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Homogeneous Distributions

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Introduction

In this article the nature of an arbitrary distribution f, homogeneous of degree λ for a complex λ, is described in terms of an expansion $f=\sum b_{mn} r^{\lambda} S_{mn}$ in spherical harmonics, and the Fourier transform is shown to have the form $f\hat{}=\sum b_{mn} \gamma_m(\lambda) r^{\lambda} S_{mn}$. The form of these expansions is slightly different for certain integer values of λ. The expansion of singular integral operators in spherical harmonics as in [2] together with the discussion of homogeneous distributions in [3], form the background of this investigation.

We consider distributions on real ν-dimensional space R_ν. Points in R_ν are denoted by $x=(x_1,\dots,x_\nu)$, and $|x|^2=\sum_1^\nu x_j^2$. The spherical coordinates (r,ω) of x are determined by $r=|x|$, $x=r\omega$. The unit sphere in R_ν is denoted by Ω.

Several spaces of test functions on R_ν appear, namely $D_K \subset D \subset S$, all consisting of infinitely differentiable functions. Those in D_K vanish for $|x| \geqq K$; those in S have $p(x)q(\partial/\partial x_1,\dots,\partial/\partial x_\nu)\varphi$ bounded for each polynomial p and q; and $D=\bigcup_{K=1}^{\infty} D_K$. In S (in D_K) there is a base of neighbourhoods of zero given by $U_n=\{\varphi;\varphi$ in S (in D_K), $(1+|x|)^n\,|D^k\varphi(x)|<1/n$ for $0 \leqq k \leqq n\}$, where D^k runs over all differentiations $\partial^k/\partial x_1^{k_1}\dots\partial x_\nu^{k_\nu}$ of order k. D is usually not given a topology. At time the alternate notations $D(|x|<K)$, $D(R_\nu)$, and $S(R_\nu)$ are used.

The spaces D_K' and S' of distributions are respectively the continuous linear functionals on D_K and S; and $D'=\bigcap_{K=1}^{\infty} D_K'$. Thus $S' \subset D' \subset D_K'$. Sometimes the notations $D'(R_\nu)$ and $S'(R_\nu)$ are used. The value of the distribution f on the test function φ is $\langle f,\varphi\rangle$.

$D(\Omega)$ is the space of C^{∞} functions on the unit sphere Ω, with a base of neighbourhoods of zero given by $U_n=\{\psi;|D^k\psi(x/|x|)<1/n$, for $0 \leqq k \leqq n$ and $|x| \geqq 1\}$. $D'(\Omega)$

is then the space of continuous functionals on $D(\Omega)$.

For φ in S, φ_t is defined by $\varphi_t(x) = \varphi(t\,x)$. Since for a continuous function f homogeneous of degree λ, with $Re(\lambda) > -\nu$, we have $\int f(x)\,\varphi(x)dx = t^{\lambda+\nu}\int f(x)\varphi(t\,x)dx$, the following definition (given in [3]) is natural.

Definition 1. The distribution f in $D'(R_\nu)$ is homogeneous of degree λ if and only if, for each $t > 0$, $\langle f,\varphi\rangle = t^{\lambda+\nu}\langle f,\varphi_t\rangle$.

The steps to the main theorem are as follows: § 1 obtains for $Re(\lambda) > -\nu$ a representation $f = r^\lambda F$, where F is in $D'(\Omega)$; § 2 discusses the convergence of the expansion in spherical harmonics of a distribution F in $D'(\Omega)$; § 3 computes the Fourier transform of the individual terms in the expansion of f; § 4 combines these into the theorem, and makes a few applications.

§ 1 Here we establish Lemma 2, and the following corollary: if λ is any complex number, and f is in D' and homogeneous of degree λ, then f has an extension in S'; i.e. f is continuous on the larger space S.

Definition 2. Let $Re(\lambda) > -\nu$, and φ be in S. Then $P_\lambda\varphi$ is the function on Ω defined by $(P_\lambda\varphi)(\omega) = \int_0^\infty t^{\lambda+\nu-1}\varphi(t\omega)dt$.

P_λ is continuous from S to $D(\Omega)$, since $\int_0^\infty t^{\lambda+\nu-1}D^n\varphi(tx/|x|)dt$ can be estimated in terms of the supremum of $(1+|x|)^m|D^k\varphi(x)|$ for $k \leqq n$ and sufficiently large m.

Definition 3. If F is in $D'(\Omega)$, and $Re(\lambda) > -\nu$, then $r^\lambda F$ is the distribution in $S'(R_\nu)$ defined by $\langle r^\lambda F,\varphi\rangle = \langle F, P_\lambda\varphi\rangle$.

Since P_λ is continuous from S to $D(\Omega)$, the composition of F and P_λ is a continuous linear functional on S. Informally written, $\langle r^\lambda F,\varphi\rangle = \int_0^\infty \langle r^\lambda F(\omega), \varphi(r\omega)\rangle\, r^{-1}dr$.

Definition 4. Let a(t) be a non-negative C^∞ function on R_1 with support in $1/2 \leqq t \leqq 2$. Then for ψ in $D(\Omega)$, $A_\lambda\psi$ is defined by $(A_\lambda\psi)(x) = a(|x|)|x|^{-\lambda-\nu+1}\psi(x/|x|)/\int_0^\infty a(t)dt$.

Thus A_λ depends on the arbitrarily chosen function a(t); but since we consider a fixed a(t) this dependence is not indicated in the notation. It is clear that A_λ is continuous from $D(\Omega)$ to $D(|x|<2)$.

Lemma 1. Let $Re(\lambda) > -\nu$, and f in D' be homogeneous of degree λ. Then $\langle f,\varphi\rangle = \langle f, A_\lambda P\varphi\rangle$ for each φ in D.

Proof. The basic calculation is

$$1) \quad \left(\int_0^\infty a\right) \langle f, A_\lambda P_\lambda\varphi\rangle = \langle f(x), |x|a(|x|) \int_0^\infty s^{\lambda+\nu-1}\varphi(sx)ds\rangle$$

$$= \int_0^\infty s^{\lambda+\nu-1}\langle f(x), [\varphi(x)a(|x|/s)|x|/s]_s\rangle ds$$

$$= \int_0^\infty s^{-2}\langle f(x), \varphi(x)a(|x|/s)|x|\rangle ds$$

$$= \int_0^\infty \langle f(x), \varphi(x)a(t|x|)|x|\rangle dt$$

$$= \langle f(x), \varphi(x)\int_0^\infty a(t|x|)d(t|x|)\rangle$$

$$= \left(\int_0^\infty a\right) \langle f,\varphi\rangle.$$

The interchange of $\int$ and $\langle\ ,\ \rangle$ seems to be difficult to justify unless φ vanishes in a neighbourhood of the origin, so we first consider a φ with $\varphi=0$ for $|x|\leqq\epsilon\leqq 1/2$ and $|x|\geqq M\geqq 2$. Then the interchanges can be justified by showing that if ψ and ψ_1 are C^∞ functions vanishing for $|x|\leqq\epsilon$ and $|x|\geqq M$, and μ is any complex number then, in the topology of $D(|x|<M)$, $\psi_1(x)\int_0^A s^\mu\ (sx)ds \longrightarrow \psi_1(x)\int_0^\infty s^\mu\psi(sx)ds$ and $\psi_1(x)(A/N)\sum_{n=1}^N (An/N)^\mu\psi(Anx/N) \longrightarrow \int_0^A s^\mu\psi(sx)ds$. Since the derivatives of each of these expressions are linear combinations of the same type, it suffices to show that $\int_0^A s^\mu\psi(sx)ds \longrightarrow \int_0^\infty$ and $(A/N)\sum_1^N (An/N)^\mu\psi(Anx/N)\longrightarrow\int_0^A$ uniformly in $\epsilon\leqq|x|\leqq M$, for each μ.

For the first we have in $|x|\geqq\epsilon$ that $\left|\int_A^\infty s^\mu\psi(sx)ds\right|$

$$\leqq \sup_{t>gx} (1+|x|t)^k|\psi(tx)|(1+t)^k(1+\epsilon t)^{-k}\int_A^\infty s^{Re(\mu)}(1+s)^{-k}ds\ ;$$

choosing $k > \mathrm{Re}(\mu)-1$ yields the result. For the convergence of the Riemann sums, we have

$$\left|\int_0^A -(A/N)\sum_1^N\right| \leq (A^2/N) \max_{\epsilon\leq|x|\leq M} |\, ds^{\mu}\psi(sx)/ds\,| \;;$$

since $\psi(y)$ vanishes for $|y|\leq\epsilon$, we need only consider $\epsilon/M \leq s \leq A$, and $\max_{\substack{|x|\leq M\\ \epsilon/M\leq s\leq A}} |\, ds^{\mu}\psi(sx)/ds\,|$ is finite.

This justifies formula (1), and hence completes the lemma, for functions φ vanishing in a neighbourhood of the origin. Thus if g is defined by $\langle g,\varphi\rangle = \langle f, A_\lambda P_\lambda \varphi\rangle$, then f-g has support $\{x = 0\}$, and hence (see [4], p.99) $f-g=\sum_1^N L_n\delta$, where L_n is a homogeneous differential operator of order n with constant coefficients.
Now the n^{th} term of this sum is homogeneous of degree $-\nu-n$, and g is easily shown to be homogeneous of degree λ. Since distributions which are homogeneous of different degrees are linearly independent (see [3], p.86), we are led to f=g, and Lemma 1 is proved.

Now it is easy to define an F in $D'(\Omega)$ such that $f=r^{\lambda}F$, i.e. such that $\langle f,\varphi\rangle = \langle F,P_\lambda\varphi\rangle$: set $\langle F,\psi\rangle = \langle f,A_\lambda\psi\rangle$. Then $\langle f,\varphi\rangle = \langle f,A_\lambda P_\lambda\varphi\rangle = \langle F,P_\lambda\varphi\rangle$ as desired. In spite of the arbitrariness in A_λ, F is unique; for if ψ is in $D(\Omega)$ we have $\psi=P_\lambda A_\lambda\psi$, so that $r^{\lambda}G=f$ implies $\langle G,\psi\rangle = \langle G,P_\lambda A_\lambda\psi\rangle = \langle f,A_\lambda\psi\rangle = \langle F,\psi\rangle$ for each ψ.
Thus we have established

Lemma 2. If $\mathrm{Re}(\lambda) > -\nu$, and f in $D'(R_\nu)$ is homogeneous of degree λ, then there is a unique F in $D'(\Omega)$ such that $f=r^{\lambda}F$.

Corollary. If f is in $D'(R_\nu)$, homogeneous of any complex degree λ, then f has an extension in S'.

Proof. If $\mathrm{Re}(\lambda) > -\nu$, this follows from Lemma 1; for $A_\lambda P_\lambda$ is bounded from S to $D(|x|<2)$, so $\langle f,\varphi\rangle = \langle f,A_\lambda P_\lambda\varphi\rangle$ defines the extension. If $\mathrm{Re}(\lambda) \leq -\nu$, choose an integer k so that $2k+\mathrm{Re}(\lambda) > -\nu$. It is easy to check $|x|^{2k}f$ is homogeneous of degree $2k+\lambda$, hence continuous on S; and if $\chi(|x|)$ is a C^{∞}

cut-off function such that $\chi(|x|)=1$ for $|x| \leqq 1$, $\chi(|x|)=0$ for $|x| \geqq 2$, we have

$$2) \quad \langle f,\varphi\rangle = \langle f,\chi\varphi\rangle + \langle |x|^{2k} f, |x|^{-2k}(1-\chi)\varphi\rangle .$$

Here $\varphi \to \chi\varphi$ is continuous from S to D($|x|<2$), and $\varphi \to |x|^{-2k}(1-\chi)\varphi$ is continuous from S to S, so the right hand side of (2) is continuous on S.

§ 2 The-spherical harmonic expansion in D'(Ω).

Let S_m denote a real-valued normalized spherical harmonic of degree m; thus S_m is the restriction to Ω of a homogeneous harmonic polynomial of degree m, and $\int_\Omega |S_m|^2=1$. Let $\{S_{mn}\}$ denote an orthonormal basis for $L^2(\Omega)$ consisting of such spherical harmonics, S_{mn} being of degree m and n running from 1 to $(2m+\nu-2)(m+\nu-3)!/m!(\nu-2)!$ (see [1],p.237). If we define an operator L on D(Ω) by $L\psi$ = the restriction to Ω of $\Delta\psi(x/|x|)$, we have from [2] that

$$3) \qquad -m(m+\nu-2)\int_\Omega S_{mn}\psi = \int_\Omega S_{mn} L\psi .$$

The same reference shows that there are constants $C_{k,m}$ such that

$$4) \qquad D^k S_{mn}(x/|x|) \leqq C_{k,m}\, m^{k-1+\nu/2} \text{ in } |x| \geqq 1,$$

where D^k is an arbitrary differentiation of order k.

Each ψ in D(Ω) has an expansion $\psi = \sum a_{mn} S_{mn}$, with $a_{mn} = \int_\Omega S_{mn}\psi$. The estimate (3) shows that $a_{mn}=O(m^{-k})$ for every k. Taking into account the number of S_{mn} for each m, estimate (4) then shows that $\sum a_{mn} S_{mn}$ and all its derivatives converge uniformly in $|x| \geqq 1$, so that the series converges in D(Ω) to ψ.

To each F in D'(Ω) there corresponds a sequence of coefficients $b_{mn} = \langle F,S_{mn}\rangle$. If we set $F_M = \sum_{m \leqq M} \sum_n b_{mn} S_{mn}$, then F_M converges weakly to F:

$\langle F_M,\psi\rangle = \sum_{m\leqq M} \sum_n b_{mn} a_{mn} = \langle F,\psi\rangle \to \langle F,\psi\rangle$ for each ψ. Since

$\lim_{M\to\infty} \sum_{m\leq M} \sum_{n} b_{mn} a_{mn}$ exists for each $\{a_{mn}\}$ such that $a_{mn}=O(m^{-k})$ for all k, it follows that $b_{mn}=O(m^{k})$ for some k.

We now have the expansion described in Theorem 1 below for the case $Re(\lambda) > -\nu$. Expend the F of Lemma 2 as $F=\sum\sum b_{mn} S_{mn}$. Then $\sum\sum b_{mn} r^{\lambda} S_{mn}$ converges weakly (in $S'(R_\nu)$) to f, since $\lim_{M\to\infty} \sum_{m=M} \sum_{n} b_{mn} \langle r^{\lambda} S_{mn}, \varphi\rangle =$ $\lim \langle F_M, P_\lambda \varphi\rangle = \langle F, P_\lambda \varphi\rangle = \langle f, \varphi\rangle$.

§ 3 Fourier transforms.

For $Re(\lambda) > -\nu$, $r^{\lambda} S_m(\omega)$ is a locally integrable function on R with polynomial growth at ∞, hence defines a distribution on S, homogeneous of degree λ . Here we compute its Fourier transform $(r^{\lambda} S_m)^{\wedge}$, and consider the analytic extension to $Re(\lambda) \leq -\nu$. The method of calculation is borrowed from [2].

The Fourier transform of φ in S is the function $\varphi^{\wedge}$ defined by $\varphi^{\wedge}(y)=\int e^{ix\cdot y} \varphi(x)dx$; this is a continuous 1-1 transformation on S, whose inverse is given by $\varphi^{\vee}(x)=(2\pi)^{-\nu}\int e^{-ix\cdot y} \varphi(y)dy$. (See [5], p.105). The Fourier transform of f in S' is the distribution $f^{\wedge}$ in S' given by $\langle f^{\wedge}, \varphi\rangle = \langle f, \varphi^{\wedge}\rangle$. Thus $^{\wedge}$ and $^{\vee}$ are reciprocal isomorphisms on S'. One has, immediately, for an arbitrary polynomial P, that

5) $$[P(x)]^{\wedge}=(2\pi)^{\nu} P(-i\,\partial/\partial x_1, \ldots, -i\,\partial/\partial x_\nu)\delta$$

and

6) $$[P(\partial/\partial x_1, \ldots, \partial/\partial x_\nu)\delta]^{\wedge}=P(-ix_1, \ldots, -ix_\nu).$$

It is easy to see that the distribution $(r^{\lambda} S_m)^{\wedge}$ corresponds to the function $(r^{\lambda} S_m)^{\wedge}(y)= \lim_{\epsilon\to 0} \int_{\epsilon\leq|x|\leq 1/\epsilon} |x|^{\lambda} e^{-ix\cdot y} S_m(x/|x|)dx$,

for all values of λ such that this limit exists uniformly in each ball $|y|\leq K$. This turns out to include the strip $-\nu < Re(\lambda) < (1-\nu)/2$. The analytic expression obtained in this strip is then valid for all values of λ , by analytic continuation.

Consider thus $-\nu < \mathrm{Re}(\lambda) < (1-\nu)/2$, and set $x=r\omega$, $y=\rho\sigma$ $(|\omega|=|\sigma|=1)$; then $(r^{\lambda} S_m)^{\wedge}(\rho\sigma)=$

$$\lim_{\epsilon\to 0} \int_{\epsilon}^{1/\epsilon} r^{\lambda+\nu-1} \int_{\Omega} e^{i\rho r\sigma.\omega} S_m(\omega)d\Omega\, dr=$$

$$\lim \rho^{-\lambda-\nu} \int_{\epsilon}^{1/\epsilon} s^{\lambda+\nu-1} \int_{\Omega} e^{-is\sigma.\omega} S_m(\omega)d\Omega\, ds.$$

Further calculation depends on the formulas

7) $$e^{is\cos\varphi} = 2^a\,\Gamma(a)s^{-a} \sum_{k=0}^{\infty} (i)^k(k+a)\, J_{k+a}(s)C_k^a\,(\cos\varphi)$$

([1], p.213),

where $C_k^a(t)$ is a Gegenbauer polynomial;

8) $$\int_{\Omega} C_j^{\,a}(\sigma.\omega)S_m(\omega)d\Omega = \delta_{jm}\, S_m(\sigma)4\pi^{1+a}/(2m+\nu-2)\,\Gamma(a)$$

([1],p.247);

and

9) $$\int_0^{\infty} t^{\lambda+\nu/2}\, J_{m-1+\nu/2}(t)dt =$$

$$2^{\lambda+\nu/2}\,\Gamma((m+\nu+\lambda)/2)/\,\Gamma((m-\lambda)/2),$$

for $-m-\nu < \mathrm{Re}(\lambda) < (1-\nu)/2$. ([1], p.49).

Thus setting the letter a in formulas (7) and (8) equal to $(\nu/2)-1$, we obtain

10) $$\int_{\Omega} e^{-is\sigma.\omega} S_m(\omega)d\Omega =$$

$$2\pi^{\nu/2}(-i)^m(s/2)^{1-\nu/2}\, J_{m-1+\nu/2}(s)\, S_m(\sigma),$$

and

11) $$(r^{\lambda} S_m)^{\wedge}(\rho\sigma) = \rho^{-\lambda-\nu} S_m(\sigma)(-i)^m\pi^{/2}2^{\lambda+\nu}\Gamma((m+\nu+\lambda)/2)/$$

$$\Gamma(m-\lambda)/2),$$

for $-\nu < \mathrm{Re}(\lambda) < (1-\nu)/2$. It follows easily that for the same values of λ

12) $$(r^{\lambda}S_m)^{\vee}(\rho\sigma) = \rho^{-\lambda-\nu}S_m(\sigma)(i)^m\pi^{-\nu/2}2^{\lambda}\Gamma((m+\nu+\lambda)/2)/\Gamma((m-\lambda)/2$$

Then, if $r^\lambda S_m$ is defined for all λ (except possible poles) as the analytic extension from $Re(\lambda) > -\nu$, we have for all λ

13) $$[1/\Gamma((m+\nu+\lambda)/2)]\ (r^\lambda S_m)\hat{}=$$
$$[(-i)^m\ \pi^{\nu/2}\ 2^{\lambda+\nu}/\Gamma((m-\lambda)/2)]\ r^{-\lambda-\nu} S_m.$$

Since for any λ either $Re(\lambda) > -\nu$ or $Re(-\lambda-\nu) > -\nu$, at least one of $r^\lambda S_m$ and $r^{-\lambda-\nu} S_m$ is always defined as a regular distribution. Formula (13) then defines the other of these as a Fourier transform or inverse Fourier Transform, except for the values of λ which yield a pole of the gamma functions occurring in (13). In this way $r^\lambda S_m$ is defined by formula (13) as a distribution in S'(R) except for $\lambda=-\nu-m-2k$, $k=0,1,2,\ldots$ The fact that this extension is possible can also be checked directly by a using a Taylor expansion of the test functions; this is done for the case m=0 in [3].

Since at least one side in (13) is always non-zero, the poles of the gamma function do not correspond to the zero distribution, but rather to distributions concentrated at the origin. In fact, if $\lambda=m+2k$ then $r^\lambda S_m$ is $r^{2k}H_m$, where H_m is a harmonic polynomial. Thus from (5) we have

14) $$(r^{m+2k}\ S_m)\hat{}=(2\pi)^\nu(-i)^{m+2k}\ \Delta^k H_m(\partial/\partial x_1,\ldots,\partial/\partial x_\nu)\delta$$

and

15) $$(r^{m+2k}\ S_m)\check{}=(i)^{m+2k}\ \Delta^k\ H_m(\partial/\partial x_1,\ldots,\partial/\partial x_\nu)\delta\ ,$$

where Δ is the Laplacian, and the δ is Dirac's.

Thus $r^\lambda S_m$, defined for $Re(\lambda) > -\nu$ as a regular distribution, has an analytic extension to the whole complex λ-plane except for poles at $\lambda=-m-2k$. Its Fourier transform is given either by (13) or by (14), and the inverse transform by (13) or by (15).

§ 4 The expansion and transform of a homogeneous distribution

Theorem 1. Let f be a distribution in $D'(R_\nu)$, homogeneous of degree λ, and $r^\lambda S_{mn}$ be defined for $\lambda \neq -\nu, -\nu-1, \dots$ by analytic continuation from $\mathrm{Re}(\lambda) > -\nu$.

i) If λ is not an integer of the form $-\nu-k$ $(k=0,1,\dots)$, then $f = \sum_m \sum_n b_{mn} r^\lambda S_{mn}$, where $b_{mn} = O(m^k)$ for some k, and the series is weakly convergent in $S'(R_\nu)$.

ii) If $\lambda = -\nu-N$ for some $N=0,1,\dots$, then $f = P_N(\partial/\partial x_1, \dots, \partial/\partial x_\nu)\delta + \sum_m^* \sum_n b_{mn} r^{-\nu-N} S_{mn}$, where P_N is a homogeneous polynomial of degree N, δ is the Dirac δ, and $\sum_m^*$ is the sum over all $m \geqq 0$ such that $m \neq N-2k$ for a $k=0,1,2,\dots$. The series converges as in (i).

iii) The Fourier transform of f is obtained by term-by-term application of (13), (14), or (6).

Proof. Part (iii) follows from the fact that the Fourier transform is continuous in the weak topology of distributions. Part (i) follows, for $\mathrm{Re}(\lambda) > -\nu$, from Lemma 2 and the last paragraph of § 2. If $\mathrm{Re}(\lambda) \leqq -\nu$, then a trivial check shows that $f^\wedge$ is homogeneous of degree $-\lambda-\nu$; and $\mathrm{Re}(-\lambda-\nu) \geqq 0$, so $f^\wedge$ may be expanded as in (i). Applying the inverse Fourier transform term-by-term yields the expansion for f.

An immediate consequence is

Corollary 1. Any distribution homogeneous of degree λ, $\lambda \neq -\nu-N$, has the form $r^\lambda F$, where F is in $D'(\Omega)$. If $\lambda = -\nu-N$, then $f = P_N(\partial/\partial x)\delta + r^{-\nu-N} F$, where F is orthogonal to all S_m with $m=N, N-2, \dots$.

We say that a distribution F in $D'(\Omega)$ corresponds to a distribution $f_\lambda = r^\lambda F$ in $S'(R_\nu)$ if and only if, for each φ vanishing in a neighbourhood of the origin, $\langle f, \varphi \rangle = \int_0^\infty r^{\lambda+\nu-1} \langle F, \varphi_r \rangle \, dr$. Thus if $\lambda = -\nu-N$, the f_λ above is not uniquely determined by F.

Corollary 2. For each F in $D'(\Omega)$, and for $\lambda \neq -\nu-N$, there is a corresponding unique distribution $f_\lambda = r^\lambda F$.

If $\lambda=-\nu-N$, there is a corresponding f_{λ} if and only if $\langle F, S_{mn}\rangle =0$ for $m=N, N-2, \ldots$.

Proof. Let $F=\sum b_{mn} S_{mn}$. If $\lambda \neq -\nu-N$, then f_{λ} is uniquely determined as $f_{\lambda} =\sum \sum b_{mn} r^{\lambda} S_{mn}$. If $\lambda= -\nu-N$, and $f_{-\nu-N}$ corresponds to F, we can expand $f_{-\nu-N}=$

$\sum^{*}\sum c_{mn} r^{-\nu-N} S_{mn}+P_{N}(\partial/\partial x)\delta$.

Applying $f_{-\nu-N}$ to $A_{\lambda} S_{mn}$ we find that $c_{mn}=b_{mn}$ for all m,n, and that b_{mn} vanishes for those m not occuring in $\sum^{*}$. The polynomial P_{N} is thus arbitrary, and the rest determined by F.

It is easy to show that, when it exists, $r^{\lambda} \circ F$ is the analytic extension of $r^{\lambda} F$ from $Re(\lambda) > -\nu$.

Applying Corollary 2 to regular (integrable) distributions in $D'(\Omega)$, we have

Corollary 3. If f is a function homogeneous of degree λ and locally integrable in $|x| \geqq 1$, and $\lambda \neq -\nu-N$, then f corresponds to a unique distribution homogeneous of degree λ . If $\lambda=-\nu-N$, then f corresponds to a distribution homogeneous of degree λ if and only if $\int_{\Omega} f(\omega)S_{mn}(\omega)d\Omega$ for all $m=N, N-2, \ldots \geqq 0$.

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