{1}{8\pi}\int_{-\infty}^{\infty} \exp(-pr_o\cosh u_o)\exp[pr\cos(\vartheta-\vartheta_o-iu_o)]\,\mathrm{erfc}\,[\sqrt{2pr}\cos\tfrac{1}{2}(\vartheta-\vartheta_o-iu_o)]\,du_o,$$

or, introducing (4.12) in the latter formula,

$$g = -\frac{1}{16\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp[-p(r\cosh u+r_o\cosh u_o]}{\cosh\frac{1}{2}(u+u_o+i\vartheta-i\vartheta_o)}\,du\,du_o. \tag{4.14}$$

By means of the substitutions $u=x+y$, $u_o=x-y$ (4.14) becomes

$$g = -\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\frac{1}{\cosh[x+\frac{1}{2}i(\vartheta-\vartheta_o)]}\int_{-\infty}^{\infty}\exp[-pr\cosh(x+y)-pr_o\cosh(x-y)]\,dy\,dx.$$

The integral over y is of type (4.4) and yields a modified Bessel function. In effect

$$g = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty}K_o[p\sqrt{(r+r_o)^2+4rr_o\sinh^2x}]\,\frac{dx}{\cosh[x+\frac{1}{2}i(\vartheta-\vartheta_o)]}. \tag{4.15}$$

It is easily seen that replacing x by -x in (4.15) is aequivalent to replacing $\vartheta-\vartheta_o$ by $-\vartheta+\vartheta_o$. The imaginary part of g, hence, vanishes. Taking the real part we find

$$g = -\frac{1}{2\pi^2}\int_0^{\infty}K_o[p\sqrt{(r+r_o)^2+4rr_o\sinh^2x}]\,\frac{\cos\frac{1}{2}(\vartheta-\vartheta_o)\cosh x}{\sinh^2x+\cos^2\frac{1}{2}(\vartheta-\vartheta_o)}\,dx.$$

The further substitutions $\sinh x = t\cos\frac{1}{2}(\vartheta-\vartheta_o)$, $r+r_o=R\cosh\beta$, where $R^2 = r^2+r_o^2-2rr_o\cos(\vartheta-\vartheta_o)$, yield

$$g = -\frac{1}{2\pi^2}\int_0^{\infty}K_o[pR\sqrt{t^2\sinh^2\beta+\cosh^2\beta}]\,\frac{dt}{t^2+1}, \tag{4.16}$$

and, by differentiation of g with respect to β,

$$\frac{\partial g}{\partial\beta} = \frac{pR}{2\pi^2}\,\mathrm{sh}\beta\,\mathrm{ch}\beta\int_0^{\infty}\frac{K_1[pR\sqrt{t^2\sin^2h^2\beta+\cosh^2\beta}]}{\sqrt{t^2\sinh^2\beta+\cosh^2\beta}}\,dt. \tag{4.17}$$

By means of the known integral

$$\int_0^{\infty}\frac{K_1(\sqrt{a^2t^2+b^2})}{\sqrt{a^2t^2+b^2}} = \frac{\pi\exp(-b)}{2ab},\quad a>0,\ b>0, \tag{4.18}$$

it is seen that $\partial g/\partial\beta$ equals

$$\frac{\partial g}{\partial \beta} = \frac{1}{4\pi} \exp(-pR \cosh \beta), \qquad \beta > 0,$$

or,

$$g = -\frac{1}{4\pi} \int_{\beta}^{\infty} \exp(-pR \cosh t)dt. \qquad (4.19)$$

Substitution of (4.19) in (4.13) finally gives

$$G = \frac{1}{4\pi} \int_{-\infty}^{\beta} \exp(-pR \cosh t)dt, \qquad (4.20)$$

with $\beta = \operatorname{ar\,cosh} \frac{r+r_0}{R} = \operatorname{arsinh} 2(rr_0)^{\frac{1}{2}}R^{-1}\cos \frac{1}{2}(\vartheta - \vartheta_0)$.

The integral (4.18) is a special case of a formula of Sonine; a simple direct derivation has been given by Lauwerier [5].
The result (4.20) has been obtained under the restriction $|\vartheta - \vartheta_0| < \pi$. However, it is seen from (4.10) that G is an analytic function of $\vartheta - \vartheta_0$. It then follows from the principle of analytic continuation that (4.20) is valid for all values of $\vartheta - \vartheta_0$.

5. Green's function. Second method

The modified Helmholtz equation reads in polar coordinates $x = r\cos\vartheta$, $y = r\sin\vartheta$,

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\vartheta^2} - p^2\varphi = 0. \qquad (5.1)$$

The Riemannian space is delimited by $0 < r < \infty$, $-2\pi < \vartheta < 2\pi$. A Green's function $G(r,\vartheta,r_0,\vartheta_0)$ with its singularity in (r_0,ϑ_0) satisfies the inhomogeneous equation

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial^2 G}{\partial\vartheta^2} - p^2 G = -\frac{1}{r}\,\delta(r-r_0)\delta(\vartheta-\vartheta_0), \qquad (5.2)$$

or, putting $\vartheta = \alpha + \vartheta_0$,

$$r\frac{\partial^2 G}{\partial r^2} + \frac{\partial G}{\partial r} + \frac{1}{r}\frac{\partial^2 G}{\partial\alpha^2} - p^2 rG = -\delta(r-r_0)\delta(\alpha). \qquad (5.3)$$

It follows from (5.3) that G is an even function of α. We now try a trigonometric expansion for G. Since G is an even function of α, only cosine term can occur. Also, since in the Riemannian space the two pairs of coordinates (r,ϑ) and $(r,\vartheta+4\pi)$ represent the same point, the cosines must have periods 4π. The expansion, hence, must be of the form

$$G = \sum_0^\infty A_m(r)\cos\tfrac{1}{2}m\alpha\ . \qquad (5.4)$$

Substitution of (5.4) in (5.3) gives

$$\sum_0^\infty \left[rA_m'' + A_m' - \frac{m^2}{4r}A_m - p^2 r\,A_m\right]\cos\tfrac{1}{2}m\alpha = -\delta(r-r_0)\delta(\alpha).$$

Multiplication by $\cos\frac{1}{2}n\alpha$ and integration over the interval $(-2\pi, 2\pi)$ yields

$$rA_m'' + A_m' - \frac{m^2}{4r}A_m - p^2 r\,A_m = -\frac{\varepsilon_m}{4\pi}\,\delta(r-r_0),$$

where $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ $(m \geqslant 1)$.

The solution of this inhomogeneous modified Bessel equation is given by

$$A_m = \begin{cases} \dfrac{\varepsilon_m}{4\pi}\, I_{\frac{1}{2}m}(pr)\; K_{\frac{1}{2}m}(pr_0), & r \leqslant r_0, \\[2mm] \dfrac{\varepsilon_m}{4\pi}\, K_{\frac{1}{2}m}(pr)\; I_{\frac{1}{2}m}(pr_0), & r \geqslant r_0\ . \end{cases} \qquad (5.5)$$

First consider the case $r \leqslant r_0$. Substitution of (5.5) in (5.4) yields

$$G = \frac{1}{4\pi}\sum_{m=0}^\infty \varepsilon_m\, I_{\frac{1}{2}m}(pr)\; K_{\frac{1}{2}m}(pr_0)\cos\tfrac{1}{2}m\alpha\ ,$$

or,

$$G = \frac{1}{4\pi}\sum_{m=0}^{\infty} \varepsilon_m\, I_m(pr)K_m(pr_o)\cos m\alpha + \frac{1}{2\pi}\sum_{m=0}^{\infty} I_{m+\frac{1}{2}}(pr)K_{m+\frac{1}{2}}(pr_o)\cos(m+\tfrac{1}{2})\alpha\ . \tag{5.6}$$

The first series in (5.6) is a well-known representation of $K_o(pR)$ where $R^2=r^2+r_o^2-2rr_o\cos\alpha$. Denoting the second series by $\frac{1}{2}\bar{F}(\alpha)$, (5.6) can be written in the form

$$G = \frac{1}{4\pi}\left[K_o(pR) + \bar{F}(\alpha)\right] = \frac{1}{4\pi}\left[\int_{-\infty}^{\infty} \exp(-pR\cosh y)+\bar{F}(\alpha)\right], \tag{5.7}$$

$$\bar{F}(\alpha) = 2\sum_{m=0}^{\infty} I_{m+\frac{1}{2}}(pr)K_{m+\frac{1}{2}}(pr_o)\cos(m+\tfrac{1}{2})\alpha\ . \tag{5.8}$$

If we put $p=\sqrt{s}$, a single term

$$2I_{m+\frac{1}{2}}(r\sqrt{s})K_{m+\frac{1}{2}}(r_o\sqrt{s}), \qquad r < r_o\ , \tag{5.9}$$

is the Laplace transform of

$$\frac{1}{r}\, I_{m+\frac{1}{2}}\left(\frac{rr_o}{2t}\right)\exp\left(-\frac{r^2+r_o^2}{4t}\right)\ . \tag{5.10}$$

Since the case $r \geqslant r_o$ can be obtained from the case $r \leqslant r_o$ by interchanging r with r_o, it follows that we obtain the same inverse Laplace transform if $r \geqslant r_o$.

By means of the known integral representation

$$I_{m+\frac{1}{2}}(z) = \frac{(\frac{1}{2}z)^{m+\frac{1}{2}}}{m!\sqrt{\pi}}\int_{-1}^{1}(1-u^2)^m \exp(-zu)du, \tag{5.11}$$

we can write (5.10) in the form

$$\frac{1}{m!t\sqrt{\pi}}\exp\left(-\frac{r^2+r_o^2}{4t}\right)\left(\frac{rr_o}{4t}\right)^{m+\frac{1}{2}}\int_{-1}^{1}(1-u^2)^m\exp\left(-\frac{rr_ou}{2t}\right)du =$$
$$\frac{1}{t}\sqrt{\frac{rr_o}{4\pi t}}\exp\left(-\frac{r^2+r_o^2}{4t}\right)\int_{-1}^{1}\left[\frac{rr_o}{4t}(1-u^2)\right]^m\exp\left(\frac{rr_ou}{2t}\right)\frac{du}{m!}\ .$$

Hence $\bar{F}(\alpha,s)$ is the Laplace transform of

$$F(\alpha,t) = \frac{1}{t}\sqrt{\frac{rr_o}{4\pi t}}\exp -\left(\frac{r^2+r_o^2}{ut}\right)\sum_{m=0}^{\infty}\int_{-1}^{1}\left[\frac{rr_o}{4t}(1-u^2)\right]^m \exp\left(-\frac{rr_ou}{2t}\right)\frac{du}{m!}\cos(m+\tfrac{1}{2})\alpha$$
$$= \frac{1}{t}\sqrt{\frac{rr_o}{4\pi t}}\exp\left(-\frac{r^2+r_o^2}{4t}\right)$$
$$\mathrm{Re}\left[e^{\frac{1}{2}i\alpha}\sum_{m=0}^{\infty}\int_{-1}^{1}\left\{\frac{rr_o}{4t}(1-u^2)e^{i\alpha}\right\}^m\exp\left(-\frac{rr_ou}{2t}\right)\frac{du}{m!}\right].$$

Interchanging integration and summation and performing the summation gives

$$F(\alpha,t) = \frac{1}{t}\sqrt{\frac{rr_o}{4\pi t}}\exp(-\frac{R^2}{4t})\mathrm{Re}\left[e^{\frac{1}{2}i\alpha}\int_{-1}^{1}\exp\left\{-\frac{rr_o}{4t}(ue^{i\alpha}+e^{-i\alpha})^2\right\}du\right]. \qquad (5.12)$$

By means of the substitution

$$ue^{\frac{1}{2}i\alpha} + e^{-\frac{1}{2}i\alpha} = \frac{R}{\sqrt{rr_o}}\sinh y,$$

(5.12) is transformed in

$$F(\alpha,t) = \frac{R}{2t\sqrt{\pi t}}\exp(-\frac{R^2}{4t})\mathrm{Re}\int_{-i\gamma}^{\beta}\exp(-\frac{R^2\sinh^2 y}{4t})\cosh y\,dy,$$

where $\beta = \operatorname{arsinh}\left[2R^{-1}(rr_o)^{\frac{1}{2}}\cos\frac{1}{2}\alpha\right]$ and $\gamma = \arcsin\left[2R^{-1}(rr_o)^{\frac{1}{2}}\sin\frac{1}{2}\alpha\right]$. Deforming the path of integration in such a way that it runs along the imaginary axis from $-i\gamma$ to the origin and along the real axis from the origin to β, it is easily seen that these two parts determine the imaginary and the real part of the integral respectively. Hence

$$F(\alpha,t) = \frac{R}{2t\sqrt{\pi t}}\exp(-\frac{R^2}{4t})\int_0^{\beta}\exp(-\frac{R^2\sinh^2 y}{4t})\cosh y\,dy. \qquad (5.13)$$

Returning to the function $\overline{F}(\alpha,s)$ we have by Laplace transformation of (5.13)

$$\overline{F}(\alpha,s) = \frac{1}{2}\pi R\int_0^{\infty}(\pi t)^{-1\frac{1}{2}}\exp(-ts)\exp(-\frac{R^2}{4t})\int_0^{\beta}\exp(-\frac{R^2\sinh^2 y}{4t})\cosh y\,dy\,dt$$

$$= \frac{1}{2}\pi R\int_0^{\infty}\int_0^{\beta}(\pi t)^{-1\frac{1}{2}}\exp(-ts-\frac{R^2\sinh^2 y}{4t})\cosh y\,dy\,dt.$$

Now interchange the integrations and put

$$t = \frac{R}{2\sqrt{s}}e^{x}\cosh y.$$

The result is

$$\overline{F}(\alpha,s) = \sqrt{\frac{R\sqrt{s}}{2\pi}}\int_0^{\beta}\cosh^{\frac{1}{2}}y\int_{-\infty}^{\infty}\exp(-R\sqrt{s}\cosh y\cosh x)e^{-\frac{1}{2}x}\,dx\,dy$$

$$= \sqrt{\frac{2R\sqrt{s}}{\pi}}\int_0^{\beta}\cosh^{\frac{1}{2}}y\int_0^{\infty}\cosh\tfrac{1}{2}x\exp(-R\sqrt{s}\cosh y\cosh x)\,dx\,dy,$$

or, by means of the integral representation

$$K_\nu(z) = \int_0^{\infty}\cosh\nu x\exp(-z\cosh x)\,dx,$$

$$\overline{F}(\alpha,s) = \sqrt{\frac{2R\sqrt{s}}{\pi}}\int_0^{\beta}\cosh^{\frac{1}{2}}y\,K_{\frac{1}{2}}(R\sqrt{s}\cosh y)\,dy.$$

Since

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z},$$

(5.14) equals

$$\overline{F}(\alpha,s) = \int_{0}^{\beta} \exp(-R\sqrt{s}\ \cosh y)dy.$$

Remembering that $s=p^2$ it follows from (5.7) that Green's function becomes

$$G = \frac{1}{4\pi} \int_{-\infty}^{\beta} \exp(-p\ R\ \cosh y)dy,$$

in accordance with (4.9).

6. Commentary

In the present paper the two-dimensional half-plane diffraction problem with Dirichlet or Neumann conditions along the half plane has been solved by means of a Green's function for a two-valued Riemannian plane. Such a Green's function has one logarithmic singularity and vanishes at the boundary of the region, i.e. at infinity.

In the usual direct approach a Green's function is sought for a region which extends to infinity, but is also bounded by the half-plane. Besides the usual boundary conditions and the prescribed behaviour at the singularity this Green's function is also required to satisfy a condition of quadratic integrability. The question then may be raised whether such a condition would not have been necessary for the determination of the Green's function for the Riemannian plane, in other words, whether the Green's function obtained in this paper is unique for the used conditions. Although the conditions are known to suffice for a one-valued plane in the case of a modified Helmholtz equation, it might be surmised that the presence of a branch-point would introduce a difficulty. That this is not the case can be shown in the following way.

The Helmholtz equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} - p^2\varphi = 0$$

reads in parabolic coordinates $x=\frac{1}{2}(u^2-v^2)$; $y=uv$:

$$\frac{\partial^2\varphi}{\partial u^2} + \frac{\partial^2\varphi}{\partial v^2} - p^2(u^2+v^2)\varphi = 0.$$

The corresponding equation for Green's function reads

$$\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2} - p^2(u^2+v^2)G = -\delta(u-u_0)\delta(v-v_0).$$

The parabolic transformation serves to map the two-valued x-y plane on a one-valued u-v plane. In the latter plane, however, G is known to be uniquely determined by the condition that it vanishes at infinity. It, hence, is also uniquely determined in the original two-valued x-y plane.

Considering the two m thods which in the present paper were used for obtaining the Green's function the first method constructs it out of plane waves with a variable angle of incidence. In the second method a general technique for obtaining a series representation of a Green's function is used. An analysis, akin to that of Mac Donald

[6], then transforms this series in an integral.
Both methods have in common that, although use is made of a two-valued Riemannian plane, paths of integration which occur during the evaluations, always lie in a one-valued complex plane. In this respect they differ from Sommerfeld's original method [2], [7] which uses paths of integration which lie in multi-valued complex Riemannian planes.

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