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# THE NECESSARY AND SUFFICIENT REQUIREMENTS FOR THE EXISTENCE OF DISSIPATIVE STRUCTURES IN (BIO)CHEMICAL SYSTEMS

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# THE NECESSARY AND SUFFICIENT REQUIREMENTS FOR THE EXISTENCE OF DISSIPATIVE STRUCTURES IN (BIO)CHEMICAL SYSTEMS *)

H.C. Hemker **) and G.M. Willems

SUMMARY

For systems in which only the usual (biochemical) transport phenomena occur, i.e. diffusion and chemical interconversion, it is shown that stationary and stable situations in which one reactant is unevenly distributed along a space coordinate are impossible.

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*) This paper is not for review; it is meant for publication in a journal.

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# 1. One-dimensional one-morphogen systems

*Introduction*

Spatial constraints imposed upon molecules of a chemical system by macroscopic structures play an important role in theoretical biochemistry: transport across membranes, reactions at interphases, reactions confined to compartments, etc., are commonly considered. The reversal: macroscopic structure arising from chemical interactions in an originally homogeneous medium (i.e., morphogenesis in the sense of Turing) is less often the subject of investigation.

Turing (1952) was the first to draw attention to this possibility. He showed that on the basis of a certain chemical mechanism - conceived *ad hoc* in his 1952 article - space-dependent concentration changes would arise. We will define a "Turing system" as an open system allowing chemical and diffusional concentration changes and tending to the formation of macroscopic structure.

Prigogine and coworkers (see Prigogine, 1971) pointed out that the apparent violation of the second law of thermodynamics involved in the genesis of structure in a previously homogeneous medium could be explained in terms of symmetry-breaking instabilities on basis of the fact that the structures are dissipative. Dissipative structures, unlike equilibrium structures, cannot exist unless there is an exchange of matter or energy with the surroundings. It has been proven on the basis of graph theory (Rosen, 1958) that a biochemical system of the type found in all living matter contains at least one "non-regenerative component", which means that they need at least one component supplied from the surround-

ings to continue functioning. It is also known that no living structure will remain intact under adiabatic conditions. It thus seems a necessary condition for a living structure to be a dissipative structure.

The transport phenomena occuring in a dissipative structure may differ widely, the most important in living matter being chemical interconversion and diffusion.

There is no conclusive evidence yet that stable chemical dissipative structures (i.e. stable Turing systems) exist except in living matter. To investigate the hypothesis that a necessary and sufficient condition for living matter is to be a stable dissipative structure, it is compulsory to determine the minimal conditions for the existence of a "Turing system".

There are several other reasons for being interested in the minimal and sufficient requirements for the existence of Turing systems:

a. Since no living system is known that is not a dissipative structure, in the origin of life a Turing system must have occured somewhere under abiogenic conditions. It is of interest to know what the minimal and sufficient conditions for this happening were.

b. A "Turing system" implies the existence of concentration gradients that tend to be stable. It therefore permits the transport of molecules - by a chemical mechanism - against a concentration gradient. This suggests that problems of so-called "active transport" might be strongly related to "Turing systems".

c. Last, but not least, the thermodynamicist remains interested in the conditions to be fulfilled if a flow of matter or energy is to give rise to "symmetry-breaking instabilities" and the establishment of stable dissipative structures.

It can be shown that by providing suitable boundary conditions, e.g. by creating sources and sinks, dissipative structures can be maintained with very simple chemical reaction schemes (Hermens, 1971). The thermodynamic treatment of dissipative structures given thus far only considered this situation, i.e. the system is assumed to satisfy fixed boundary

conditions. Such boundary conditions presuppose the existence of structures outside the system that are so to speak reflected inside the system. Although this situation might occur, the three points mentioned above show that a number of fundamental questions require a more general treatment.

Therefore, we will analyse systems without fixed boundary conditions or with boundary conditions compatible with systems enclosed in vessels with impermeable walls.

*Definitions*

We define the following terms:

*Reactants* are all chemical species involved in the system. Following Turing, we will call *morphogens* the reactants whose concentrations may vary as a function of the time and space coordinates.

We define the *chemical order* of the system as the highest order of a reaction in which one or more of the morphogens is involved. The order of a reaction is defined in the usual way as: n in $\frac{dM}{dt} = M^n + f(M)$, f(M) containing only terms of an order < n (M being the concentration of a morphogen).

The *dimension* of the system equals the number of space coordinates along which the concentration of the morphogen is thought to vary. All real systems are of course three-dimensional. Experimental conditions can be chosen such that only one or two dimensions are of practical importance. A long narrow tube or a container in the form of a ring are examples of conditions that approach one-dimensional systems.

Finally, the number of morphogens should be taken into consideration. Interacting morphogens will of course add to the complexity of the system, because one morphogen can (counter)act catalytically the formation or disappearance of another.

*Scope of the problem*

Theoretical treatments of Turing systems have been carried out only

on the basis of a given chemical reaction scheme. We set out to investigate a more general question: What are the necessary and sufficient conditions for a chemical system to give rise to a dissipative structure? This article is the first of a series in which this problem will be treated.

For a dissipative structure to be of interest it must be stationary, i.e. the concentration(s) must be constant in time. Also it must be stable, i.e. it must react to perturbations by returning to its original state. The stability and stationarity of each Turing system under consideration must therefore be investigated.

Eventually we want to be able to consider three-dimensional multimorphogen systems of any desired order. In this paper we start by investigating one-dimensional one-morphogen systems of any order.

## 2. The mathematical model

The state of a one-dimensional one-morphogen system is described by a formula that gives the concentration as a function of the time and space coordinates. The variation of the concentration at a given point results from chemical interconversion and diffusion. The chemical term depends upon the chemical make-up of the system. Diffusion is governed by Fick's law, hence

$$\frac{\partial}{\partial t} U(x,t) = f(U(x,t)) + d \frac{\partial^2}{\partial x^2} U(x,t), \tag{1}$$

in which

U is the concentration of the morphogen,
x is the space coordinate,
t is the time,
d is the diffusion constant,
f is the source term generated by chemical reations.

A physically realistic approximation of a one-dimensional system is

a long narrow trough or tube (length $\pi$) with impermeable walls. For such a system the boundary conditions

$$\frac{\partial}{\partial x} U(0,t) = \frac{\partial}{\partial x} U(\pi,t) = 0. \tag{2a}$$

should be considered. Another possibility is a narrow circular trough or tube of length $2\pi$. This setup is due to Turing (1952). The periodicity is expressed by the boundary conditions

$$U(0,t) = U(2\pi,t),$$
$$\frac{\partial}{\partial x} U(0,t) = \frac{\partial}{\partial x} U(2\pi,t). \tag{2b}$$

As pointed out in the introduction we are primarily interested in the stationary (i.e. time-dependent) solutions $U = U_s$ of equation (1), which corresponds to the stationary states of the physical system. A stationary solution satisfies the equation

$$0 = f(U_s) + d \frac{\partial^2}{\partial x^2} U_s, \tag{3a}$$

and the boundary conditions

$$\frac{\partial}{\partial x} U_s(0) = \frac{\partial}{\partial x} U_s(\pi) = 0, \tag{3b}$$

or

$$U_s(0) = U_s(2\pi), \ \frac{\partial}{\partial x} U_s(0) = \frac{\partial}{\partial x} U_s(2\pi). \tag{3c}$$

In the next section we will show that for a large class of functions the boundary-value problem (3) does have a solution $U_s$ that is space dependent.

## 3. The existence of a steady state

We first consider equation (3a) with boundary conditions (3c). By integration of (3a) after multiplication by $\frac{d}{dx} U_s$ we obtain

(4a) $$E(U_s, \frac{dU_s}{dx}) = E_0 = \text{constant along each solution of (3a)},$$

in which

(4b) $$E(U_s, \frac{dU_s}{dx}) = \frac{d}{2}(\frac{d}{dx} U_s)^2 + F(U_s), \quad F(U_s) = \int_0^{U_s} f(v)\, dv.$$

Thus E is conserved. At this point it may be useful to note that (3a) also describes the one-dimensional dynamics of a point mass in a conservative force field $f(U_s)$. In this case x stands for time, d for the mass of the point, and $U_s$ for the space coordinate. Now the E defined by (4) is the usual mechanical energy, which is known to be conserved.

The trajectories $(U_s(x), \frac{d}{dx} U_s(x))$ of solutions $U_s$ of (3a) are given by (4). These trajectories are the curves in the $(U_s, \frac{d}{dx} U_s)$-plane determined by the equation

$$E(U_s, \frac{d}{dx} U_s) = E_0,$$

for some constant $E_0$.

A solution $U_s$ of (3a,c) corresponds to a closed trajectory of period $\frac{2\pi}{k}$, k=1,2,... . In figure 1 the trajectories are sketched for the case

$$f(U) = (U-1)(U-2).$$

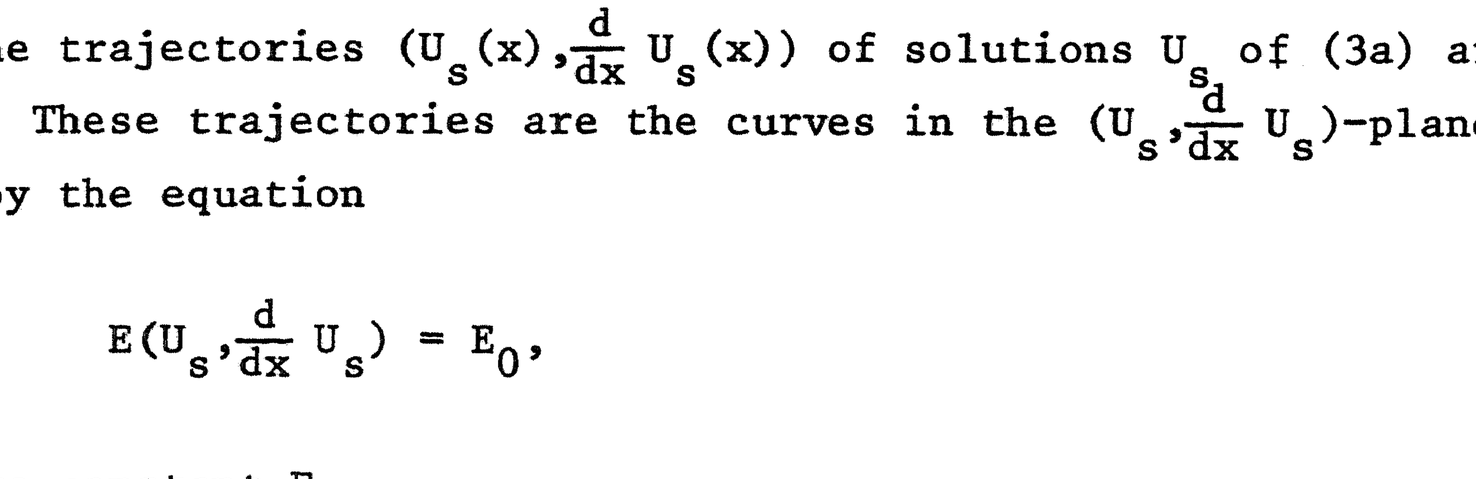


fig. 1

It is easily recognized that there exists a one-parameter family of closed trajectories surrounding every point $(U_0,0)$ such that

$$f(U_0) = 0, \qquad \text{i.e. } U_0 \text{ is a homogeneous steady state of (1)}$$

and

$$\frac{d}{dU} f(U_0) > 0, \quad \text{i.e. } U_0 \text{ is a local minimum of F.}$$

On closer examination it turns out that to every point $(v,0)$ with v in the interval $[U_1,U_2]$ on the $U_s$-axis there is associated a closed trajectory given by the equation $E_0 = F(v)$, provided that the interval $[U_1,U_2]$ has the properties:

$U_1 < U_0 < U_2$,
F is monotonic on $[U_1,U_0]$ and $[U_0,U_2]$, and
$F(U_1) = F(U_2)$.

Next, it turns out that the trajectory associated to $(v,0)$, say $B_v$, with $v \approx U_0$ has a period P(v)

$$P(v) = \int_{B_v} dx = \int_{B_v} \frac{dx}{dU_s}\, dU_s = \int_{B_v} \pm \frac{d}{2(F(v)-F(U_s))}\, dU_s. \tag{5}$$

which is approximately equal to

$$2\pi \frac{d}{\frac{d}{dU} f(U_0)}.$$

If we have for some $v \in (U_0,U_2)$

$$P(v) > \frac{2\pi}{k}, \quad k = 1,2,\ldots, \tag{6a}$$

and if the inequality

$$2\pi \frac{d}{\frac{d}{du} f(U_0)} < \frac{2\pi}{k} \tag{6b}$$

holds for the same k, then it follows from the continuity of P(v) that there exists a $v_0 \in (U_0, v)$ such that the associated trajectory has a period

$$P(v_0) = \frac{2\pi}{k} .$$

Hence, equation (3a,c) has a non-homogeneous solution if conditions (6a) and (6b) are satisfied. If F(u) has a (local) maximum in $U_1 > U_0$ and does not have an extremum in the interval $(U_0, U_1)$, then we obtain from (5)

$$\lim_{v \uparrow U_1} P(v) = \infty.$$

So in that case (6a) is certainly satisfied.

From the foregoing we may conclude that (3a,c) has a solution for a large class of functions f. It also follows that in general for non-linear functions f a small perturbation of f is reflected in a small perturbation of the steady state $U_s$. This in contrast with the linear case, where a small perturbation of the linear function f causes the inhomogeneous steady state to disappear. As a last remark we should mention that from (5) it follows that for decreasing diffusion constant the period of solutions $U_s$ of (3a) also decreases. So for small d we may expect rapidly oscillating steady states.

For solution $U_s$ of the boundary value problem (3a,b) the trajectory $(U_s(x), \frac{d}{dx} U_s(x))$ must have the property that it intersects the $U_s$-axis for $x = 0$ and $x = \pi$.

From the fact that $\frac{d}{dx} U_s$ appears only in the quadratic term in (4), it follows that this occurs if (3a) possesses a $2\pi$-periodic solution. Conditions similar to (6) yield the existence of a solution of (3a,b). In a certain way the problems (3a,b) and (3a,c) are equivalent.

## 4. Stability of the steady states

As pointed out in the introduction, we are primarily interested in

stable steady states.

A steady state $U_s$ of (1) is said to be *stable* in the sense of Liapunov in the norm $\| \cdot \|$, if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution v of the initial-boundary value problem (1,2a) or (1,2b) with the property

$$\| v(.,0) - U_s(.) \| < \delta ,$$

satisfies

$$\| v(.,t) - U_s(.) \| < \varepsilon ,$$

for all $t > 0$ (see [5], p. 314). We give two examples of an appropriate norm for the initial-boundary value problem (1,2b).

a. the norm $\| \cdot \|_\infty$ on the space $C_{2\pi}$ of $2\pi$-periodic continuous (real-valued) functions defined by

$$\| f \|_\infty = \sup_{x \in [0,2\pi]} |f(x)| .$$

b. The norm $\| \cdot \|_2$ on the space of square integrable functions on the interval $[0,2\pi]$ defined by

$$\| f \|_2 = \left( \int_0^{2\pi} |f(x)|^2 \, dx \right)^{\frac{1}{2}} .$$

The steady state $U_s$ is said to be *unstable* if it is not stable.

As usual our analysis of the stability of $U_s$ proceeds by considering the linearization in $U = U_s$ of equation (1). We see that (small) perturbations of $v(x,t)$ of the steady state $U_s(x)$ approximately satisfy

$$(8) \qquad \frac{\partial}{\partial t} v = \frac{d}{dU} f(U_s) v + d \frac{\partial^2}{\partial x^2} v ,$$

and the boundary conditions (2a) or (2b).

The method of separation of variables is used to solve the linear

equation (8). We try to represent v by a linear combination of functions

$$(9) \qquad v_\lambda(x)e^{\lambda t}.$$

From substitution of (9) in (8), it follows that $\lambda$ and $v_\lambda$ must satisfy

$$(10) \qquad \lambda v_\lambda = \frac{d}{dU} f(U_s)v_\lambda + d\,\frac{d^2}{dx^2} v_\lambda$$

and the appropriate boundary conditions. Thus $\lambda$ is an eigenvalue and $v_\lambda$ is the associated eigenfunction of the differential operator

$$(11) \qquad d\,\frac{d^2}{dx^2}\,\cdot + \frac{d}{dU} f(U_s).$$

defined on the space of twice differentiable functions satisfying boundary conditions (2a) or (2b). Since the operator (11) is self-adjoint, it follows that the eigenvalues $\lambda$ are real and the eigenfunctions $v_\lambda$ corresponding to different eigenvalues are orthogonal. Moreover, it is well-known that every twice-differentiable function g satisfying the boundary conditions can be represented by a uniform convergent series of the form

$$g = \sum_\lambda a_\lambda v_\lambda,$$

(These results on self-adjoint differential operators can e.g. be found in [5], Ch. 7).

From the foregoing it is clear that the solution of (8) with initial condition

$$v(x,0) = g(x)$$

is given by

$$(12) \qquad v(x,t) = \sum_\lambda a_\lambda\, e^{\lambda t}\, v_\lambda(x).$$

From (12) it can be seen that the stability of the zero solution of

(8) is determined by the sign of the maximal eigenvalue $\lambda_m$ of (11); $\lambda_m$ positive implies instability, $\lambda_m$ negative implies stability. The steady state $U_s$ of (1) is said to be *exponentially stable* (*unstable*) if $\lambda_m$ is negative (positive).

The common belief that exponential stability (instability) implies stability (instability) is in the case justified by

Theorem 1. a. If the steady state $U_s$ of (1), (2a) or (2b) is exponentially stable then it is also stable.

b. If the steady state $U_s$ of (1), (2a) or (2b) is exponentially unstable then it is also unstable.

Proof. The proof of theorem 1.b is given in Appendix A. Theorem 1.a will not be used in this paper, but can be proven in the same way as theorem 1.b. □

Thus we are led to the problem of the determination of the sign of the maximal eigenvalue $\lambda_m$ of the self-adjoint operator (11). We will use the following characterization of the maximal eigenvalue $\lambda_m$,

$$\lambda_m = \sup_{v \neq 0} \frac{\int [\frac{d}{dU} f(U_s) v^2 - (\frac{d}{dx} v)^2] dx}{\int v^2 \, dx}, \tag{13}$$

where the supremum is taken over the class of all differentiable functions satisfying the boundary conditions (2a,b) (see e.g. [5], p. 195).

Using (13), one can prove:

Theorem 2. If the steady state $U_s$, considered as function of x, is not a constant, then the maximal eigenvalue $\lambda_m$ of (10) is positive, i.e. $U_s$ is exponentially unstable.

Proof. See Appendix B. □

From theorems 1 and 2 it is easily seen that equation (1) with boundary conditions (2a,b) does not have inhomogeneous (non-constant) stable steady states.

It thus appears that no matter what chemical reaction scheme applies, it is impossible for one component of a chemical system to show stable variations of concentration along a space coordinate if the other components are constant. This is not in contradiction with the results of Turing (1952), because in his system two morphogens are considered.

The results given here are limited to one-dimensional systems. From investigations being carried out we expect that application of functional analysis will make it possible to extend this statement to systems of three (or indeed any number of) dimensions.

## Appendix A

For the proof of theorem 1.b it is sufficient to show that there exists an $\varepsilon > 0$ and a sequence of solutions $w_n(x,t)$ of (1) with boundary conditions (2a,b) satisfying

$$(A_1') \qquad \| U_s(\cdot) - w_n(\cdot,0) \| < 1/n$$

and

$$(A_1') \qquad \sup_{t>0} \| u_s(\cdot) - w_n(\cdot,t) \| > \varepsilon.$$

Let $\lambda_m$ be the largest eigenvalue of (11) with eigenfunction $v_{\lambda_m}$, $(\lambda_m > 0)$, $w_n(x,t)$ the solution of (1) with initial conditions

$$w_n(x,0) = U_s(x) + \frac{1}{n} v_{\lambda_m}(x).$$

Then the function $v_n$ defined by

$$v_n(x,t) = w_n(x,t) - U_s(x)$$

is a solution of the differential equation

$$(A_2) \qquad \frac{\partial}{\partial t} v_n(x,t) = \frac{d}{dU} f(U_s(x))\, v_n(x,t) + d \frac{\partial^2}{\partial x^2} v_n(x,t) + R_n(x,t),$$

$$R_n(x,t) = 0(|v_n(x,t)|^2),$$

with initial conditions

$$v_n(x,0) = \frac{1}{n} v_{\lambda_m}(x).$$

Let $r_n(x,t)$ be defined as

$$(A_3) \qquad r_n(x,t) = v_n(x,t) - \frac{1}{n} e^{\lambda_m t} v_{\lambda_m}(x),$$

where $\frac{1}{n} e^{\lambda_m t} v_{\lambda_m}(x)$ is a solution of the linearized equation $(A_2)$. Now we

shall estimate $r_n(x,t)$. From $(A_2)$ and $(A_3)$ it follows that $r_n$ satisfies

$$(A_4) \qquad \frac{\partial}{\partial t} r_n(x,t) = \frac{d}{dU} f(U_s(x))\, r_n(x,t) + d \frac{\partial^2}{\partial x^2} r_n(x,t) + \tilde{R}_n(x,t),$$

$$\tilde{R}_n(x,t) = O(\frac{1}{n^2} e^{2\lambda_m t} + (r_n(x,t))^2),$$

and the initial condition

$$(A_5) \qquad r_n(x,0) = 0.$$

From the proof in Appendix B (or from [5], p. 214) it follows that the eigenfunction $v_{\lambda_m}$ associated to the maximal eigenvalue $\lambda_m$ is of constant sign, say positive. Hence there exists a positive constant $C_1$ such that

$$\min_x v_{\lambda_m}(x) = C_1$$

Let $C_2$ be a positive constant with the property that the inequality

$$|\tilde{R}_n(x,t)| \le C_2(\frac{1}{n^2} e^{2\lambda_m t} + r_n^2(x,t))$$

holds for all x and t with

$$|\frac{1}{n} e^{\lambda_m t}|\ t\ |r_n(x,t)| < \varepsilon_0.$$

Then by using a maximum principle (see [6], p. 187) it is easily deduced that the solution $\bar{r}_n(x,t)$ of the initial value problem

$$\frac{\partial}{\partial t}\,\bar{r}_n(x,t) = \frac{d}{dU}\,f(U_s(x))\;\bar{r}_n(x,t) + d\,\frac{\partial^2}{\partial x^2}\,\bar{r}_n(x,t) +$$

$$(A_6) \qquad + \frac{\lambda_m}{2}\;\bar{r}_n(x,t) + \frac{C_2}{C_1}\;\frac{1}{n^2}\,e^{2\lambda_m t}\,v_{\lambda_m}(x),$$

$$\bar{r}_n(x,0) = 0$$

has the property

$$(A_7) \qquad \bar{r}_n > r_n > -\bar{r}_n$$

for all t with

$$(A_8) \qquad \sup_{0\le\tau\le t}\ \sup_{x}\ \left(\left|\bar{r}_n(x,t)\right|\ t\ \left|\frac{1}{n}\,e^{\lambda_m t}\right|\right) < \min\left(\epsilon_0, \frac{\lambda_m}{2C_2}\right).$$

Substitution of

$$\bar{r}_n(x,t) = h(t)\;v_{\lambda_m}(x)$$

in $A_5$ gives

$$\frac{d}{dt}\,h = \frac{3}{2}\,\lambda_m\,h + \frac{C_2}{C_1}\;\frac{1}{n^2}\,e^{2\lambda_m t}$$

$$h(0) = 0.$$

Hence, we can estimate $\bar{r}_n$ by

$$(A_9) \qquad 0 \le \bar{r}_n \le 2\,\frac{C_2}{C_1\lambda_m n^2}\;\left(e^{2\lambda_m t} - e^{\frac{3}{2}\lambda_m t}\right)\;v_{\lambda_m}.$$

Inequalities $A_6$ and $A_9$ imply

$$(A_{10}) \qquad \|\,v_n(x,t)\,\| \ge f_n(t) = \left\{\frac{e}{n}^{\lambda_m t} - 2\,\frac{C_2}{C_1\lambda_m n^2}\,\left(e^{2\lambda_m t} - e^{\frac{3}{2}\lambda_m t}\right)\right\}\,\|\,v_{\lambda_m}\,\|$$

for all t satisfying $A_8$. It is easily seen that there exists an $\varepsilon > 0$ such that

$$\lim_{n\to\infty} \sup_{0\le t\le t_n} f_n(t) > \varepsilon > 0,$$

where $t_n$ is the largest positive t satisfying $A_8$. Thus we have found an $\varepsilon > 0$ and a sequence $w_n$ such that $A_1$ holds.

## Appendix B

We first prove Theorem 2 for the boundary conditions (2b). It is assumed that $U_s$ considered as function of x is not a constant (otherwise there is nothing to prove).
From (3) it follows that

$$v(x) = \frac{d}{dx} U_s(x)$$

satisfies the equation

$$(B_1) \qquad d \frac{d^2}{dx^2} v + \frac{d}{dU} f(U_s(x))\ v = 0,$$

and the boundary conditions (3c). Thus (9) has an eigenvalue zero and v is the corresponding eigenfunction. From the fact, that $U_s$ is $2\pi$-periodic and not a constant, it follows that there are at least two zeros of $v = \frac{d}{dx} U_s$ in the interval $[0,2\pi]$. Since v is a solution of the second order linear differential equation $(B_1)$, all its zeros have to be simple unless v itself is identically zero. This is seen in the following manner. From the unique solvability of the initial value problem $(B_1)$ with initial conditions

$$(B_2) \qquad v(x_0) = 0,\ \frac{d}{dx} v(x_0) = 0,$$

we deduce that $(B_2)$ implies

$$v \equiv 0.$$

Without affecting the generality it is assumed that there exist two consecutive zeros $x_0, x_1$ of v with the properties (see fig. $B_1$)

$$0 < x_0 < x_1 < 2\pi,$$

$$\frac{d}{dx} v(x_0) < 0, \quad \frac{d}{dx} v(x_1) > 0,$$

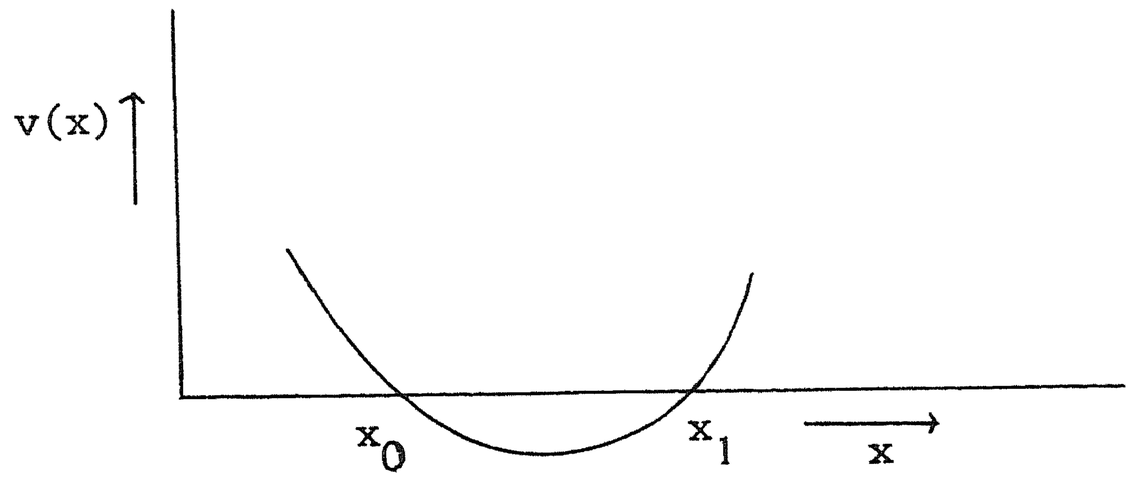


fig. $B_1$

Now the function w given by

$$w(x) = v(x), \quad x\in[0,2\pi], \quad x\notin[x_0,x_1],$$
$$w(x) = -v(x), \quad x\in[x_0,x_1],$$

(see fig. $B_2$) is considered.

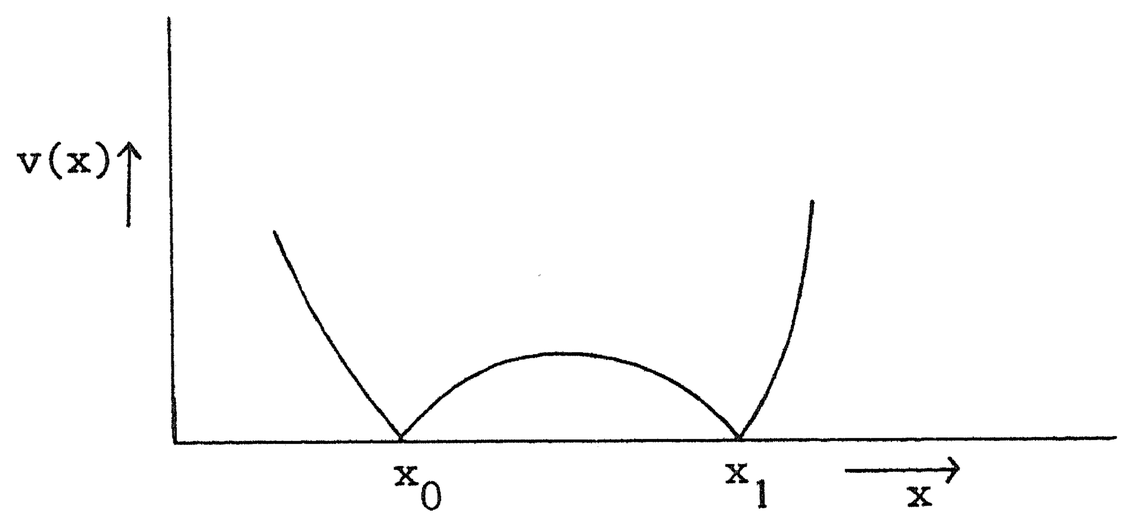


fig. $B_2$

We denote the functional

$$\int_a^b [\frac{d}{dU} f(U_s)z^2 - d(\frac{d}{dx} z)^2] \, dx,$$

defined on the space of continuous differentiable functions on the interval (a,b) by

$$I(z,a,b).$$

An easy computation using $B_1$ and partial integration shows that

$$(B_4) \qquad I(w,0,x_0) + I(w,x_0,x_1) + I(w,x_1,2\pi) = 0$$

The idea is to regularize the function w in such a way that the regularized function $\tilde{w}$ is continuous differentiable on $[0,2\pi]$ and that

$$I(\tilde{w},0,2\pi) > 0.$$

Using (9) we then can conclude that the maximal eigenvalue $\lambda_m$ is positive. The functions $\psi_\varepsilon(x)$ defined by

$$\psi_\varepsilon(x) = 0, \qquad x < -2\varepsilon,$$

$$\psi_\varepsilon(x) = \int_{-2\varepsilon}^{x} g_\varepsilon(x)dx, \quad -2\varepsilon \le x \le 0,$$

$$\psi_\varepsilon(x) = \psi_\varepsilon(-x), \qquad x > 0,$$

with

$$g_\varepsilon(x) = \frac{1}{\varepsilon}(x+2\varepsilon), \qquad -2\varepsilon < x < -\varepsilon$$

$$g_\varepsilon(x) = 1 \qquad , \qquad -\varepsilon \le x \le 0$$

have the properties (see fig. $B_3$)

$$\psi_\varepsilon(x) = 0, \quad |x| > 2\varepsilon$$

$$(B_5) \qquad 0 < \psi_\varepsilon(x) < 2\varepsilon,$$

$$\left|\frac{d}{dx}\psi_\varepsilon(x)\right| \le 1,$$

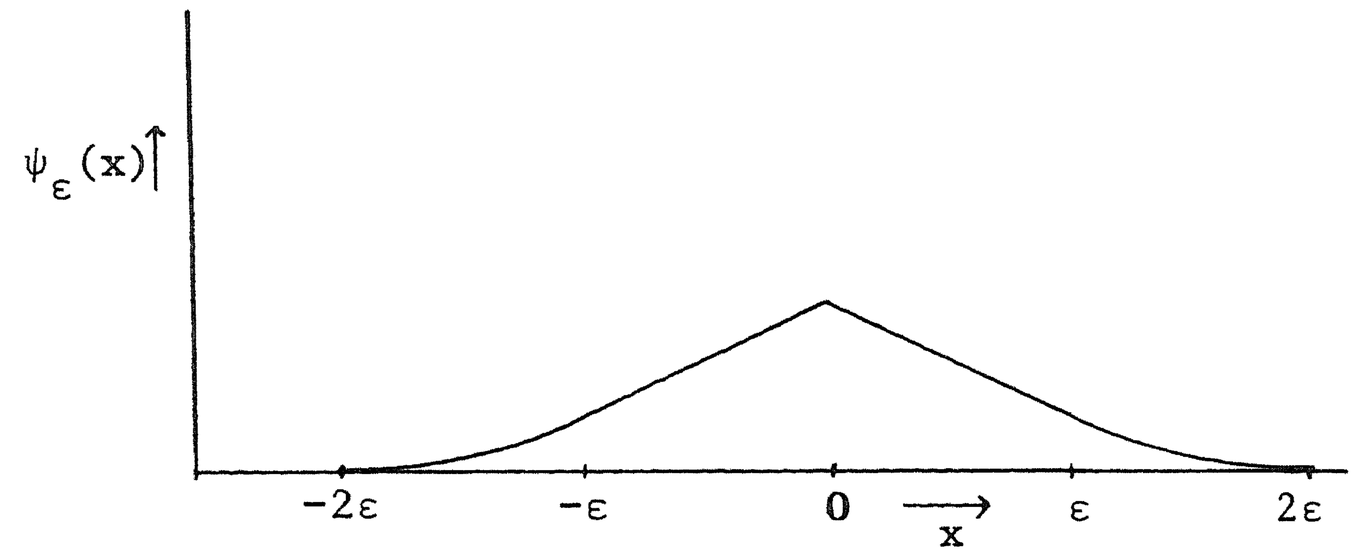


fig. $B_3$

The regularization $w_\varepsilon$ of w is defined by

$$(B_6) \qquad w_\varepsilon(x) = w(x) + \left|\frac{d}{dx}\, v(x_0)\right| \psi_\varepsilon(x-x_0) + \left|\frac{d}{dx}\, v(x_1)\right| \psi_\varepsilon(x-x_1).$$

Finally, we prove the existence of an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the inequality

$$(B_7) \qquad I(w_\varepsilon, 0, 2\pi) > 0$$

holds. From $B_2$ and $B_5$ it follows that it is sufficient to prove

$$(B_8) \qquad \begin{aligned} I(w_\varepsilon, x_0-2\varepsilon, x_0) &> I(w, x_0-2\varepsilon, x_0), \\ I(w_\varepsilon, x_0, x_0+2\varepsilon) &> I(w, x_0, x_0+2\varepsilon), \\ I(w_\varepsilon, x_1-2\varepsilon, x_1) &> I(w, x_1-2\varepsilon, x_1), \\ I(w_\varepsilon, x_1, x_1+2\varepsilon) &> I(w, x_1, x_1+2\varepsilon). \end{aligned}$$

Since the treatment of these four cases is entirely identical, we need only to consider the first inequality. For this we need the following estimates:

$$(B_{9a}) \qquad \sup_{x\in[0,2\pi]} \left|\frac{d}{dU} f(U_s(x))\right| < M ,$$

$$(B_{9b}) \qquad \left|\frac{d}{dx} w(x)\right| > \frac{1}{2}\left|\frac{d}{dx} w(x_0)\right|, \qquad |x-x_0| \le \varepsilon,$$

$$(B_{9c}) \qquad \left|\frac{d}{dx} w(x)\right| > \left|\frac{d}{dx} w_\varepsilon(x)\right| , \qquad -2\varepsilon \le x-x_0 < 0,$$

$$(B_{9d}) \qquad \left|\frac{d}{dx} w_\varepsilon(x)\right| < C_1 \varepsilon , \qquad -\varepsilon \le x-x_0 < 0,$$

$$(B_{9e}) \qquad |w_\varepsilon(x)| < C_2 \varepsilon , \qquad -2\varepsilon \le x-x_0 \le 0,$$

$$(B_{9f}) \qquad |w_\varepsilon(x)| \le C_2 \varepsilon , \qquad -2\varepsilon \le x-x_0 \le 0.$$

All estimates hold for $0 < \varepsilon < \varepsilon_0$ sufficiently small. Using these estimates, we find

$$(B_{10a}) \qquad \begin{aligned} I(w_\varepsilon, x_0-2\varepsilon, x_0) &= \int_{x_0-2\varepsilon}^{x_0} [\frac{d}{dU} f(U_s) w_\varepsilon^2 - d(\frac{d}{dx} w_\varepsilon)^2]dx \ge \\ &\ge -M C_2^2 \varepsilon^3 - dC_1^2 \varepsilon^3 - d \int_{x_0-2\varepsilon}^{x_0-\varepsilon} (\frac{d}{dx} w_\varepsilon)^2 dx \end{aligned}$$

and

$$(B_{10b}) \qquad \begin{aligned} I(w, x_0-2\varepsilon, x_0) &= \int_{x_0-2\varepsilon}^{x_0} [\frac{d}{dU} f(U_s) w^2 - d(\frac{d}{dx} w)^2]dx < \\ &< M C_2^2 \varepsilon^2 - \frac{d}{4}\left|\frac{d}{dx} v(x_0)\right|^2 \varepsilon - d \int_{x_0-2\varepsilon}^{x_0-\varepsilon} (\frac{d}{dx} w_\varepsilon)^2 dx. \end{aligned}$$

From $(B_{9c})$, $(B_{10a})$ and $(B_{10b})$ we deduce that

$$\begin{aligned} I(w_\varepsilon, x_0-2\varepsilon, x_0) &= \int_{x_0-2\varepsilon}^{x_0} [\frac{d}{dU} f(U_s) w_\varepsilon^2 - d(\frac{d}{dx} w_\varepsilon)^2]dx > \\ &> \int_{x_0-2\varepsilon}^{x_0} [\frac{d}{dU} f(U_s) w^2 - d(\frac{d}{dx} w)^2]dx = I(w, x_0-2\varepsilon, x_0) \end{aligned}$$

holds for $\varepsilon$ sufficiently small. So we have proved

$$\int_0^{2\pi} [\frac{d}{dU} f(U_s)w_\varepsilon^2 - d(\frac{d}{dx} w_\varepsilon)^2]dx > 0$$

for sufficiently small $\varepsilon$. Since, for $0 < \varepsilon < \varepsilon_0$, $w_\varepsilon$ satisfies the boundary conditions (3c), the characterization (12) of the largest eigenvalue gives us the desired result that the eigenvalue problem (9) has an eigenvalue $\lambda_m > 0$.

The proof of theorem 2 for the boundary conditions (2a) is entirely analogous. In this case we again use the fact that

$$v(x) = \frac{d}{dx} U_s(x)$$

satisfies

$$\frac{d^2}{dx^2} v + \frac{d}{dU} f(U_s)\ v = 0, \quad 0 < x < \pi.$$

Using partial integration we see that the equality

$$\int_0^{\pi} [\frac{d}{dU} f(U_s)v^2 - d(\frac{d}{dx} v)^2]dx = 0$$

holds. Because of (3b) we have

$$v(0) = v(\pi) = 0.$$

In the same way as in the preceding proof, v is modified in an epsilon neighbourhood of $x = 0$ and $x = \pi$ to satisfy the boundary conditions (see fig. $B_4$)

$$\frac{d}{dx} v_\varepsilon(0) = \frac{d}{dx} v_\varepsilon(\pi),$$

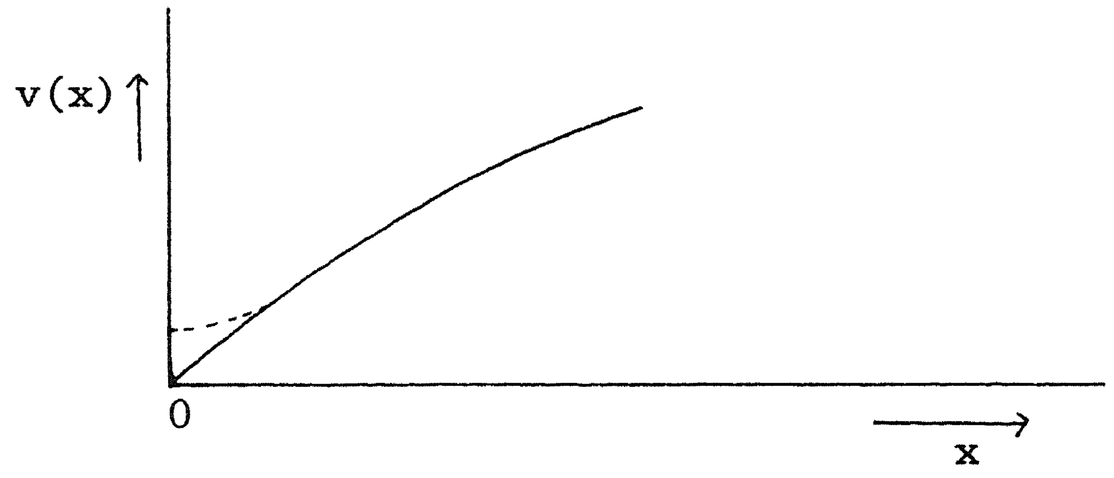


fig. $B_4$

It then also turns out that for sufficiently small $\varepsilon > 0$ the inequality

$$\int_0^{\pi} [\frac{d}{dU} f(U_s)v_\varepsilon^2 - d(\frac{d}{dx} v_\varepsilon)^2]dx > \int_0^{\pi} [\frac{d}{dU} f(U_s)v^2 - d(\frac{d}{dx} v_\varepsilon)^2]dx = 0$$

holds. Hence, it follows from (12) that the eigenvalue problem (9) has an eigenvalue $\lambda_m > 0$.

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