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The diffraction of a cylindrical pulse by a half-plane

by

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1. Introduction

In this paper we shall consider the well-known problem of diffraction of a cylindrical pulse by an absorbing half plane. Although this problem and related problems have been discussed by many authors it seems to be justified to present a solution which is obtained in a relatively simple and natural way by using the methods developed by the author in previous publications on this subject. In particular we mention the papers of the series "Solutions of the equation of Helmholtz" in which a generalized version of Sommerfeld's diffraction problem is treated. In this paper an independent treatment will be given.

The problem to be considered here may be formulated as follows

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial t^2} - 2\pi\,\delta(t)\,\delta(x-x_0)\,\delta(y-y_0) \tag{1.1}$$

with

$$f=0 \quad \text{at} \quad y=0\,, \quad x<0. \tag{1.2}$$

Without loosing generality it will be assumed that $y_0 \geq 0$.

Thus we have the wave equation in two dimensions with an instantaneous point source at (x_0, y_0). The negative part of the X-axis acts as a semi-infinite absorbing barrier.

As far as we know L. Cagniard (1935) was the first who gave a complete solution of the closely related problem with a reflecting barrier*). Afterwards an incomplete solution of the problem with the absorbing barrier was given by R.D. Turner (1956). For further details concerning problems of this type we refer to the last-mentioned paper and to the monograph of F.G. Friedlaender (1958).

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*) This was brought to the attention of the author by H. Levine.

2. The mathematical tools

In the following we need a few properties of the error function and related functions. The auxiliary function $\psi(r,z)$ is defined for $r \geq 0$ and all complex values of z by

$$(2.1) \qquad \psi(r,z) = e^{r \operatorname{ch} z} \operatorname{erfc}(\sqrt{2r} \operatorname{ch}\tfrac{1}{2} z)$$

This function is even and periodic in z with the period $4\pi i$. It satisfies the symmetry relation

$$(2.2) \qquad \psi(r,z+i\pi) + \psi(r,z-i\pi) = 2e^{-r \operatorname{ch} z}.$$

It can easily be verified that for $|\operatorname{Im} z| < \pi$

$$(2.3) \qquad \psi(r,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-r \operatorname{ch} u} \frac{1}{\operatorname{ch}\frac{1}{2}(u+z)} du.$$

The auxiliary function $\chi(a,b,\gamma)$ is defined for $a \geq 0$, $b \geq 0$ and all γ by

$$(2.4) \qquad \chi(a,b,\gamma) = \tfrac{1}{2} \int_{-\infty}^{\infty} e^{-a \operatorname{ch} u} \psi(b,u+i\gamma) du .$$

Substitution of (2.3) gives the expression

$$(2.5) \qquad \chi(a,b,\gamma) = \frac{1}{4\pi} \iint_{-\infty}^{\infty} e^{-a \operatorname{ch} u - b \operatorname{ch} v} \frac{}{\operatorname{ch}\frac{1}{2}(u+v+i\gamma)} du\, dv,$$

valid for $|\operatorname{Re}\gamma| < \pi$,
which shows that χ is symmetric in a and b. The symmetry relation (2.2) leads to a similar relation for χ viz.

$$(2.6) \quad \chi(a,b,\gamma+2\pi) + \chi(a,b,\gamma) = 2 K_0(\sqrt{a^2+b^2-2ab\cos\gamma})$$

It is not difficult to prove that for $|\operatorname{Re}\gamma| < \pi$

$$(2.7) \qquad \chi(a,b,\gamma) = \int_{a+b}^{\infty} \frac{e^{-t}}{\sqrt{t^2-R^2}} dt ,$$

where $R^2=a^2+b^2-2ab\cos\gamma$. An elementary proof is given in Lauwerier (VI).

In the following we also need the inverse Laplace transforms of $K_o(pR)$ and $\chi(pa,pb,\gamma)$. It is a well-known result that

(2.8) $$K_o(pR)=\int_R^\infty \frac{e^{-pt}}{\sqrt{t^2-R^2}}\,dt,$$

and it follows at once from (2.7) that

(2.9) $$\chi(pa,pb,\gamma)=\int_{a+b}^\infty \frac{e^{-pt}}{\sqrt{t^2-R^2}}\,dt.$$

Therefore we have the two pairs

(2.10) $$K_o(pR)\doteqdot\frac{\theta(t-R)}{\sqrt{t^2-R^2}}\quad;\quad\chi(pa,pb,\gamma)\doteqdot\frac{\theta(t-a-b)}{\sqrt{t^2-R^2}}\;,$$

with $|\mathrm{Re}\,\gamma|<\pi$

where $\theta(t)$ denotes the unit-step function.

3. Reduction to a Hilbert problem

Laplace transformation of (1.1) by means of

(3.1) $$\bar{f}(x,y,p)=\int_{-\infty}^{\infty} e^{-pt} f(x,y,t)dt \quad ,$$

gives

(3.2) $$\frac{\partial^2 \bar{f}}{\partial x^2}+\frac{\partial^2 \bar{f}}{\partial y^2}-p^2\bar{f} = -2\pi\,\delta(x-x_o)\,\delta(y-y_o) \quad ,$$

with

(3.3) $$\bar{f}=0 \qquad \text{for } y=0 \ , \quad x<0.$$

The equation (3.2) has the fundamental solution

(3.4) $$\bar{f}_o=K_o(pR) = \tfrac{1}{2}\int_{-\infty}^{\infty} \exp\{-ip(x-x_o)\mathrm{sh}u-p|y-y_o|\,\mathrm{ch}\,u\}du,$$

where R is given by

(3.5) $$R = \sqrt{(x-x_o)^2+(y-y_o)^2}$$

According to (2.10) we have

(3.6) $$f_o = \frac{\theta(t-R)}{\sqrt{t^2-R^2}}$$

which describes the radial propogation of the disturbance at (x_o,y_o).

In view of (3.4) the desired solution of (3.2) which satisfies the boundary condition (3.3) will be sought in the following form

(3.7) $$\bar{f}=\bar{f}_o + \tfrac{1}{2}\int_{-\infty}^{\infty} \exp\{-ip\,x\,\mathrm{sh}u-p|y|\ \mathrm{ch}\,u\}\ g(u)du \quad ,$$

where g(u) is some analytic function of u.
We note that by this representation the continuity of $\bar{f}$ at y=0 is automatically fulfilled.
It will be found convenient to use polar coordinates by substituting

(3.8) $$x=r\cos\varphi \ , \ y=r\sin\varphi \ , \ x_o=r_o\cos\varphi_o, \ y_o=r_o\sin\varphi_o.$$

Then (3.7) may be written as

(3.9) $$\overline{f}=\overline{f}_o+\frac{1}{2}\int_{-\infty}^{\infty}\exp\left\{-ipr\ \mathrm{sh}(w-i|\varphi|)\right\}g(w)dw.$$

The continuity of $\frac{\partial\overline{f}}{\partial y}$ at the positive x-axis requires that

(3.10) $$\int_{-\infty}^{\infty}e^{-ipx\ \mathrm{sh}\ w}g(w)\mathrm{ch}\ w\ dw=0 \quad \text{for } x>0.$$

Therefore we may put

(3.11) $$g(w)=\phi^{-}(w),$$

Where $\phi^{-}(w)$ denotes a sectionally holomorphic function which is holomorphic in the lower strip $-\pi<\mathrm{Im}\ w<0$ and which is symmetric with respect to $-\frac{1}{2}i\pi$. The boundary condition at the negative x-axis requires that

(3.12) $$\int_{-\infty}^{\infty}e^{-ipx\ \mathrm{sh}w}\left\{\exp\left\{ipr_o\mathrm{sh}(w+i\varphi_o)\right\}+g(w)\right\}dw=0$$

for $x<0$.

Therefore we may put

(3.13) $$g(w)+\exp\left\{ipr_o\mathrm{sh}(w+i\varphi_o)\right\}=\mathrm{ch}\ w\ \phi^{+}(w),$$

where $\phi^{+}(w)$ denotes a sectionally holomorphic function which is holomorphic in the upper strip $0<\mathrm{Im}w<\pi$ and which is symmetric with respect to $\frac{1}{2}i\pi$.

From (3.11) and (3.13) we obtain the following Hilbert problem on the real w-axis

(3.14) $$\mathrm{ch}\ u\ \phi^{+}(u)-\phi^{-}(u)=\exp\left\{ipr_o\mathrm{sh}(u+i\varphi_o)\right\}.$$

It may be pointed out that this Hilbert problem is aequivalent to a similar problem on the real axis of the full complex z(x,y) plane. In fact by making the transformation $z=\mathrm{sh}w$ by means of which the (x,y)-plane is a double map of the strip $-\pi<\mathrm{Im}\ w<\pi$ we obtain for real z

(3.15) $$\sqrt{1+z^2}\ \psi^{+}(z)-\psi^{-}(z)=\exp\left\{p(i\,x_oz-y_o\sqrt{1+z^2})\right\}.$$

4. Solution of the Hilbert problem

The Hilbert problem (3.14) can be solved very easily by using the relation (2.2). An obvious substitution gives

$$\exp\left\{-p\, r_o \operatorname{ch}(u-\tfrac{1}{2}i\pi+i\varphi_o)\right\}=\tfrac{1}{2}\psi(p\, r_o, u+\tfrac{1}{2}i\pi+i\varphi_o)+ \tag{4.1}$$
$$+\tfrac{1}{2}\psi(p\, r_o, u-\tfrac{3}{2}i\pi+i\varphi_o).$$

According to (3.14) the right-hand side has to be separated in two parts the first of which is antisymmetric with respect to $\frac{1}{2}i\pi$ and the second of which is symmetric with respect to $-\frac{1}{2}i\pi$.
This reflection principle which is merely Sommerfeld's well-known reflection principle in disguised form gives at once

$$-\emptyset^-(w)=\tfrac{1}{2}\psi(pr_o, w+\tfrac{1}{2}i\pi+i\varphi_o)+\tfrac{1}{2}\psi(pr_o, w+\tfrac{1}{2}i\pi-i\varphi_o) \tag{4.2}$$
$$\operatorname{ch} w\ \emptyset^+(w)=\tfrac{1}{2}\psi(pr_o, w-\tfrac{1}{2}i\pi+i\varphi_o)-\tfrac{1}{2}\psi(pr_o, w+\tfrac{1}{2}i\pi-i\varphi_o)$$

Substituting this result in e.g.

$$\bar{f}=K_o(pR)+\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-p\, r \operatorname{ch} u}\ \emptyset^-(u-\tfrac{1}{2}i\pi+i\varphi)\,du\ , \tag{4.3}$$

which is valid for $-\frac{1}{2}\pi<\varphi<\frac{1}{2}\pi$, we find at once

$$\bar{f}=K_o(pR)-\tfrac{1}{2}\chi(pr, pr_o, \varphi-\varphi_o)-\tfrac{1}{2}\chi(pr, pr_o, \varphi+\varphi_o). \tag{4.4}$$

The latter expression holds of course for all φ and φ_o. In view of (2.6) an equivalent expression would be

$$\bar{f}=\tfrac{1}{2}\chi(pr, pr_o, \varphi-\varphi_o+2\pi)-\tfrac{1}{2}\chi(pr, pr_o, \varphi+\varphi_o), \tag{4.5}$$

or

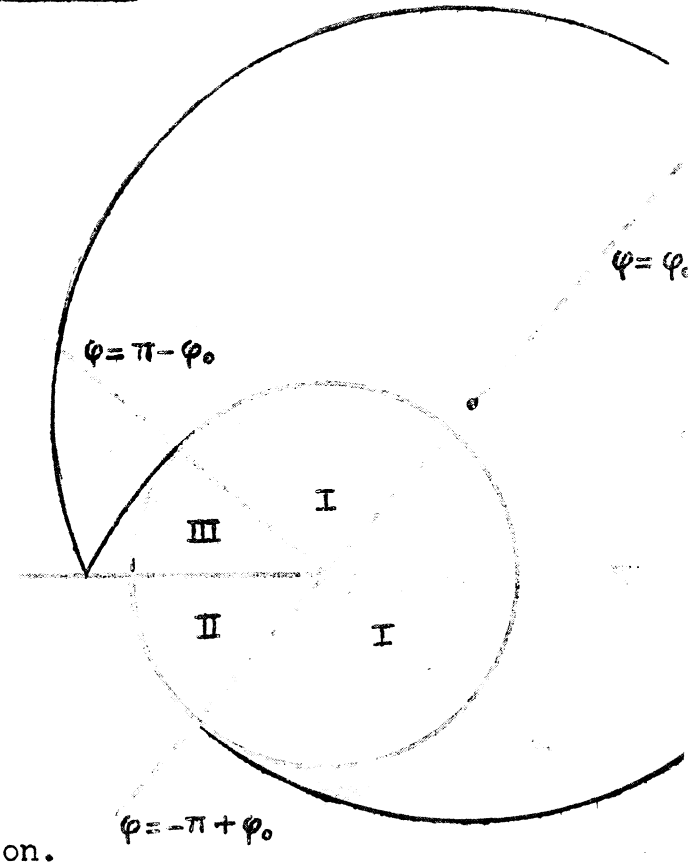
$$\bar{f}=K_o(pR)-K_o(pR')-\tfrac{1}{2}\chi(pr, pr_o, \varphi-\varphi_o)+\tfrac{1}{2}\chi(pr, pr_o, \varphi+\varphi_o-2\pi), \tag{4.6}$$

where

$$R'=\sqrt{(x-x_o)^2+(y+y_o)^2}\ . \tag{4.7}$$

5. Solution of the diffraction problem

The solution $f(r,\varphi,t)$ can be easily obtained from either (4.4), (4.5) or (4.6) by applying (2.10).
It has now become necessary to distinguish the following three regions



I $\quad -\pi + \varphi_0 < \varphi < \pi - \varphi_0$,
direct disturbance and diffraction.

II $\quad \varphi < -\pi + \varphi_0$,
diffraction (shadow region).

III $\quad \varphi > \pi - \varphi_0$
direct and reflected disturbance plus diffraction.

For each region a different inverse transform is found in view of the fact that (2.9) only holds for $-\pi < \gamma < \pi$. Therefore the inverse transforms in the three regions is obtained from respectively (4.4), (4.5) and (4.6). We find at once

$$
(5.1)\begin{cases}
\text{I} & f = \dfrac{\theta(t-R)}{\sqrt{t^2-R^2}} - \frac{1}{2}\left(\dfrac{1}{\sqrt{t^2-R^2}} + \dfrac{1}{\sqrt{t^2-R'^2}}\right)\theta(t-r-r_0) \\
\text{II} & f = \frac{1}{2}\left(\dfrac{1}{\sqrt{t^2-R^2}} - \dfrac{1}{\sqrt{t^2-R'^2}}\right)\theta(t-r-r_0) \\
\text{III} & f = \dfrac{\theta(t-R)}{\sqrt{t^2-R^2}} - \dfrac{\theta(t-R')}{\sqrt{t^2-R'^2}} - \frac{1}{2}\left(\dfrac{1}{\sqrt{t^2-R^2}} - \dfrac{1}{\sqrt{t^2-R'^2}}\right)\theta(t-r-r_0)
\end{cases}
$$

We note that in particular for $t > r+r_0$

$$
(5.2) \qquad f = \frac{1}{2}\left(\frac{1}{\sqrt{t^2-R^2}} + \frac{1}{\sqrt{t^2-R'^2}}\right) .
$$

6. A harmonic disturbance

We consider in this section the influence of a harmonic pulse starting at t=0

$$2\pi \; e^{-i\sigma t} \quad \theta(t)\delta(x-x_0)\delta(y-y_0).$$

The solution of this problem can be derived from that for an instantaneous pulse by using the superposition principle. Thus we have

$$(6.1) \qquad \begin{aligned} f_{harm} &= f_{inst} * e^{-i\sigma t} \quad \theta(t) = \\ &= e^{-i\sigma t} \int_0^t e^{i\sigma\tau} f_{inst}(x,y,\tau)d\tau . \end{aligned}$$

If the barrier is absent the solution is obtained from (3.6) as

$$(6.2) \qquad f_{harm} = e^{-i\sigma t} \int_R^t e^{i\sigma\tau} (\tau^2-R^2)^{-1/2} d\tau .$$

For $t \to \infty$ we obtain the stationary solution

$$(6.3) \qquad e^{-i\sigma t} \int_R^\infty e^{i\sigma\tau} (\tau^2-R^2)^{-1/2} d\tau = \tfrac{1}{2}\pi i \; e^{-i\sigma t} H_0^{(1)}(\sigma R)$$

which is the usual form of an outgoing harmonic wave in cylindrical coordinates.

The stationary solution may also be obtained by taking the residue of

$$(6.4) \qquad \frac{e^{pt}}{p+i\sigma} \; \bar{f}_{inst}(x,y,p) .$$

In fact we obtain from (3.4)

$$(6.5) \qquad f_{stat} = e^{-i\sigma t} K_0(-i\sigma R),$$

which is identical to the expression (6.3).

In the presence of the barrier the stationary solution can be derived from (4.4) as

$$f_{stat}=e^{-i\sigma t}\left\{K_o(-i\sigma R)-\tfrac{1}{2}\chi(-i\sigma r,-i\sigma r_o,\varphi-\varphi_o)-\right.$$

(6.6)

$$\left.-\tfrac{1}{2}\chi(-i\sigma r,-i\sigma r_o,\varphi+\varphi_o)\right\} \quad .$$

In view of (2.7) this may be written as

$$(6.7)\quad f_{stat}=e^{-i\sigma t}\left\{\tfrac{1}{2}\pi i\, H_o^{(1)}(\sigma R)-\tfrac{1}{2}\int_{\frac{r+r_o}{R}}^{\infty} e^{i\sigma Rt}(t^2-1)^{1/2}dt\right.$$

$$\left.-\tfrac{1}{2}\int_{\frac{r+r_o}{R'}}^{\infty} e^{i\sigma R't}(t^2-1)^{-1/2}\right\},$$

which holds in the region I. Similar expressions can easily be obtained for the other regions from (4.5) and (4.6).

7. An infinite harmonic source

What happens if the source recedes to infinity as

$$(7.1)\qquad x_o = r_o \cos\alpha \ , \ y_o = r_o \sin\alpha \ , \ r_o \longrightarrow \infty$$

may be seen by considering the stationary solution of a harmonic disturbance (6.5).
Since

$$(7.2)\qquad \begin{aligned} R &= \left\{ r_o^2 - 2r_o(x\cos\alpha + y\sin\alpha) + r^2 \right\}^{1/2} = \\ &= r_o - (x\cos\alpha + y\sin\alpha) + O(r_o^{-1}), \end{aligned}$$

we have by using the asymptotic behaviour of the Bessel function

$$(7.3)\qquad \begin{aligned} f_{stat} &\approx \left(\frac{1/2\,\pi i}{\epsilon R}\right)^{1/2} e^{-i\epsilon t + i\epsilon R} \\ &\approx \left(\frac{1/2\,\pi i}{\epsilon r_o}\right)^{1/2} e^{i\epsilon r_o} \exp i\epsilon(t + x\cos\alpha + y\sin\alpha) \end{aligned}$$

Thus by removing the factor

$$(7.4)\qquad \left(\frac{1/2\,\pi i}{\epsilon r_o}\right)^{1/2} e^{i\epsilon r_o}$$

we arrive at the familiar expression describing the propagation of a plane wave viz.

$$(7.5)\qquad f_{stat} = \exp -i\epsilon(t + x\cos\alpha + y\sin\alpha)$$

We shall now apply the same process on the solution (6.6) or (6.7) where the effect of a semi-infinite barrier is taken into account. It follows at once from (2.4) that for $a \to \infty$

$$(7.6)\qquad \chi(a,b,\gamma) \approx \psi(b,i\gamma)\,\tfrac{1}{2}\int_{-\infty}^{\infty} e^{-a\,\mathrm{ch}\,u}\,du = \psi(b,i\gamma)K_o(a).$$

Therefore we may derive from (6.6) that

$$f_{stat} \approx e^{-i\sigma t}\left\{K_o(-i\sigma R)-\tfrac{1}{2}\psi(-i\sigma r,i\varphi-i\alpha)K_o(-i\sigma r_o)-\right.$$

(7.7)

$$\left.-\tfrac{1}{2}\psi(-i\sigma r,i\varphi+i\alpha)K_o(-i\sigma r_o)\right\} .$$

Carrying out the asymptotic evaluations we find after removal of the factor (7.4) the well-known Sommerfeld expression

$$f_{stat} = \left\{ \exp -i\sigma(x\cos\alpha+y\sin\alpha)-\tfrac{1}{2}\psi(-i\sigma r,i\varphi-i\alpha)-\right.$$

(7.8)

$$-\tfrac{1}{2}\psi(-i\sigma r,i\varphi+i\alpha)e^{-i\sigma t}$$

or written in a slightly different way

$$f_{stat}=\left\{\exp\left\{-i\sigma r\cos(\varphi-\alpha)\right\} - \tfrac{1}{2}\exp\left\{-i\sigma r\cos(\varphi-\alpha)\right. \right. .$$

(7.9)

$$.\mathrm{erfc}(\sqrt{-2i\sigma r}\cos\tfrac{1}{2}(\varphi-\alpha))-\tfrac{1}{2}\exp\left\{-i\sigma r\cos(\varphi+\alpha)\right\} .$$

$$\left. .\mathrm{erfc}(\sqrt{-2i\sigma r}\cos\tfrac{1}{2}(\varphi+\alpha))\right\} e^{-i\sigma t} .$$

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