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# Solutions of the Equation of Helmholtz in an Angle V.

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(The Case of a Halfplane)
in boundary conditions

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MATHEMATICS

# SOLUTIONS OF THE EQUATION OF HELMHOLTZ IN AN ANGLE *). V

## THE CASE OF A HALFPLANE

BY

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## 1. *Introduction*

This paper is an elaboration of the preceding paper for the special case of a halfplane. Therefore, it may be considered as a companion to the third paper in this series in which the same problem has been solved with simpler boundary conditions.

The problem to be considered here may be formulated in Cartesian coordinates by

$$(1.1) \qquad y>0 \qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-1\right)G(x, y, x_0, y_0) = -\delta(x-x_0)\delta(y-y_0),$$

$$(1.2) \qquad y=0,\ x>0 \qquad \cos\gamma_1\frac{\partial G}{\partial y}-\sin\gamma_1\frac{\partial G}{\partial x}-\operatorname{ch}\alpha_1\, G = 0,$$

$$(1.3) \qquad y=0,\ x<0 \qquad \cos\gamma_2\frac{\partial G}{\partial y}-\sin\gamma_2\frac{\partial G}{\partial x}-\operatorname{ch}\alpha_2\, G = 0.$$

It will be assumed that for $j=1, 2$

$$(1.4) \qquad -\tfrac{1}{2}\pi<\operatorname{Re}\gamma_j\leqq\tfrac{1}{2}\pi \quad , \quad 0\leqq\operatorname{Im}\alpha_j<\pi.$$

The notation differs slightly from that of the previous paper where in the boundary conditions (1.2) and (1.3) $\operatorname{ch}\alpha_j$ was replaced by $\sin\beta_j$. However, this means that whenever comparison is needed we may perform the substitution $\alpha_j=(\tfrac{1}{2}\pi-\beta_j)\mathrm{i}$.

The treatment of this paper is very similar to that of the general problem in IV. The Hilbert problem which was studied there reduces now to a Wiener-Hopf problem of the ordinary kind.

The factorization of the kernel function solves the homogeneous problem which is the subject of section 2. The finding of a Green's function which is considered in section 3 is now an easy matter. In the final section the particular case is studied which arises when there is a single boundary

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condition for the total $X$-axis. However, the solution of the latter problem does not generally admit of a simple geometrical interpretation as that given in II section 4.

It is intended to end this series with an extension of the well-known Sommerfield problem of diffraction round an edge.

2. *The F-problem*

Putting

$$F(x, y) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp \{-(\mathrm{i}x \operatorname{sh} w + y \operatorname{ch} w)\} f(w) dw, \tag{2.1}$$

the boundary conditions (1.2) and (1.3) give

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x \operatorname{sh} w} f_j(w) \operatorname{ch} w \, dw = 0, \tag{2.2}$$

with $j=1$ for $x>0$ and $j=2$ for $x<0$, and where $f_j(w)$ is defined by

$$\operatorname{ch} w \, f_j(w) \stackrel{\text{def}}{=} \{\operatorname{ch}(w - \mathrm{i}\gamma_j) + \operatorname{ch} \alpha_j\} f(w). \tag{2.3}$$

The function $f_1(w)$ should be holomorphic in the lower strip $-\pi < \operatorname{Im} w < 0$ with a point of symmetry at $-\frac{1}{2}\pi\mathrm{i}$, the function $f_2(w)$ is holomorphic in the upper strip $0 < \operatorname{Im} w < \pi$ with a point of symmetry at $\frac{1}{2}\pi\mathrm{i}$.

Elimination of $f(w)$ yields a homogeneous Hilbert problem

$$\frac{f_2(w)}{\operatorname{ch}(w - \mathrm{i}\gamma_2) + \operatorname{ch}\alpha_2} = \frac{f_1(w)}{\operatorname{ch}(w - \mathrm{i}\gamma_1) + \operatorname{ch}\alpha_1} \tag{2.4}$$

where the equality holds for real $w$. We note that (2.4) represents a Hilbert problem of the more conventional type in the full $z$-plane which is obtained from the $w$-plane by the transformation $z = \operatorname{sh} w$.

This problem has the solution (cf. III 2.8)

$$f_j(w) = (\operatorname{sh} w)^{\frac{\gamma_2-\gamma_1}{\pi}} \exp\left\{-\mathrm{i}\gamma_j - \frac{1}{2\pi\mathrm{i}} \int_{\infty}^{\infty} \ln Q(t) \frac{\operatorname{ch} t}{\operatorname{sh} t - \operatorname{sh} w} dt\right\}, \tag{2.5}$$

where

$$Q(t) = \frac{\operatorname{ch}(t - \mathrm{i}\gamma_1) + \operatorname{ch}\alpha_1}{\operatorname{ch}(t - \mathrm{i}\gamma_2) + \operatorname{ch}\alpha_2} \exp\{\mathrm{i}(\gamma_1 - \gamma_2) \operatorname{sgn} t\}. \tag{2.6}$$

In order to get a better representation of say $f_2(w)$ we differentiate (2.5) logarithmically and proceed as in III section 2. We have at first

$$\frac{d}{dw} \ln f_2(w) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \left\{\frac{\operatorname{sh}(t - \mathrm{i}\gamma_2)}{\operatorname{ch}(t - \mathrm{i}\gamma_2) + \operatorname{ch}\alpha_2} - \frac{\operatorname{sh}(t - \mathrm{i}\gamma_1)}{\operatorname{ch}(t - \mathrm{i}\gamma_1) + \operatorname{ch}\alpha_1}\right\} \cdot \frac{\operatorname{ch} w}{\operatorname{sh} t - \operatorname{sh} w} dt. \tag{2.7}$$

Then applying the Fourier transform III (2.11) and using the following generalization of III (2.12)

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{itu}\frac{\operatorname{sh}(t-i\gamma)}{\operatorname{ch}(t-i\gamma)+\operatorname{ch}\alpha}\,dt = \frac{e^{-\gamma u}\cos\alpha u}{\operatorname{sh}\pi u}, \tag{2.8}$$

valid for $|\operatorname{Re}\gamma|+|\operatorname{Im}\alpha|<\pi$ we obtain

$$\left\{\begin{aligned} &\frac{d}{dw}\ln f_2(w) = \int_{-\infty}^{\infty}\sin\{u(\tfrac{1}{2}\pi i-w)\}\cdot \\ &\quad\cdot\frac{\cos\alpha_2 u\exp\{-(\tfrac{1}{2}\pi+\gamma_2)u\}-\cos\alpha_1 u\exp\{-(\tfrac{1}{2}\pi+\gamma_1)u\}}{\operatorname{sh}^2\pi u}\,du.\end{aligned}\right. \tag{2.9}$$

If the following function is introduced

$$e^*(z,\gamma,\alpha)\overset{\text{def}}{=}\exp\int_{-\infty}^{\infty}\frac{1-\cos tz}{t}\,\frac{\operatorname{sh}\gamma t\cos\alpha t}{\operatorname{sh}^2\pi t}\,dt, \tag{2.10}$$

where $|\operatorname{Im} z|<2\pi-|\operatorname{Re}\gamma|-|\operatorname{Im}\alpha|$, we may write

$$f_2(w)=\frac{e^*(\tfrac{1}{2}\pi i-w,\ \tfrac{1}{2}\pi+\gamma_2,\ \alpha_2)}{e^*(\tfrac{1}{2}\pi i-w,\ \tfrac{1}{2}\pi+\gamma_1,\ \alpha_1)}, \tag{2.11}$$

and likewise

$$f_1(w)=\frac{e^*(\tfrac{1}{2}\pi i+w,\ -\tfrac{1}{2}\pi+\gamma_2,\alpha_2)}{e^*(\tfrac{1}{2}\pi i+w,\ -\tfrac{1}{2}\pi+\gamma_1,\alpha_1)}. \tag{2.12}$$

The function introduced by (2.10) is closely related to the function $e(z,\gamma,\beta)$ which has been used in the previous paper (cf. IV 2.11).

We note the obvious relations $e^*(0,\gamma,\alpha)=1$, and

$$e^*(-z,\gamma,\alpha)=e^*(z,\gamma,\alpha),\qquad e^*(z,-\gamma,\alpha)\,e^*(z,\gamma,\alpha)=1. \tag{2.13}$$

Further we have the functional relations

$$\frac{e^*(z+i\pi,\gamma,\alpha)}{e^*(z-i\pi,\gamma,\alpha)}=\frac{\operatorname{ch}(z+i\gamma)+\operatorname{ch}\alpha}{\operatorname{ch}(z-i\gamma)+\operatorname{ch}\alpha}, \tag{2.14}$$

and

$$\frac{e^*(z-\tfrac{1}{2}\pi i,\ \gamma+\tfrac{1}{2}\pi,\ \alpha)}{e^*(z+\tfrac{1}{2}\pi i,\ \gamma-\tfrac{1}{2}\pi,\ \alpha)}=C\{\operatorname{ch}(z-i\gamma)+\operatorname{ch}\alpha\}. \tag{2.15}$$

In view of (2.13) the relation (2.14) follows easily from (2.15). The latter relation may be proved by means of (2.10) by applying logarithmic differentiation. For the left-hand side of (2.14) this operation gives

$$\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(z-i\gamma)t\cos\alpha t}{\operatorname{sh}\pi t}\,dt,$$

which by using III (6.2) equals

$$\tfrac{1}{2}\{\operatorname{th}\tfrac{1}{2}(z-i\gamma+\alpha)+\operatorname{th}\tfrac{1}{2}(z-i\gamma-\alpha)\}=\frac{\operatorname{sh}(z-i\gamma)}{\operatorname{ch}(z-i\gamma)+\operatorname{ch}\alpha},$$

which is rightly the logarithmic derivative of the right-hand side of (2.14).

Substitution of (2.11) and (2.12) in (2.4) and using the functional relation (2.14) shows that both sides of (2.4) can be represented by the function

(2.16) $$K(w) \stackrel{\text{def}}{=} \frac{e^*(w+\frac{1}{2}\pi i,\ \gamma_2-\frac{1}{2}\pi,\ \chi_2)}{e^*(w-\frac{1}{2}\pi i,\ \gamma_1+\frac{1}{2}\pi,\ \chi_1)}.$$

The poles and the zeros of $e^*(z, \gamma, \chi)$ cannot easily be derived from those of $e(z, \gamma, \beta)$ for arbitrary $\theta$ since the special case $\theta = \pi$ gives rise to confluences. The following derivation shows in fact that the poles and zeros appear with an increasing multiplicity. From (2.10) there follows

$$\ln e^*(z, \gamma, \chi) = 2\int_0^\infty \frac{1-\cos tz}{t} e^{-2\pi t}(e^{(\gamma+i\chi)t} - e^{-(\gamma+i\chi)t})(1-e^{-2\pi t})^{-2}\, dt =$$

$$= 2\int_0^\infty \frac{1-\cos tz}{t} \sum_{n=1}^\infty (ne^{-(2n\pi-\gamma-i\chi)t} - ne^{-(2n\pi+\gamma+i\chi)t})\, dt =$$

$$= \sum_{n=1}^\infty n \ln\left(1 - \frac{z^2}{(2n\pi-\gamma-i\chi)^2}\right)\left(1 + \frac{z^2}{(2n\pi+\gamma+i\chi)^2}\right)^{-1},$$

so that finally (at least formally)

(2.17) $$e^*(z, \gamma, \chi) = \prod_{n=1}^\infty \left\{\frac{z^2+(2n\pi-\gamma-i\chi)^2}{z^2+(2n\pi+\gamma+i\chi)^2}\right\}^n.$$

From (2.10) the following asymptotic behaviour can be derived for $\mathrm{Re}\, z \to +\infty$

(2.18) $$\ln e^*(z, \gamma, \chi) = \frac{\gamma z}{\pi} + C + O(ze^{-z}),$$

where $C$ is some constant.
Hence we obtain from (2.16) the estimation

(2.19) $$\ln K(w) = \frac{\gamma_2-\gamma_1-\pi}{\pi} w + O(1), \qquad \mathrm{Re}\, w \to +\infty.$$

Since the solution (2.1) may be written as

(2.20) $$F(x, y) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp\{-(ix\,\mathrm{sh}w + y\,\mathrm{ch}w)\}\,\mathrm{ch}w\, K(w)\, dw,$$

it follows that there is a regular solution with a finite limit at the origin for $\mathrm{Re}\,\gamma_1 > \mathrm{Re}\,\gamma_2$. If $\mathrm{Re}\,\gamma_1 = \mathrm{Re}\,\gamma_2$ the solution (2.20) is logarithmic at the origin. The "higher" solutions $F_m(x, y)$ are of the form (cf. III 2.19) with $m = 1, 2, 3 \ldots$.

(2.21) $$F_m(x, y) = \tfrac{1}{2} \int_{-\infty}^{\infty} \exp\{-(ix\,\mathrm{sh}w + y\,\mathrm{ch}w)\}\,\mathrm{ch}w\,\mathrm{sh}^m w\, K(w)\, dw.$$

3. *The G-problem*

A solution of the $G$-problem may be represented by (cf. III 3.1)

$$(3.1)\quad \begin{cases} 2\pi G(x, y, x_0, y_0) = K_0(\sqrt{(x-x_0)^2+(y-y_0)^2}) + \\ \qquad + \frac{1}{2}\int_{-\infty}^{\infty} \exp - (ix\,\mathrm{sh}w + y\,\mathrm{ch}w)\, g(w)dw. \end{cases}$$

By using the formula III (3.2) the boundary conditions require the existence of holomorphic functions $g_j(w)$ where $g_1(w)$ is holomorphic in the lower strip $-\pi<\mathrm{Im}\,w<0$ and symmetric with respect to $-\frac{1}{2}\pi i$, and where $g_2(w)$ is holomorphic in the upper strip $0<\mathrm{Im}\,w<\pi$ and symmetric with respect to $\frac{1}{2}\pi i$. These functions are defined by (cf. 2.3 and III 3.4)

$$(3.2)\quad \begin{cases} \mathrm{ch}w\, g_j(w) = \{\mathrm{ch}\,(w-i\gamma_j)+\mathrm{ch}\,\alpha_j\}\, g(w) - \{\mathrm{ch}\,(w+i\gamma_j)-\mathrm{ch}\,\alpha_j\} \\ \qquad \exp\,(ix_0\,\mathrm{sh}w - y_0\,\mathrm{ch}w). \end{cases}$$

Elimination of $g(w)$ leads to the non-homogeneous Hilbert problem

$$(3.3)\quad \begin{cases} \dfrac{\mathrm{ch}w\, g_2(w)}{\mathrm{ch}\,(w-i\gamma_2)+\mathrm{ch}\,\alpha_2} - \dfrac{\mathrm{ch}w\, g_1(w)}{\mathrm{ch}\,(w-i\gamma_1)+\mathrm{ch}\,\alpha_1} = \\ = \left\{\dfrac{\mathrm{ch}\,(w+i\gamma_1)-\mathrm{ch}\,\alpha_1}{\mathrm{ch}\,(w-i\gamma_1)+\mathrm{ch}\,\alpha_1} - \dfrac{\mathrm{ch}(w+i\gamma_2)-\mathrm{ch}\,\alpha_2}{\mathrm{ch}(w-i\gamma_2)+\mathrm{ch}\,\alpha_2}\right\} \exp\,(ix_0\mathrm{sh}w - y_0\mathrm{ch}w). \end{cases}$$

Introducing the functions $f_j(w)$ defined by (2.11) and (2.12) and using the function $K(w)$ defined by (2.16) we may write (3.3) in the essentially simpler form

$$(3.4)\qquad \frac{g_2(w)}{f_2(w)} - \frac{g_1(w)}{f_1(w)} = h(w),$$

where $h(w)$ is defined by

$$(3.5)\quad \begin{cases} \mathrm{ch}w\, K(w)\, h(w) = \\ = \left\{\dfrac{\mathrm{ch}(w+i\gamma_1)-\mathrm{ch}\alpha_1}{\mathrm{ch}(w-i\gamma_1)+\mathrm{ch}\alpha_1} - \dfrac{\mathrm{ch}(w+i\gamma_2)-\mathrm{ch}\alpha_2}{\mathrm{ch}(w-i\gamma_2)+\mathrm{ch}\alpha_2}\right\} \exp\,(ix_0\,\mathrm{sh}w - y_0\,\mathrm{ch}w). \end{cases}$$

Since $g_j(w)/f_j(w)$ is holomorphic in the appropriate strip and symmetric with respect to $-\frac{1}{2}\pi i$ for $j=1$ and $\frac{1}{2}\pi i$ for $j=2$ the problem (3.4) has the solution (cf. III 3.8)

$$(3.6)\qquad g_j(w) = \frac{1}{2\pi i} f_j(w) \int_{-\infty}^{\infty} h(w_0) \frac{d\,\mathrm{sh}w_0}{\mathrm{sh}w_0 - \mathrm{sh}w},$$

where $-\pi<\mathrm{Im}\,w<0$ for $j=1$ and $0<\mathrm{Im}\,w<\pi$ for $j=2$.

The function $g(w)$ appearing in the integrand on the righthand side of (3.1) is determined by (3.2) with either $j=1$ or $j=2$ and by (3.6). By using Plemelj's formula we may write (cf. III 3.10) for real $w$

$$(3.7)\quad \begin{cases} g(w) = \frac{1}{2}\left\{\frac{\mathrm{ch}(w+\mathrm{i}\gamma_1)-\mathrm{ch}\alpha_1}{\mathrm{ch}(w-\mathrm{i}\gamma_1)+\mathrm{ch}\alpha_1}+\frac{\mathrm{ch}(w+\mathrm{i}\gamma_2)-\mathrm{ch}\alpha_2}{\mathrm{ch}(w-\mathrm{i}\gamma_2)+\mathrm{ch}\alpha_2}\right\}\exp\,(\mathrm{i}x_0\,\mathrm{sh}w - y_0\,\mathrm{ch}w) + \\ \qquad + \frac{1}{2\pi\mathrm{i}}\,\mathrm{ch}w\,K(w)\int_{-\infty}^{\infty} h(w_0)\frac{d\,\mathrm{sh}w_0}{\mathrm{sh}w_0-\mathrm{sh}w}. \end{cases}$$

We note that in view of (2.19) $g(w)$ behaves asymptotically as

$$(3.8)\qquad g(w) = O\{(\mathrm{ch}w)^{\frac{\gamma_2-\gamma_1}{\pi}-1}\}.$$

Therefore the Green's function (3.1) is always finite at the origin.

4. *A single boundary condition*

If there is a single boundary condition along the $x$-axis

$$(4.1)\qquad \left((\cos\gamma\frac{\partial}{\partial y} - \sin\gamma\frac{\partial}{\partial x} - \mathrm{ch}\,\alpha\right)G = 0,$$

then in view of the fact that $h(w)\equiv 0$ the solution of the $G$-problem reduces to the following simple form

$$(4.2)\quad \begin{cases} 2\pi G(x, y, x_0, y_0) = K_0(\sqrt{(x-x_0)^2+(y-y_0)^2}) + \\ \qquad + \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{ch}\,(w+\mathrm{i}\gamma)-\mathrm{ch}\,\alpha}{\mathrm{ch}\,(w-\mathrm{i}\gamma)+\mathrm{ch}\,\alpha}\exp\,\{-\mathrm{i}(x-x_0)\,\mathrm{sh}w-(y+y_0)\,\mathrm{ch}w\}\,dw. \end{cases}$$

If we define

$$(4.3)\qquad R(x, y) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{ch}\,(w+\mathrm{i}\gamma)-\mathrm{ch}\,\alpha}{\mathrm{ch}\,(w-\mathrm{i}\gamma)+\mathrm{ch}\,\alpha}\exp - (\mathrm{i}x\,\mathrm{sh}w + y\,\mathrm{ch}w)\,dw,$$

we may write (4.2) in the form (cf. II 5.2)

$$(4.4)\qquad G(x, y, x_0, y_0) = G_0(x, y, x_0, y_0) + (2\pi)^{-1}R(x-x_0, y+y_0),$$

where the first term on the right-hand side is the Green's function in the full $x, y$-plane and where the second term represents some reflection at $(x_0, -y_0)$. It is easily seen that [1])

$$(4.5)\quad \begin{cases} \left(\cos\gamma\frac{\partial}{\partial y} - \sin\gamma\frac{\partial}{\partial x} - \mathrm{ch}\,\alpha\right)R(x, y) = \left(\cos\gamma\frac{\partial}{\partial y} + \sin\gamma\frac{\partial}{\partial x} + \mathrm{ch}\,\alpha\right)\cdot \\ \qquad \cdot K_0(\sqrt{x^2+y^2}), \end{cases}$$

which may give a geometrical interpretation to the kind of reflection. However, this is generally not of such a simple nature as in the sub-case of II section 5 where $\beta = 0$. If one tries to continue $R(x, y)$ outside its

[1]) The corresponding formula II 5.5 contains a misprint.

region of definition difficulties are also experienced. In polar coordinates we may write

$$(4.6)\quad R(r,\varphi)=\tfrac{1}{2}\int_{-\infty+ic}^{\infty+ic}\frac{\operatorname{ch}(w+i\gamma)-\operatorname{ch}\alpha}{\operatorname{ch}(w-i\gamma)+\operatorname{ch}\alpha}\exp\{-ir\operatorname{sh}(w-i\varphi)\}\,dw,$$

which is convergent in the half-plane sector

$$(4.7)\qquad c<\varphi<c+\pi,$$

provided the integrand has no pole. Continuation is permitted until

$$(4.8)\qquad c=\operatorname{Re}\gamma\pm\operatorname{Im}\alpha\pm\pi.$$

Therefore continuation is possible in the wider sector

$$(4.9)\qquad \operatorname{Re}\gamma+\operatorname{Im}\alpha-\pi<\varphi<2\pi+\operatorname{Re}\gamma-\operatorname{Im}\alpha.$$

Now we see that in the subcase $\alpha=\frac{1}{2}\pi i$ $(\beta=0)$ there is just a full plane with a slit at the dipole line $\varphi=\operatorname{Re}\gamma-\frac{1}{2}\pi$. In the general case the situation is obviously more complicated.

## REFERENCES

DANTZIG, D. VAN, Solutions of the equation of Helmholtz in an angle with vanishing directional derivatives along each side. I, II. Kon. Ned. Akad. v. Wet. Proc. A **61**, 4, 384–398 (1958).

LAUWERIER, H. A., A note on the problem of the sloping beach. Ibid. A **62**, 3, 229–240 (1959).

———, Solutions of the equation of Helmholtz in an angle. I, II, III, IV. Ibid. **A 62**, 5, 475–488 (1959); **A 63**, 4, 355–372 (1960); **A 64**, 1, 123–140 (1961); **A 64**, 3, 348–359 (1961).