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Some recent work of the Amsterdam Mathematical Centre on the hydrodynamics of the North Sea

by

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Summary

Storm surges of the North Sea are studied by means of a simplified mathematical model which is described by linear partial differential equations that can be solved by analytical methods. The problem is also solved numerically by using difference equations. The results obtained by both methods are in perfect agreement, which provides us with a check of the adequacy of either method.

The numerical method is also applied to more complicated models.

The research on the hydrodynamic behaviour of the North Sea was started by the Amsterdam Mathematical Centre shortly after the renowned storm surge of February 1st 1953. Under the supervision of the late Van Dantzig the applied mathematics departmant concentrated all efforts on the study of simplified analytical models. The foundations of this research were laid by H. Poincaré (1) and G.I. Taylor (2). In a number of papers and reports the Mathematical Centre gave a further development and extension of the theory. We mention a report given by Van Dantzig (5) at the international congress of mathematicians at Amsterdam 1954 and a paper by the same author (6) offered at the GAMM conference at Saarbrücken 1958. Work of the Mathematical Centre is also contained in the final report of the Delta Committee (7). A systematic account of the research of the Mathematical Centre is given in a series of papers in the Proceedings of the Kon.Ned. Ak. v. Wet. of Amsterdam (also published in the Indagationes Mathematicae) under the general heading "The North Sea Problem". (Cf. the bibliography at the end of this paper.)

In this paper we shall consider some recent results obtained in connection with the analytical model of a rectangular shallow bay which at its open end is connected to an ocean and which is bounded by three coasts. In the analytical model the depth is taken constant. Further the ocean is assumed to have an infinite depth. As we shall see this model may give a fair description of what is going on in the North Sea when it is subjected to a stormfield. Of course local effects such as the influence of the Channel and the Kattegat are left out of consideration.
Also the astronomical tide is disregarded.
The analytical model may be described by the well-known linearized hydrodynamic equations

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \lambda u - \Omega v + gh\dfrac{\partial \zeta}{\partial x} = U(x,y,t) \\[2ex] \dfrac{\partial v}{\partial t} + \lambda v + \Omega u + gh\dfrac{\partial \zeta}{\partial y} = V(x,y,t), \end{cases}$$

and the equation of continuity

$$(2)\qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0.$$

The region is described by $0 \leq x \leq a$, $0 \leq y \leq b$ where $x=0$, $x=a$, $y=0$ are coasts and $y=b$ is the open end at the ocean.
The boundary conditions are

$$(3)\qquad \begin{cases} u=0 & \text{for} \quad x=0 \quad \text{and } x=a, \\ v=0 & \text{for} \quad y=0 \\ \zeta=0 & \text{for} \quad y=b \end{cases}$$

Here we shall consider only the case of a uniform "northern" wind

$$(4)\qquad U=0 \qquad\qquad V=V(t).$$

It is assumed that for $t \to -\infty$ all dependent variables tend to zero exponentially with the time. Then we may apply the technique of Laplace transformation in the sense of

$$(5)\qquad \bar{\zeta}(x,y,p) = \int_{-\infty}^{\infty} e^{-pt}\,\zeta(x,y,t)\,dt.$$

In view of the linearity of the model we may take

$$(6)\qquad V(t) = -\,\delta(t)$$

without loss of generality.
Physically this is a momentary disturbance of the sea surface by a sudden stress from the north.
Postponing the details for a while we eventually arrive at a solution $\bar{\zeta}_o(x,y,p)$ holding of course for the particular windfield (4) and (6). However, if we take

(7) $$V(t) = -\varphi(t)$$

with an arbitrary time function the solution becomes simply $\bar{\zeta} = \bar{\zeta}_o(x,y,p)\ \bar{\varphi}(p)$. Inversion of (5) gives

(8) $$\zeta(x,y,t) = \frac{1}{2\pi i} \int e^{pt}\ \bar{\zeta}_o(x,y,p)\ \bar{\varphi}(p)dp,$$

with a vertical path of integration in the complex p-plane. By this inversion formula the determination of ζ is reduced to the calculation of the poles of $\bar{\zeta}_o$ and $\bar{\varphi}$ and the corresponding residues.

In this paper we shall consider the following cases

a) Exponential windfield

(9) $$\varphi(t) = c_1\ e^{p_1 t} - c_2\ e^{p_2 t},$$

with positive constants c_1, c_2, p_1, p_2.

b) Unit-step windfield

(10) $$\varphi(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

c) Step-sine windfield

(11) $$\varphi(t) = \begin{cases} 0 & \text{for } t < 0 \\ \sin\omega t & \text{for } t > 0 \end{cases}$$

In the first case the inversion formula gives simply

(12) $$\zeta(x,y,t) = c_1\ \bar{\zeta}_o(x,y,p_1)e^{p_1 t} - c_2\ \bar{\zeta}_o(x,y,p_2)e^{p_2 t}$$

so that we need to determine the Laplace transform of ζ for two particular values of p only. As we shall show below this calculation can be performed quite accurately.
In the second and the third case, however, we need some knowledge of the analytic behaviour of $\bar{\zeta}_o(x,y,p)$. Unfortunately it is not possible to obtain an explicit expression of this function. However, it has been found possible by a process which is described in detail in N.S.P. VI to obtain a fairly good approximate analytical expression for $\bar{\zeta}_o(\frac{1}{2}a,0,p)$ i.e. at the

mid-point of the "Dutch" coast y=0 corresponding roughly to the position of Den Helder. The analytic behaviour of this approximation which we shall denote by Z(p) is very similar to that of $\bar{\zeta}_0(\frac{1}{2}a,0,p)$, the agreement being best near p=0. In particular the analytical approximation enables us to determine the lowest eigenvalue of the rectangular bay. It turns out that this eigenvalue is a small real negative number. This means that the lowest free motion of this North Sea model is not a (slightly damped) oscillation but a pure damping.

The analytical solution has been worked out with the following numerical constants corresponding to realistic physical data in connection with storm surges of the North Sea

$$a = 400 \text{ km} \qquad \Omega = 0.44 \ \text{hr}^{-1}$$
$$b = 800 \text{ km} \qquad \lambda = 0.088 \ \text{hr}^{-1}$$

For the constant depth we take the harmonic mean of the depth of the North Sea which equals approximately

$$h = 65 \text{ m}.$$

In our calculations we have introduced a natural unit of length a/π and a natural unit of time

$$\frac{a}{\pi\sqrt{gh}} = 1.4 \text{ hr}.$$

This means that a and b can be replaced by π and 2π. Further the corresponding values of Ω and λ can be taken as $\Omega = 0.6$ and $\lambda = 0.12$. Unless specified otherwise we shall use from now on only these natural units. Then we have also $\sqrt{gh}=1$ i.e. the unit of velocity is determined by the speed of propagation of the free waves. As regards the windfields a,b,c we always take $\max|V| = 1$. In order to see what this means we take the uniform and stationary windfield U=0 V=-1. The corresponding stationary situation is easily obtained from the equations (1) by putting u=v=0. The solution is simply

$$\zeta_{st}(x,y) = 2\pi - y. \tag{13}$$

Hence at the "Dutch" coast an elevation of 6.28 units is obtained. If we connect the stress at the surface with the velocity of the wind at the surface V_s in m/sec it can be derived (cf. NSP VI section 2) that for the value of V_s=41 m/sec the unit of the elevation is exactly one meter. In other words formula (13) gives at y=0

$$(14) \qquad \zeta_{st}(x,0) = 6.28 \quad (V_s/41)^2 \quad \text{meter.}$$

For a realistic storm with V_s = 35 m/sec we obtain e.g. ζ_{st} = 3.60 m.

For the first windfield we take the numerical case

$$c_1 = 0.27 \qquad c_2 = 0.2\, c_1$$
$$p_1 = 0.12 \qquad p_2 = 0.18.$$

This windfield is represented graphically in figure 1. It represents a storm lasting for about 1½ days with a slow rise and a sharp decline. Beyond t = 27 the function (9) assumes very large negative values so that all physical reality is lost there. Therefore all calculations carried out for this windfield have to be stopped at this point or soon thereafter.

The second windfield may tell us in what way the stationary situation is approached when at t = 0 everything is at rest.

For the third windfield we take ω = 0.1. A semi-period of the sine represents a storm lasting about 44 hrs. By this a very reasonable model of a storm is obtained. Later on when discussing the numerical solution of the problem obtained by purely numerical methods we shall see what happens when ω is varied.

We shall now outline the way in which the analytical solution is obtained. Starting from the equations (1) and (2) we find by performing the Laplace transformation (5) the set of equations

$$(15) \qquad \begin{cases} (p+\lambda)\bar{u} & -\Omega\bar{v} + \bar{\zeta}_x = \bar{U} \\ (p+\lambda)\bar{v} & +\Omega\bar{u} + \bar{\zeta}_y = \bar{V} \\ & \bar{u}_x + \bar{v}_y + p\bar{\zeta} = 0. \end{cases}$$

Taking the momentary windfield (4) and (6) meaning that $\bar{U}$ = 0 and $\bar{V}$ =-1 we obtain by elimination of $\bar{u}$ and $\bar{v}$ the Helmholtz equation

$$(16) \qquad \bar{\zeta}_{xx} + \bar{\zeta}_{yy} - \kappa^2\bar{\zeta} = 0$$

where

$$(17) \qquad \kappa^2 = p(p+\lambda) + \Omega^2 \frac{p}{p+\lambda} \;,$$

with the boundary conditions

$$(18)\quad \begin{cases} (p+\lambda)\,\bar{\zeta}_x + \Omega(\bar{\zeta}_y+1) = 0 & \text{for } x=0 \text{ and } x=\pi \\ -\Omega\,\bar{\zeta}_x + (p+\lambda)\,(\bar{\zeta}_y+1) = 0 & \text{for } y=0 \\ \bar{\zeta} = 0 & \text{for } y=2\pi \end{cases}$$

It is possible to solve this problem by an expansion which contains two Kelvin waves and two infinite sets of Poincaré waves

$$(19)\quad \begin{aligned} \bar{\zeta}(x,y,p) &= \frac{\Omega}{p+\lambda}\,\frac{\operatorname{sh}\kappa(\frac{1}{2}\pi-x)}{\kappa\operatorname{ch}\frac{1}{2}\kappa\pi} + A_o\operatorname{sh}\left\{s(x-\tfrac{1}{2}\pi)-qy\right\} + \\ &+ B_o\operatorname{ch}\left\{s(x-\tfrac{1}{2}\pi)-qy\right\} + \sum_{n=1}^{\infty} A_n(\sin nx+\theta_n\cos nx)e^{-\nu_n y} + \\ &+ \sum_{n=1}^{\infty} B_n(\sin nx-\theta_n\cos nx)e^{-\nu_n(2\pi-y)}, \end{aligned}$$

where

$$(20)\quad s = \Omega\sqrt{\frac{p}{p+\lambda}},\quad q=\sqrt{p^2+\lambda p},\quad \nu_n=\sqrt{n^2+\kappa^2},\quad \theta_n=\frac{p+\lambda}{\Omega}\,\frac{n}{\nu_n}.$$

The unknown coefficients A_n and B_n are determined by the conditions at y=0 and y=b. They lead to two sets of linear equations of the type

$$(21)\quad \begin{cases} A_n = \phi_n(B_o) + O(e^{-2\pi}) \\ B_n = \psi_n(A_o) + O(e^{-2\pi}), \end{cases}$$

where the order terms containing the contribution of the Poincaré waves at the opposite boundary are almost negligible. These equations can be solved only numerically. In NSP VII the calculations are carried out explicitly for the two values $p_1 = 0.12$ and $p_2 = 0.18$.

Eventually we arrive at the following values at the midpoint of the "Dutch" coast

$$\bar{\zeta}(\tfrac{1}{2}\pi,0,p_1) = 4.14 \qquad \bar{\zeta}(\tfrac{1}{2}\pi,0,p_2) = 3.46.$$

Substitution of these values into (12) gives the elevation at $(\frac{1}{2}\pi,0)$ as a function of time. This is shown in figure 1 where it is compared with the quasi-stationary elevation $2\pi\,\varphi(t)$ i.e. the elevation which we would obtain if at any moment the sea is in its equilibrium position with respect to the windstress at that time. The same figure contains also the elevation for $\Omega=0$, i.e. with absence of the Coriolis force. We note that for $\Omega = 0$ the following explicit expression can be derived quite easily

$$(22)\quad \bar{\zeta}(x,0,p) = \frac{\tanh 2\pi q}{q}$$

Using the expression (19) a great number of values of $\zeta(x,y,t)$ have been calculated. From these data a number of pictures (figure 2) have been drawn showing lines of equal elevation at the instant $t = 0\ (\pi)\ 9\pi$. It is of interest to note that this rather simplified model gives, at least qualitatively, a rather good picture of the true pattern of the North Sea surface. (Cf. Corkan (3) and Rossiter (4)).

It has already been said that the equations (21) cannot be solved analytically. However, it has been found possible to obtain approximate analytical expressions for the coefficients A_n ($n=0,1,2,\ldots$) and B_o. Substitution of these expressions in (19) gives for $y=0$ the following result

$$(23)\qquad \bar{\zeta}(x,0,p) \approx \frac{\mathrm{sh}\,\frac{1}{2}\pi(s+4q) - \mathrm{sh}\,\frac{1}{2}\pi s}{q\ \mathrm{ch}\,\frac{1}{2}\pi(s+4q)} \stackrel{\text{def}}{=} Z(p)$$

It can be shown that this approximation is best at the mid-point $(\frac{1}{2}\pi, o)$. The lowest eigenvalue of the model corresponds with the lowest pole of $Z(p)$ i.e. $p_o = -\ 0.074$ or $-0.053\ \mathrm{hr}^{-1}$.

By using (8) with the approximation (23) the response at $y=0$ (say at $x = \frac{1}{2}\pi$) to the windfields (10) and (11) can be determined by standard methods. For the step-function windfield we have obtained

$$(24)\qquad \zeta(\tfrac{1}{2}\pi,0,t) \approx 2\pi - 0.77\ e^{-0.074t} + \ldots$$

The graph of this function is given in figure 3 together with the corresponding graphs for the cases $\Omega = 0$ and $\lambda = \Omega = 0$. For the step-sine windfield we have obtained (see figure 4)

$$(25)\qquad \zeta(\tfrac{1}{2}\pi,0,t) \approx 5.83\ \sin\omega(t-3.9) + 0.37\ e^{-0.074t} + \ldots$$

Figures 1 and 4 lead to the assumption that a fair prediction of the maximum elevation at $(\frac{1}{2}\pi,0)$ is obtained by taking the stationary elevation 2π corresponding to the windmaximum $\max|V| = 1$. In fact the exact numerical values are as follows. For the exponential windfield we have $\max \zeta = 5.90$ and for the step-sine windfield $\max \zeta = 5.93$. Moreover, we observe that the elevation $\zeta(\frac{1}{2}\pi,0,t)$ imitates the wind-function $\varphi(t)$ with a time-lag of about 5 hours. This is in agreement with the observations in nature.

The analytical treatment has already furnished us much insight into the hydrodynamics of a storm surge, yet is deficient in two respects. First, although this is not an essential restriction of the method, our treatment considers only uniform windfields. Second, and this restriction is essential, the method applies only to the idealized case of a uniform depth.

Quite recently the North Sea problem has been attacked anew by purely numerical means. This attack was made possible by having at our disposal the electronic computer X1 which was put into use at the Mathematical Centre in the beginning of 1960.

Our first objective was to check the results obtained by the analytical method. Therefore we have considered the problem (1) (2) and (3) for an arbitrary bottom profile with a fairly general windfield depending on x,y and t, but still for a rectangular sea.

For the numerical treatment we use the system (1) (2) with the boundary conditions (3). At first the equations (1) and (2) are approximated by

$$(26)\quad \begin{cases} u(x,y,t+\Delta t) = (1-\lambda\Delta t)u(x,y,t)+\Omega\Delta t\ v(x,y,t) + \\ \qquad\qquad + \Delta t\left\{U(x,y,t)-gh\ \zeta_x(x,y,t)\right\} \\ v(x,y,t+\Delta t) = (1-\lambda\Delta t)v(x,y,t)-\Omega\Delta t\ u(x,y,t) + \\ \qquad\qquad + \Delta t\left\{V(x,y,t)-gh\ \zeta_y(x,y,t)\right\}, \end{cases}$$

and

$$(27)\quad \zeta(x,y,t+\Delta t) = \zeta(x,y,t) - \Delta t\left\{u_x(x,y,t+\Delta t)+v_y(x,y,t+\Delta t)\right\}.$$

The next approximation consists in replacing the partial derivatives with respect to x and y by suitable differences. The essential problem is to set up a difference scheme which is stable. However, the problem is in general not of the kind to which the methods of Lax and Richtmyer are easily applicable. Only for $\Omega=0$ may a simple stability criterion be derived. This criterion has been used to adapt the time interval Δt to the mesh width Δx and Δy.

In the first difference scheme which we tried central differences were used such as

$$(28)\quad \zeta_x(x,y,t) \approx \frac{\zeta(x+\Delta x,y,t)-\zeta(x-\Delta x,y,t)}{2\Delta x}.$$

When put on the computer the scheme proved to be stable for small values of Ω of the order of 0.1. However, for higher values of Ω strong instabilities developed causing almost useless results at the North Sea value of $\Omega=0.6$.

In the second scheme the stream and the elevation were calculated at different points forming two interlacing nets. Still using central differences and making appropriate changes at the boundary we obtained only slightly better results than in the previous scheme. It is not unlikely that the instability was generated at the corner points of the rectangular basin where boundaries of different nature, ocean and coast, meet each other,

The final scheme which proved to be stable even for higher values of Ω uses interlacing nets where each stream point (elevation point) is surrounded by a "square" of elevation points (stream points) (see fig.5). The partial derivatives were replaced by averaged central differences, a typical example of which is

$$(29)\quad \begin{aligned} \zeta_x(x,y,t) \approx \tfrac{1}{2}\Big\{ & \frac{\zeta(x+\Delta x, y+\Delta y, t) - \zeta(x-\Delta x, y+\Delta y, t)}{2\Delta x} + \\ & + \frac{\zeta(x+\Delta x, y-\Delta y, t) - \zeta(x-\Delta x, y-\Delta y, t)}{2\Delta x} \Big\} . \end{aligned}$$

At the boundaries we used weighted averages such as

$$(30)\ \zeta_x(x,0,t) \approx \frac{3\zeta(x+\Delta x,\Delta y,t) - 3\zeta(x-\Delta x,\Delta y,t) - \zeta(x+\Delta x,3\Delta y,t) + \zeta(x-\Delta x,3\Delta y,t)}{4\Delta x}$$

This scheme has been successfully applied to a great variety of cases. The program used meshes with $\Delta x = a/24$, $\Delta y = b/49$, i.e. with 13×25 stream points and 12×25 elevation points. The time step Δt was about 5 min. The program could accept an arbitrary depth function $h(x,y)$ and a windfield of the type

$$(31)\qquad \begin{cases} U(x,y,t) = U'(x,y)U''(t) \\ V(x,y,t) = V'(x,y)V''(t). \end{cases}$$

We first considered the analytical model discussed above. The numerical results confirmed those obtained earlier by the analytical method, which provided a check on either method. Figure 6 shows the values of $\zeta(\frac{1}{2}\pi, 0, t)$ for the step-sine windfield (11) obtained by the two methods. Next we considered the same model but with a non-uniform depth h of the type

$$(32)\qquad h = h_0 \exp \alpha y$$

with $\alpha = \frac{1}{4}$ and a harmonic mean of 65 m. so that h varies between 30 m. and 160 m.

The exponential depth function (32) gives a much better approximation of the bottom profile of the North Sea and has the advantage that it also permits analytical methods (e.g. when discussing the stationary problem, cf. N.S.P. III). Figure 6 also shows the elevation at $(\frac{1}{2}\pi, 0)$ when the exponential depth (32) is taken. The various values of max ζ are listed below

approximate analytic expression	5.93
numerical method, uniform depth	6.13
numerical method, exponential depth	6.66
stationary elevation	6.28

The numerical method enabled us to assess the influence of the damping constant λ. We have calculated $\zeta(\frac{1}{2}\pi, 0, t)$ for a number of λ-values for the model with the exponential depth (32) and the step-sine windfield (11). The results are shown in figure 7. It appears that the elevation is rather insensitive to small variations of λ.

Next we have considered a number of sine-windfields with different values of ω. We have plotted the resulting elevations at $(\frac{1}{2}\pi, 0)$ in figure 8. Some sort of resonance is shown at $\omega = 0.14$ where max $\zeta = 6.83$. This resonance is shown more clearly in figure 9 where we have plotted max $\zeta(\frac{1}{2}\pi, 0, t)$ against ω.

These calculations indicate that a storm with a sinusoidel time behaviour is most dangerous when it has a devation - i.e. the semi-period of the sine- of about 32 hours. Further calculations indicate that the worst direction for a storm is one which makes an angle of about 12° with the positive Y-axis. For the North Sea this means a deviation of some 25° from the North in Western direction.

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