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The North Sea Problem VII.

Non-Stationary Wind-Effects in a Rectangular Bay

Numerical Part

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MATHEMATICS

THE NORTH SEA PROBLEM. VII

NON-STATIONARY WIND-EFFECTS IN A RECTANGULAR BAY [1]) NUMERICAL PART

BY

H. A. LAUWERIER

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§ 1. *Introduction and Summary of the results*

The means developed in the preceding paper will be employed here to discuss a few typical numerical cases. In all cases we consider a uniform Northern wind with some time-behaviour. More explicitly in the following three sections we consider:

a an exponential windfield,
b a step-function windfield,
c a step-sine windfield.

The exponential windfield is determined by

$$V(t)=c_1 e^{p_1 t}+c_2 e^{p_2 t}, \qquad t>-\infty, \tag{1.1}$$

where c_1, c_2, p_1, p_2 are constants.

The step-function windfield is given by

$$\begin{cases} V(t)=0 & \text{for} \quad t<0, \\ V(t)=-1 & \text{for} \quad t>0. \end{cases} \tag{1.2}$$

The step-sine windfield is determined by

$$\begin{cases} V(t)=0 & \text{for } t<0, \\ V(t)=-\sin\, \omega t & \text{for } t>0. \end{cases} \tag{1.3}$$

In order to allow for comparison between the various results we take $\max |V(t)|=1$.

The numerical data are those of VI section 2. For convenience we repeat

$$0<x<\pi \qquad 0<y<2\pi,$$
$$\lambda=0.12 \qquad \Omega=0.6$$

with a time scale of 1.4 hours pro unit.

The conversion needed in order to obtain the elevation in meters for a given maximum velocity of the wind in meters pro second is given by the following table.

[1]) Report TW 73 of the Mathematical Centre, Amsterdam.

We remind the reader that the elevation depends quadratically on the windvelocity (cf. VI 2–3).

TABLE 1

max v_s	factor
29	0.50
30	0.54
35	0.75
41	1.00

By way of illustration we may take the stationary case $U=0$, $V=-1$. Then at the "Dutch" coast the stationary elevation 2π is obtained. For a wind of 30 m/sec this corresponds with an elevation of approximately 3.40 meters (cf. also II 6.13).

In the case *a* the formalism of the Laplace transformation can be used but the inverse transformation is not needed. However, the results only apply for a restricted interval of time i.e. the interval for which (1.1) has a physical meaning. For this type of windfield with the constants given by (2.2) the maximum intensity is reached at $t=20.1$ corresponding to about 28 hours.

The maximum elevation $\zeta(\frac{1}{2}\pi, 0, t)$ at the middle of the "Dutch" coast is found be

$$\zeta_{\max}=5.90 \quad \text{for} \quad t=23 \text{ (32 hr)}.$$

If the effect of the rotation of the Earth were absent we would have found

$$\zeta_{\max}=6.65 \text{ at about the same time.}$$

The graphs of $\zeta(\frac{1}{2}\pi, 0, t)$ in both cases and the wind-function are given in fig. 1.

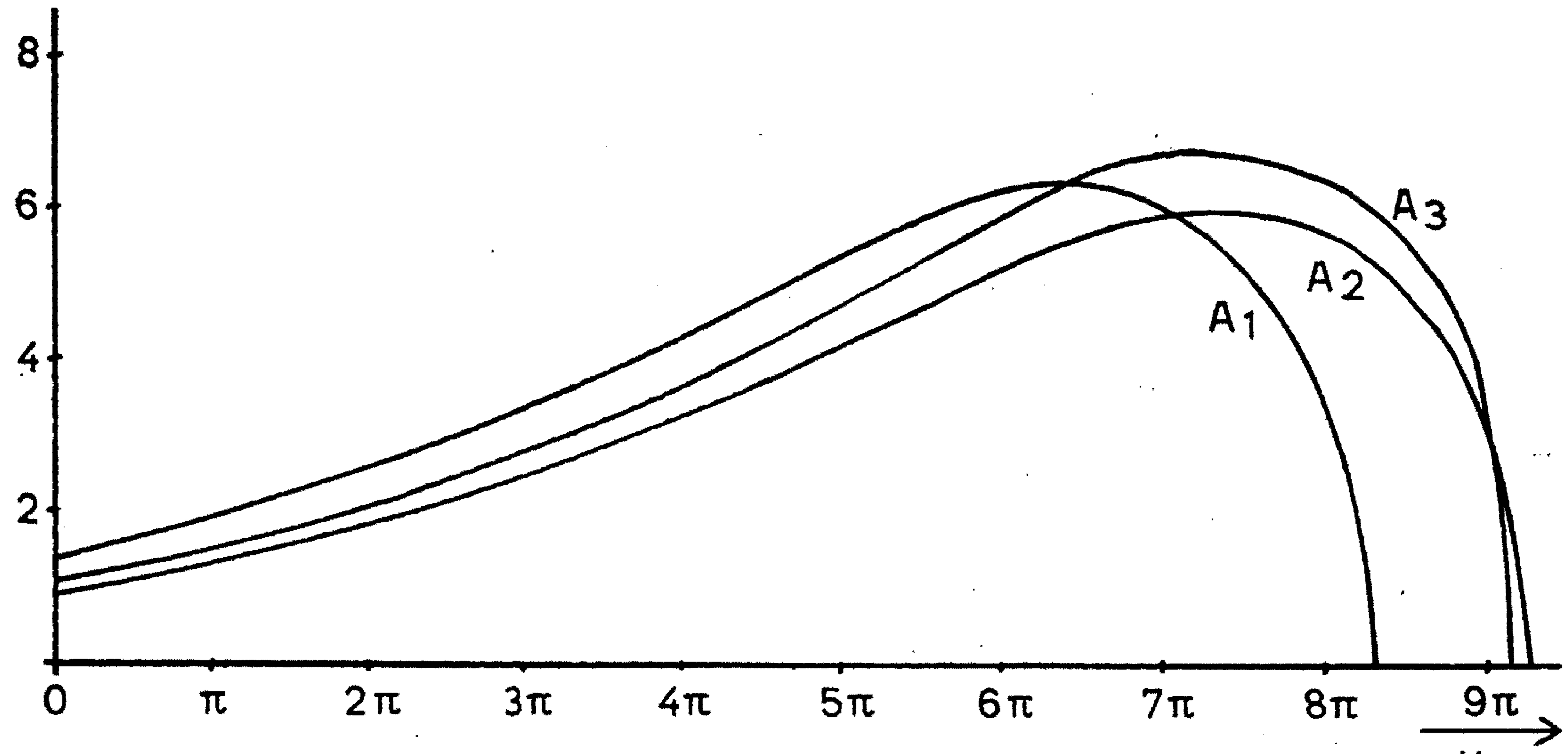

Fig. 1. Elevation at the "Dutch" coast due to the exponential windfield

$V = -0.27\,(e^{0.12t} - 0.2e^{0.18t})$

A1 Quasi-stationary elevation $-2\pi\, V\,(t)$

A2 Elevation at $(\frac{1}{2}\pi, 0)$ for $\Omega = 0.6$

A3 Elevation at $\frac{1}{2}\pi, 0)$ for $\Omega = 0$.

A great number of elevations at various instants have been calculated. They enable us to draw lines of constant elevation in the geometrical rectangular model. A number of those isohypses are given in fig. 2. Inspection leads to the following preliminary conclusion

1°. The rotation of the Earth causes chiefly an East-West skewness of the sea surface. This skewness is slight at the southern coast, but increases in northern direction right up to the ocean. Moreover it brings about a reduction of the maximum elevation.

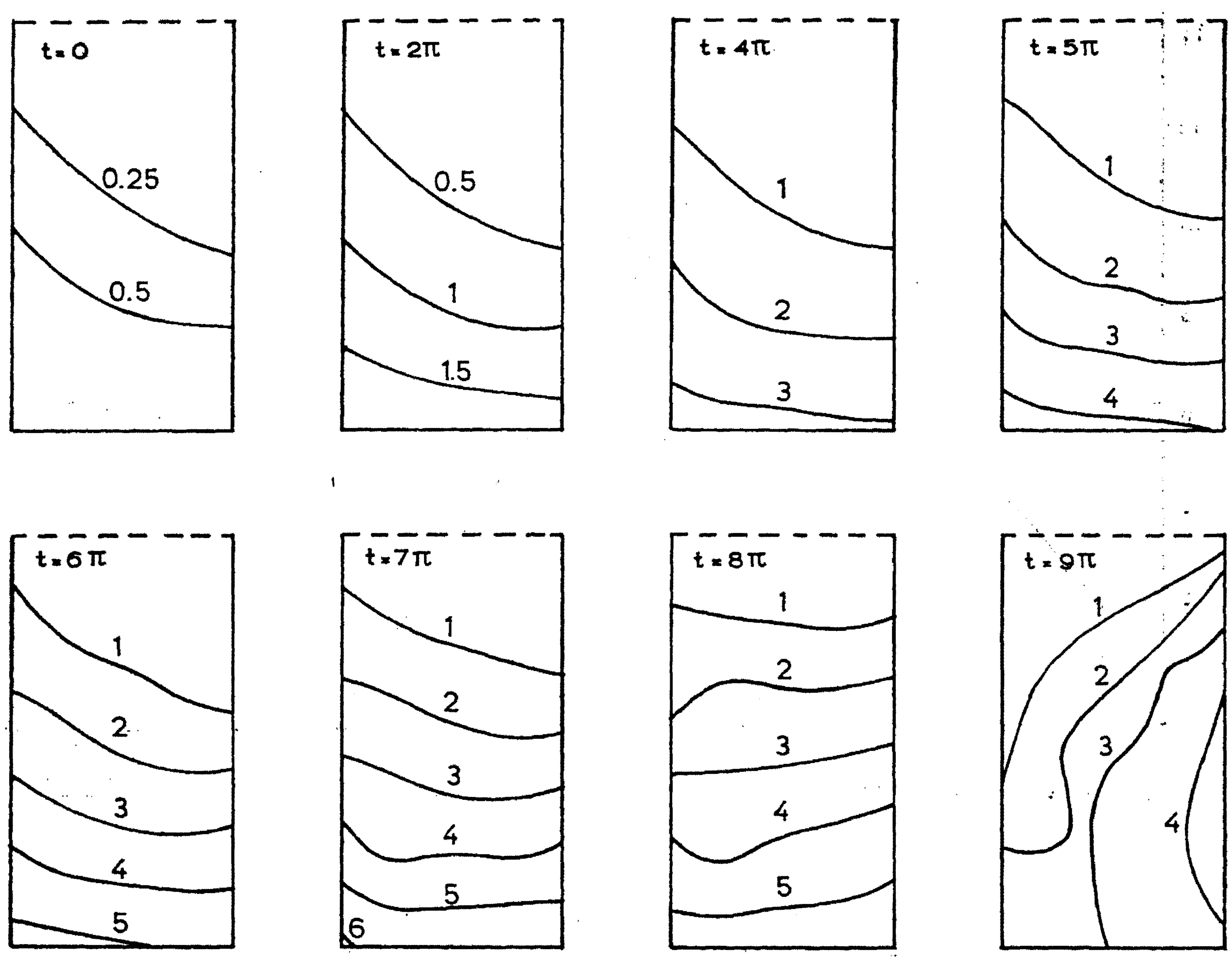

Fig. 2. Isohypses in a rectangular bay due to an exponential windfield.

In the case b we have the transient phenomenon of the discontinuity at $t=0$. Here inverse Laplace transformation is imperative. By using the approximate expression of $\zeta(x, 0, p)$ derived in section 8 of the preceding paper (formula 8.16) it is possible to compute $\zeta(x, 0, t)$ by means of the calculus of residues applied to the complex inversion formula. It is known that the best approximation is obtained at the middle $(\frac{1}{2}\pi, 0)$. Moreover, the results of the previous case indicate that $\zeta(x, 0, t)$ depends only slightly on x thereby confirming the theoretical result obtained earlier (cf. VI 8.6).

The calculations show that $\zeta(\frac{1}{2}\pi, 0, t)$ increases almost monotonically to its stationary value of 2π i.e.

$$\zeta_{\max} = 6.28 \quad \text{for} \quad t \to \infty.$$

In the absence of the Coriolis effect the elevation has a much more oscillatory character with a considerable overshoot of the stationary value at $t=4\pi$ i.e.

$$\zeta_{\max}=9.17 \quad \text{for} \quad t=12.6 \ (18 \text{ hr}).$$

Also here the damping influence of Ω is apparent. The elevation at other points of the sea has not been considered. The graphs of $\zeta(\frac{1}{2}\pi, 0, t)$ are given in fig. 3.

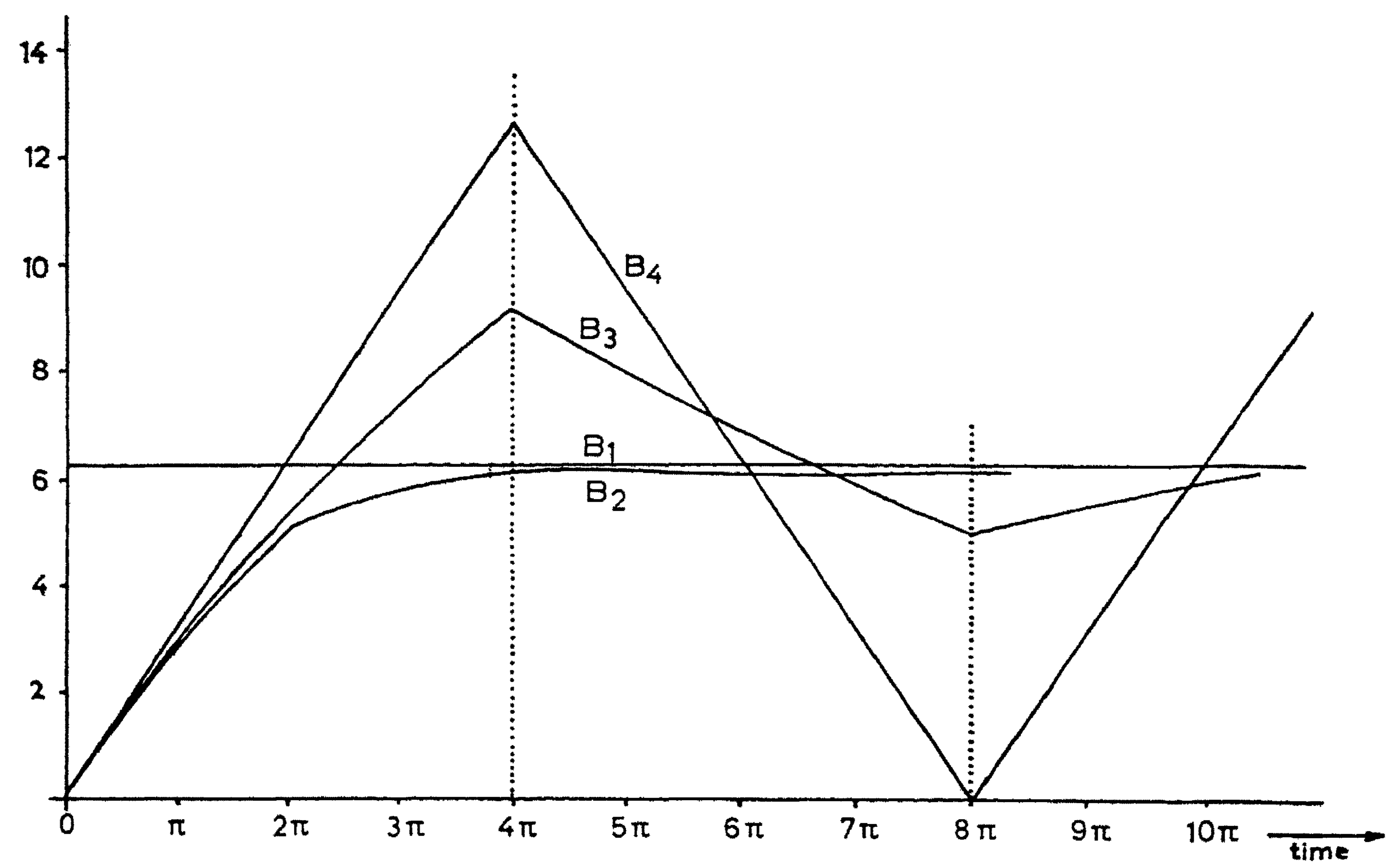

Fig. 3. Elevation at the "Dutch" coast due to the step-function windfield

$$V = \begin{cases} 0 & \text{for } t < 0 \\ -1 & \text{for } t > 0. \end{cases}$$

B1 Quasi-stationary elevation 2π
B2 Elevation at $(\frac{1}{2}\pi, 0)$ for $\Omega = 0.6$
B3 Elevation at $(\frac{1}{2}\pi, 0)$ for $\Omega = 0$
B4 Elevation at $(\frac{1}{2}\pi, 0)$ for $\lambda = \Omega = 0$.

In the case c we have taken the typical case $\omega=0.1$ so that (1.3) represents a windfield of a "symmetrical" storm which reaches its maximum at $t=5\pi$ (22 hr) and returns to zero at $t=10\pi$ (44 hr). Here the same technique has been used as in the preceding case. It appears that $\zeta(\frac{1}{2}\pi, 0, t)$ imitates the wind-function with an approximately constant shift in time viz.

$$\zeta_{\max}=5.92 \quad \text{for} \quad t=19 \ (26\tfrac{1}{2} \text{ hr}).$$

For $\Omega=0$ we would have

$$\zeta_{\max}=7.55 \quad \text{for} \quad t=17.6 \ (24\tfrac{1}{2} \text{ hr}).$$

Again there is a damping due to the Coriolis effect. The graphs of $\zeta(\frac{1}{2}\pi, 0, t)$ are given in fig. 4.

From the results obtained in the cases a and c it may be deduced that a fair prediction of the maximum elevation at $(\frac{1}{2}\pi, 0)$ can be obtained by taking the stationary elevation 2π which corresponds to the maximum of the wind. This prediction, 6.28 instead of 5.90, is on the safe side with a deviation of about 6 %. The observation that the elevation $\zeta(\frac{1}{2}\pi, 0, t)$

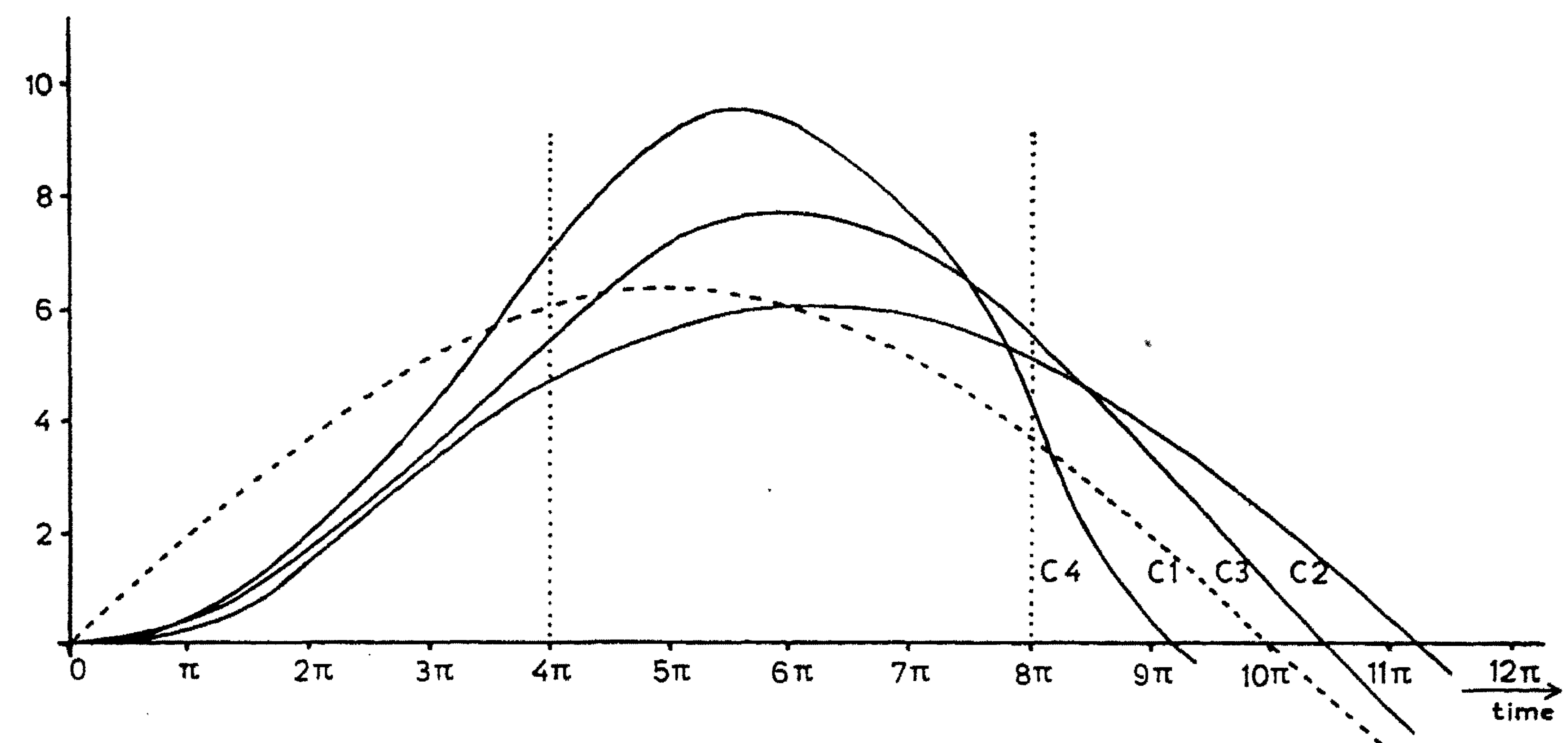

Fig. 4. Elevation at the "Dutch" coast due to the sine-windfield

$$V = \begin{cases} 0 & \text{for } t < 0 \\ -\sin 0.1\,t & \text{for } t > 0. \end{cases}$$

C1 Quasi-stationary elevation $2\pi \sin 0.1t$.
C2 Elevation at $(\frac{1}{2}\pi, 0)$ for $\Omega = 0.6$.
C3 Elevation at $(\frac{1}{2}\pi, 0)$ for $\Omega = 0$.
C4 Elevation at $(\frac{1}{2}\pi, 0)$ for $\lambda = \Omega = 0$.

as a function of t imitates the wind-function $V(t)$ may also be expressed by saying that to a certain extent the sea is in equilibrium with the windfield of a few hours earlier. This finding confirms the usual practice of the forecasting of water-levels at the Dutch coast.

Obviously we should be cautious in drawing conclusions from the few numerical applications which have been made here. Therefore the following conclusions must be taken with due reservation.

2°. For a suddenly rising storm (stepfunction windfield) the elevation at the southern coast rapidly increases and after some 12 hours already takes on about 90 % of its stationary value.

3°. For a sinusoidal storm the elevation at the southern coast equals approximately the stationary elevation due to a stationary wind with an intensity corresponding to that of the wind occurring a certain constant time earlier.

4°. The elevation at the southern coast, due to a sinusoidal storm, can be considered as the sum of the quasi-stationary elevation (forced oscillation) and a damping term which is due to the lowest negative real eigenvalue.

Two important effects are not considered, namely the effect of the non-uniformity of the depth and the inhomogeneity of the windfield. The influence of these effects will be considered in a subsequent paper in this series in which the problem is solved by purely numerical methods with the use of the X1 computer of the Mathematical Centre.

2. *An exponential windfield*

If the components of the windfield U, V are proportional to the time factor exp (pt) the original system of equations (VI 2.1) has a solution which also contains the time factor exp (pt). In particular we may write

$$\zeta(x, y, t) = \bar{\zeta}(x, y, p)\, e^{pt}, \tag{2.1}$$

and similarly for u and v.

If (2.1) and similar expressions are substituted in the equations (VI 2.1) we again find the equations (VI 2.9) which previously were obtained by Laplace transformation. Therefore the finding of a solution of the type (2.1) is equivalent to that of determining the Laplace transform of ζ. Without needing an inverse transformation by (2.1) a special solution is obtained albeit for a somewhat pathological windfield. Of course (2.1) makes sense only for $p>0$ (or more generally Re $p>0$). Then a windfield is described growing in intensity from a perfect calmness ($t \to -\infty$) to an infinite force ($t \to +\infty$).

Making use of the superposition principle, by adding solutions of the above kind more natural situations can be dealt with. We shall consider here only a very special case but generalization will appear obvious. We shall take the following "exponential" windfield (see fig. 1)

$$\begin{cases} U = 0 \\ V = -0.27\ (e^{0.12t} - 0.2e^{0.18t}). \end{cases} \tag{2.2}$$

A few values of $-V$ are given below

TABLE 2

t	$-V$	t	$-V$
0	0.22	5π	0.85
π	0.30	6π	0.98
2π	0.40	7π	0.95
3π	0.54	8π	0.53
4π	0.70	9π	-0.73

This windfield represents a uniform Northern wind which is active from $t=-\infty$ onwards and which slowly rises until it reaches its maximum 1 at $t=20.1$. Next it falls off more and more rapidly. Beyond $t=9\pi$ the model looses all physical reality.

The obvious solution to (2.2) is

$$\zeta(x, y, t) = 0.27\{\bar{\zeta}(x, y, 0.12)\, e^{0.12t} - 0.2\ \bar{\zeta}(x, y, 0.18)\, e^{0.18t}\}, \tag{2.3}$$

where $\zeta(x, y, p)$ coincides with the Laplace transform of the elevation which is the solution of the problem stated at the beginning of VI section 4. Therefore we may apply without reservation the results of the sections 5 and 6 of the preceding paper.

The first step in our calculations is the computation of the coefficients A_n, B_n ($n=0, 1, 2, \ldots$) of (VI 4.12). The system (VI 5.7) gives after only a few iterations

$$p=0.12 \qquad \begin{aligned} A_0 &= 4.805-0.022\ B_0 \\ A_1 &= 0.239\ B_0 \\ A_2 &= -0.005+0.007\ B_0 \\ A_3 &= 0.031\ B_0 \\ A_4 &= -0.001+0.002\ B_0 \\ A_5 &= 0.001\ B_0. \end{aligned}$$

$$p=0.18 \qquad \begin{aligned} A_0 &= 3.390-0.035\ B_0 \\ A_1 &= 0.289\ B_0 \\ A_2 &= -0.009+0.013\ B_0 \\ A_3 &= 0.038\ B_0 \\ A_4 &= -0.002+0.003\ B_0 \\ A_5 &= 0.014\ B_0. \end{aligned}$$

The ocean condition gives after a number of purely numerical manipulations

$$p=0.12 \qquad \begin{aligned} A_0 &= 4.731 & B_0 &= 3.366 \\ A_1 &= 0.805 & B_1 &= 0.847 \\ A_2 &= 0.019 & B_2 &= -0.721 \\ A_3 &= 0.104 & B_3 &= 0.710 \\ A_4 &= 0.006 & B_4 &= -0.654 \\ A_5 &= 0.037 & B_5 &= 0.628. \end{aligned}$$

$$p=0.18 \qquad \begin{aligned} A_0 &= 3.294 & B_0 &= 2.702 \\ A_1 &= 0.782 & B_1 &= 0.846 \\ A_2 &= 0.026 & B_2 &= -0.674 \\ A_3 &= 0.103 & B_3 &= 0.669 \\ A_4 &= 0.006 & B_4 &= -0.657 \\ A_5 &= 0.038 & B_5 &= 0.615. \end{aligned}$$

The numerical values obtained above enable us to compute $\zeta(x, y, 0.12)$ and $\zeta(x, y, 0.18)$ for any desired position (x, y). In particular at the "Dutch" coast $y=0$ we find for $x=0(\pi/8)\pi$

x	$\zeta(x, 0, 0.12)$	$\zeta(x, 0, 0.18)$
0	4.44	3.80
$1\cdot\pi/8$	4.40	3.76
$2\cdot\pi/8$	4.33	3.67
$3\cdot\pi/8$	4.23	3.56
$4\cdot\pi/8$	4.14	3.46
$5\cdot\pi/8$	4.05	3.35
$6\cdot\pi/8$	3.97	3.25
$7\cdot\pi/8$	3.90	3.17
π	3.86	3.14

Using formula (2.3) we find in particular at the middle ($\frac{1}{2}\pi$, 0) the following elevation for $t=0(\pi)9\pi$.

TABLE 3

t	$\zeta(\frac{1}{2}\pi, 0, t)$	t	$\zeta(\frac{1}{2}\pi, 0, t)$
0	0.93	5π	4.22
π	1.30	6π	5.19
2π	1.80	7π	5.90
3π	2.45	8π	5.61
4π	3.26	9π	2.99

For other points at the "Dutch" coast the result is only slightly different. We have calculated the maximum elevation at a few points. The results are given in the following table which shows that there exists a slight decrease in Eastern direction.

TABLE 4

x	$\zeta_{\max}$	x	$\zeta_{\max}$
0	6.05	$5\pi/8$	5.88
$\pi/8$	6.04	$6\pi/8$	5.85
$2\pi/8$	6.00	$7\pi/8$	5.84
$3\pi/8$	5.99	π	5.83
$4\pi/8$	5.90		

The calculations also show that the maximum at (0, 0) is at a slightly earlier time than at (π, 0). This is in accordance with the physical picture of a tidal wave travelling counter-clockwise round the basin.

A uniform stationary windfield $U=0$, $V=-1$ would have given $\zeta(x, 0)=2\pi$ for all x. This value is only slightly higher than the values of $\zeta_{\max}$ from table 4. If we associate for each t the corresponding stationary value of $\zeta(x, 0)$ to the windfield (2.2) a curve $-2\pi V(t)$ is obtained which is very similar to that of $\zeta(\frac{1}{2}\pi, 0, t)$ (see fig. 1), the main difference being that the curves appear to be shifted in time. This suggests that the elevation at the "Dutch" coast can be approximated by the stationary elevation which corresponds to the wind-velocity at an earlier instant with an approximately constant time-lag. This can also be expressed by saying that at each time the sea is in a stationary equilibrium with the vindfield of a few hours earlier.

The influence of the rotation of the Earth appears if we consider the special case $\Omega=0$. According to (VI 7.2) we have at the "Dutch" coast

$$\zeta(x, 0, t)=c_1\frac{\text{th } 2\pi q_1}{q_1}\,e^{p_1 t}-c_2\frac{\text{th } 2\pi q_2}{q_2}\,e^{p_2 t}. \tag{2.4}$$

We find q_1^{-1} th $2\pi q_1=4.64$ and q_2^{-1} th $2\pi q_2=3.86$. Then (2.4) gives (see fig. 1)

TABLE 5

t	$\zeta(\Omega=0)$	t	$\zeta(\Omega=0)$
0	1.05	5π	4.73
π	1.46	6π	5.83
2π	2.02	7π	6.63
3π	2.75	8π	6.36
4π	3.66	9π	3.45

These values are somewhat higher than those for $\Omega \neq 0$ which were given in table 3. This may point to a damping influence of the Coriolis effect.

The elevation at the middle of the "Dutch" coast has been sketched in fig. 1 for the various cases.

A_1 represents the quasi-stationary elevation $-2\pi V(t)$. At the same time it gives on a different scale the form of the exponential windfield according to table 2.

A_2 gives $\zeta(\frac{1}{2}\pi, 0, t)$ for $\Omega = 0.6$ according to table 3.

A_3 gives what the elevation would be for $\Omega = 0$ according to table 5.

By using the expressions (VI 4.11) and (VI 4.12) and the numerical values of the coefficients A_n and B_n a great number of values of $\zeta(x, y, t)$ have been calculated all over the sea for $t = 0(\pi)9\pi$. From the numerical data a number of pictures have been drawn up giving isohypses at the various instants (fig. 2). The figures attached to each line give the level in meters for a conversion factor 1, i.e. for a wind maximum of 41 m/sec. These pictures clearly show that the influence of the Coriolis effect increases in the direction of the positive Y-axis. It is of interest to note that this rather simplified model gives, at least qualitatively, a rather good picture of the true pattern of elevations [1]).

3. *A step-function windfield*

We consider the uniform windfield

$$U = 0, \qquad V = \begin{cases} 0 & \text{for} \quad t < 0 \\ -1 & \text{for} \quad t > 0. \end{cases} \tag{3.1}$$

Further it will be assumed that at $t = 0$ everything is at rest. In section 8 of the preceding paper we have derived an approximate analytic expression of the Laplace transform of the elevation at the "Dutch" coast. It has been made clear that the approximation holds particularly well at the middle of the "Dutch" coast. Repeating (VI 8.18) we have

$$\bar{\zeta}(\tfrac{1}{2}\pi, 0, p) \approx p^{-1}\bar{Z}(p), \tag{3.2}$$

where

$$\bar{Z}(p) = \frac{\operatorname{sh}(\frac{1}{2}s\pi + 2q\pi) - \operatorname{sh}\frac{1}{2}s\pi}{q \operatorname{ch}(\frac{1}{2}s\pi + qb)} \tag{3.3}$$

1) Cf. R. H. Corkan (1950) and J. R. Rossiter (1954).

with

$$q = \sqrt{p^2 + \lambda p} \quad \text{and} \quad s = \Omega \sqrt{\frac{p}{p+\lambda}}.$$

The computations of the preceding section may be used in order to check the accuracy of (3.2) since the values of $\zeta(\frac{1}{2}\pi, 0, p)$ are known for $p = 0.12$ and $p = 0.18$. Comparing the approximations $\bar{Z}(p)$ for $p = 0.12$ and $p = 0.18$ with the exact values as given by $p\zeta$ we find the following result

TABLE 6

	exact	approximation
$p = 0.12$	4.144	4.087
$p = 0.18$	3.459	3.439

Hence a very satisfactory agreement has been obtained. For smaller values of p the approximation may be expected to be even better.

The inverse transformation of the right-hand side of (3.2) can be carried out analytically by using the calculus of residues (see VI 8.19). The pole $p = 0$ due to the windfield gives the residue 2π which corresponds to the stationary elevation. The poles of $\bar{Z}(p)$ are found by solving the equation (VI 8.21). The first few poles are given in the following table.

TABLE 7

m	c_m	$a_m \pm ib_m$
0	-0.0744	$-0.233 \pm i\,0.216$
1	-0.115	$-0.213 \pm i\,0.736$
2	-0.118	$-0.211 \pm i\,1.241$
3	-0.119	$-0.210 \pm i\,1.744$

Then the inversion formula (VI 8.19) gives

$$\zeta(\tfrac{1}{2}\pi, 0, t) \approx 2\pi + \sum_{m=0}^{\infty} \{C_m e^{c_m t} + \mathrm{Re}\,(A_m + iB_m)\, e^{(a_m + ib_m)t}\}, \tag{3.4}$$

where the first few values of A_m, B_m and C_m are given by

$$\begin{array}{lll} A_0 = -4.59 & A_1 = -0.26 & A_2 = -0.24 \\ B_0 = 2.95 & B_1 = 0.84 & B_2 = -0.16 \\ C_0 = -0.77 & C_1 = -0.01 & C_2 = -0.00 \end{array}$$

In the following table the elevation $\zeta(\frac{1}{2}\pi, 0, t)$ is given for $t = 0(\pi)10\pi$

TABLE 8

t	$\zeta(\frac{1}{2}\pi, 0, t)$	t	$\zeta(\frac{1}{2}\pi, 0, t)$
0	0	6π	6.15
π	2.79	7π	6.15
2π	5.06	8π	6.16
3π	5.73	9π	6.18
4π	6.07	10π	6.20
5π	6.18	∞	6.28

We note that after an interval of 4π (i.e. appr. 18 hours) the elevation deviates less than 4 % from the stationary value.

In order to assess the influence of Ω we have also considered the case $\Omega=0$. Then we have the exact expression

$$\zeta(\tfrac{1}{2}\pi, 0, p) = \frac{\text{th } 2\pi q}{pq} = \frac{1}{pq}(1 - 2\,\mathrm{e}^{-4\pi q} + 2\,\mathrm{e}^{-8\pi q} \ldots). \tag{3.5}$$

From ERDÉLYI *et al.* Tables I (5.3.26) and (5.6.36) it can easily be derived that

$$\frac{1}{pq} \doteqdot t\,\mathrm{e}^{\frac{1}{2}\lambda t}\,\{I_0(\tfrac{1}{2}\lambda t) + I_1(\tfrac{1}{2}\lambda t)\}, \tag{3.6}$$

and

$$\frac{1}{pq}\,\mathrm{e}^{-cq} \doteqdot \int_c^t \mathrm{e}^{-\frac{1}{2}\lambda\tau}\, I_0(\tfrac{1}{2}\lambda\sqrt{\tau^2 - c^2})\,d\tau \quad \text{for} \quad t > c. \tag{3.7}$$

By using (3.6) and (3.7) the inversion of the right-hand side of (3.5) can easily be performed. We find eventually

TABLE 9

t	$\zeta(\Omega = 0)$	t	$\zeta(\Omega = 0)$
0	0	6π	6.88
π	2.87	7π	5.90
2π	5.29	8π	4.94
3π	7.37	9π	5.45
4π	9.17	10π	5.97
5π	7.97	∞	6.28

Comparison of these values with those of table 8 shows that the Coriolis effect manifests itself in a damping influence. The various cases have been sketched in fig. 3 in a similar way as in the previous section.

B_1 gives the stationary elevation 2π.

B_2 gives $\zeta(\frac{1}{2}\pi, 0, t)$ for $\Omega=0.6$ according to table 8.

B_3 gives what the elevation would be in absence of the Coriolis effect according to table 9.

B_4 is the almost trivial case $\lambda=0$ and $\Omega=0$.

4. *A step-sine windfield*

The case considered in this section is very similar to that of the previous section, the only difference being that here the following windfield is considered

$$U=0, \qquad V = \begin{cases} 0 & \text{for} \quad t<0 \\ -\sin \omega t & \text{for} \quad t>0. \end{cases} \tag{4.1}$$

The time interval $(0, \pi/\omega)$ presents a realistic picture of a storm which gradually attains to its maximum and subsides slowly afterwards.

By way of illustration we take the numerical value $\omega=0.1$. Then the velocity of the wind has its maximum $|V|=1$ at $t=5\pi$, which is approximately 22 hours.

Similarly as in (3.2) we find the approximation

$$\bar{\zeta}(\tfrac{1}{2}\pi, 0, p) \approx \frac{\omega}{p^2+\omega^2}\bar{Z}(p). \tag{4.2}$$

Laplace inversion gives

$$\zeta(\tfrac{1}{2}\pi, 0, t) \approx \mathrm{Im}\,\{e^{i\omega t}\,\bar{Z}(i\omega)\} + \text{residues of } \bar{Z}. \tag{4.3}$$

Substitution of the numerical values gives

$$\zeta(\tfrac{1}{2}\pi, 0, t) \approx (5.42 \sin \omega t - 2.18 \cos \omega t) + 0.37\, e^{-0.074t} + \ldots. \tag{4.4}$$

The contributions from the higher poles soon become very small so that say at $t=5\pi$ they can be neglected with small loss of accuracy. This may be considered as an important result. Notably it means that the elevation due to a sine-windfield of the form (4.1) is determined mainly by

a the quasi-stationary motion,

b the contribution from the lowest negative real eigenvalue.

It may be conjectured that this property is of a general kind and not the result of a coincidence of the particular numerical values chosen here.

From (4.4) the following table can be derived

TABLE 10

t	quasi-stationary motion	$\zeta(\frac{1}{2}\pi, 0, t)$
0	− 2.18	—
π	− 0.40	···
2π	1.42	1.65
3π	3.10	3.28
4π	4.48	4.62
5π	5.42	5.53
6π	5.83	5.92
7π	5.67	5.74
8π	4.95	5.01
9π	3.75	3.80
10π	2.18	2.22
11π	0.40	0.43
12π	− 1.42	− 1.40

If $\Omega=0$ we have the exact expression

$$\bar{\zeta}(\tfrac{1}{2}\pi, 0, p) = \frac{\omega}{p^2+\omega^2}\,\frac{\mathrm{th}\; 2\pi q}{q}. \tag{4.5}$$

The quasi-stationary motion can be derived from (4.5) by taking only the contributions of the main poles $p=\pm i\omega$. This gives with $\omega=0.1$

$$\zeta_{\mathrm{qsm}}(\tfrac{1}{2}\pi, 0, t) = 6.96 \sin \omega t - 1.33 \cos \omega t. \tag{4.6}$$

In order to obtain the elevation proper the contributions of the higher poles have to be added to (4.6). These poles are given by

(4.7) $$p = -\tfrac{1}{2}\lambda \pm \tfrac{1}{2}\mathrm{i}\sqrt{(n+\tfrac{1}{2})^2 - \lambda^2}, \qquad n = 1, 2, 3 \ldots.$$

It appears to be sufficient if only the contribution of the first pair of poles i.e.

$$p = -0.06 \pm \mathrm{i}\; 0.243$$

is taken into consideration. Then we obtain

(4.8) $$\zeta(\tfrac{1}{2}\pi, 0, t) = \zeta_{\mathrm{qsm}} + e^{-0.06t}(1.32 \;\cos 0.243t - 2.05 \;\sin 0.243t) + \ldots.$$

The numerical values of ζ_{qsm} and ζ according to (4.6) and (4.8) are given in the following table

TABLE 11

t	quasi-stationary motion for $\Omega = 0$	$\zeta(\Omega = 0)$
0	− 1.33	0
π	0.88	0.50
2π	3.02	1.66
3π	4.85	3.48
4π	6.21	5.36
5π	6.96	7.06
6π	7.03	7.63
7π	6.42	7.07
8π	5.17	5.54
9π	3.42	3.38
10π	1.33	1.07
11π	− 0.88	− 1.21
12π	− 3.02	− 3.21

Comparison of tables 10 and 11 again shows the damping influence of the Coriolis effect. The various cases have been sketched in fig. 4.

C_1 gives the stationary elevation $\zeta = 2\pi \sin 0.1\, t$.

C_2 gives $\zeta(\tfrac{1}{2}\pi, 0, t)$ for $\Omega = 0.6$ according to table 10.

C_3 gives $\zeta(\tfrac{1}{2}\pi, 0, t)$ for $\Omega = 0$ according to table 11.

C_4 gives the elevation in the almost trivial case $\lambda = 0$ and $\Omega = 0$.

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