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The Lorentz-invariant solutions of the Klein-Gordon equation

by

E.M. de Jager

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Introduction

The solutions, invariant under a proper Lorentz transformation, of the homogeneous and inhomogeneous Klein-Gordon equation (see e.g. lit 1 and 2)

$$(\Box - m^2)\ f(x)=0 \tag{1.1}$$

$$(\Box - m^2)\ g(x)= -\delta(x) \tag{1.2}$$

play an important role in relativistic quantum field theory.
x denotes the coordinates of a point (x_1,x_2,x_3,x_0) in Euclidean space R_4; (x_1,x_2,x_3) are space coordinates and x_0 is the time coordinate; m is the mass of the particle under consideration.
The symbol $\Box$ stands for the differential operator:

$$\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_o^2} \tag{1.3}$$

and $\delta(x)$ denotes the four-dimensional Dirac-function, concentrated in the origin of the coordinate system.
The field equations (1.1) and (1.2) reduce for m=0 (photons) to the wave equations in three-dimensional space, and the Klein-Gordon equations can be conceived as a generalization of the latter. In textbooks on quantum field theory the solutions of (1.1) and (1.2) are obtained usually in a rather formal way (see e.g. lit 1,2,3). They are determined by applying a Fourier transformation to (1.1) and (1.2). The Fourier transform of e.g. f(x), denoted by $\hat{f}(k)$, satisfies the equation

$$(k^2-m^2)\ \hat{f}(k)=0 \tag{1.4}$$

with $k^2=k_0^2-k_1^2-k_2^2-k_3^2$.
General Lorentz-invariant solutions of (1.4) are linear combinations of the Dirac functions $\delta_+(k^2-m^2)$ and $\delta_-(k^2-m^2)$ which are concentrated on the upper respectively lower sheet of the hyperboloïd $k^2-m^2=0$. The inverse transforms of $\delta_+(k^2-m^2)$ and $\delta_-(k^2-m^2)$ are obtained by purely formal calculations; for example divergent integrals are converted into convergent integrals by interchanging the operations of differentiation and integration (see e.g. lit 1, § 15.1, lit 2, § 15[b])
It is obvious that this formal way cannot claim any mathematical rigour. The difficulties arise essentially from the fact that the Dirac function

is not a function in the classical sense; it is a generalized function or distribution and has to be treated as such. To obtain the Lorentz-invariant solutions of (1.1) and (1.2) in a rigorous way one needs essentially the theory of distributions and the calculations have to be performed within the framework of this theory (see lit.4,5,6,7).
The object of this paper is to present a rigorous derivation of the Lorentz-invariant solutions of the Klein-Gordon equations (1.1) and (1.2) by making use of the theory of distributions.
This has been done earlier by P.D.Methée (lit 8,9,10), by menas of the mapping of n-dimensional space R_n on the line R, given by the transformation

$$u = x_o^2 - \sum_{i=1}^{n-1} x_i^2 \tag{1.5}$$

Using pairs of distributions defined on C^∞ functions with compact support in R, Lorentz-invariant solutions of the Klein-Gordon equations in n dimensions are derived. Asymptotic expansions in the neighbourhood of $\varepsilon=0$ of distributions, concentrated on the hyperboloid $u=\varepsilon$, play an important role in the theory.
A simplified version of this theory is due to J.E. Roos and L. Gårding (lit 11). These authors use the following transformations for the test functions $\varphi(x)$:

$$(M\varphi)\ (\tau)=\int\varphi(x)\ \delta(\tau-x^2)\,dx \tag{1.6}$$

$$(M_1\varphi)\ (\tau)=\int\varphi(x)\ \delta(\tau-x^2)\,\mathrm{sign}\ x_o\ dx \tag{1.7}$$

with $x^2=x_o^2-x_1^2-x_2^2-x_3^2$.
Linear homeomorphisms are established between the spaces of even and uneven Lorentz-invariant distributions and the duals of the spaces of the functions $(M\varphi)\ (\tau)$ respectively $(M_1\varphi)\ (\tau)$; hence the necessary calculations can be performed in these dual.spaces.
It should be remarked that the solutions obtained by lit 8 and 11 are distributions belonging to the space D', which is the dual of the space of all C^∞ functions with compact support.
In this paper we present another method of obtaining the Lorentz-invariant solutions of (1.1) and (1.2). As in the formal way of lit 1 and lit 2 we apply again a Fourier transformation to the equations (1.1) and (1.2) and

determine the Lorentz-invariant distributions satisfying:

$$(k^2-m^2)\ \hat{f}(k)=0 \qquad (1.8)$$

respectively

$$(k^2-m^2)\ \hat{g}(k)= -1 \qquad (1.9)$$

Using the principle of an alytic continuation, the inverse transforms of these distributions are consecutively determined, needless to say with close observance of the rules of the theory of distributions. Whereas Methée, Roos and Gårding use distributions belonging to D', we consider only tempered distributions. This results in the fact, that the solutions obtained in this paper, contrary to those of lit 8 and 11, do not have terms which increase exponentially at infinity. However, due to their behaviour at infinity these terms are usually disregarded by the physicists. (see also lit 10)

We have confined our treatment to four dimensions, but the theory can be extended to the general case of n-dimensions.

In the opinion of the author, the method is easily accessible for physicists, as well as for those who are not so well acquainted with the theory of distributions. Moreover, the method links up with the purely formal method generally used by physicists to obtain the Lorentz-invariant solutions of (1.1) and (1.2).

Due to these considerations and on account of interesting incidental results this paper may be of interest for both mathematicians and physicist.

For those readers not familiar with the theory of distributions, we summarize in chapter 2 some of the basic ideas of the theory; the distributions concentrated on hypersurfaces in R_4 are considered in more detail, since we use them later on.

In chapter 3 a modified Fourier transformation is introduced, such that the image of a Lorentz-invariant distribution is again Lorentz-invariant and vice versa. This Fourier transformation is applied to the Klein-Gordon equations (1.1) and (1.2) and it turns out that the modified Fourier transform of the general Lorentz-invariant solution of (1.1) can be written as:

$$\hat{f}(k)=c_+\ \delta_+(k^2-m^2)+c_-\ \delta_-(k^2-m^2) \qquad (1.10)$$

where c_+ and c_- are arbitrary constants.
A particular solution of the modified Fourier transform of the inhomogeneous Klein-Gordon equation is

$$\hat{g}(k) = \frac{-1}{k^2-m^2} \qquad (1.11)$$

The following chapters 4,5 and 6 are devoted to the Fourier transform (which is nearly the inverse modified Fourier transform) of the distributions $\frac{1}{k^2-m^2}$, $\delta_+(k^2-m^2)$, $\delta_-(k^2-m^2)$.
In chapter 4 we establish the relation:

$$\frac{1}{k^2-m^2 \pm io} = \frac{1}{k^2-m^2} \mp i\pi\,\delta(k^2-m^2) \qquad (1.12)$$

where $\delta(k^2-m^2)$ is concentrated on both sheets of the hyperboloid $(k^2-m^2)=0$; $\delta(k^2-m^2)= \delta_+(k^2-m^2)+ \delta_-(k^2-m^2)$.
The Fourier transform of $\frac{1}{k^2-m^2 \pm i0}$ is determined in chapter 5; those of $\frac{1}{k^2-m^2}$ and $\delta(k^2-m^2)$ follow now easily by virtue of (1.12).
From the Fourier transform of the distribution $\delta(k^2-m^2)$ the Fourier transforms of the distributions $\delta_+(k^2-m^2)$ and $\delta_-(k^2-m^2)$ can be derived; this is done in chapter 6.
The paper is concluded by chapter 7, where we determine in section 7.1 the following Lorentz-invariant solutions of (1.1), viz.: the Pauli-Jordan function $\Delta(x)$, its positive and negative frequency parts $\Delta^+(x)$ and $\Delta^-(x)$, and $\Delta^{(1)}(x)= \Delta^+(x)- \Delta^-(x)$.
It is shown in section 7.2 that the distribution $\Delta(x)$ is the propagator of the solutions of (1.1) from their initial values.
In section 7.3 we derive finally the following invariant solutions of the inhomogeneous Klein-Gordon equation: the advanced and retarded Green's functions $\Delta_A(x)$ and $\Delta_R(x)$, vanishing for $x_0 > 0$ respectively $x_o < 0$, and the causal Green's functions $\Delta_C(x)$ and $\Delta_{AC}(x)$.
The author is indebted to Profs.H.A.Lauwerier and R.T.Seeley for valuable discussions on the subject.

2. The theory of distributions

2.1 Test functions and distributions

Schwartz's theory of distributions generalizes the concept of function such as to include also "functions" which cannot be defined in the classical way; for instance the δ-function of Dirac which is everywhere zero except in one point, where it is infinite such that its integral equals 1.

A distribution f(x) is defined as a continuous linear functional on some linear space of test-functions $\varphi(x)$; $x=(x_1,x_2,\dots,x_n)$ denotes a point of the n-dimensional Euclidean space R_n.

In this space of test functions a system of neighbourhoods of the 0-function is defined, such that one can define the limit of a sequence $\{\varphi_n(x)\}$ of test functions.

The distribution f(x) assigns to any function $\varphi(x)$, belonging to the space of test functions, a real or complex number, denoted by $\langle f,\varphi\rangle$.

This assignation is linear and continuous; this means:

1e $\langle f, a_1\varphi_1 + a_2\varphi_2\rangle = a_1\langle f,\varphi_1\rangle + a_2\langle f,\varphi_2\rangle$ for any φ_1 and φ_2. a_1 and a_2 are arbitrary real or complex numbers.

2e $\lim\limits_{\varphi_n\to 0}\langle f,\varphi_n\rangle = 0$ for any sequence $\{\varphi_n\}$ converging to zero.

As to the space of testfunctions one can make different choices. For instance one can use the space $S(R_n)$, consisting of all n-dimensional infinitely differentiable functions $\varphi(x)$, (C^∞) functions, which decrease at infinity stronger than any negative power of x_i ; $\varphi(x)$ is a function of the independent variables $x_1\dots x_n$. This means that for any testfunction $\varphi(x)$ exist constants C_{kq}, dependent on φ, such that

$$|x^k \; D^q \varphi(x)| \leq C_{kq} \tag{2.1}$$

with $x^k = x_1^{k_1}\, x_2^{k_2} \dots x_n^{k_n}$ and $D^q = \dfrac{\partial^{q_1+q_2+\dots+q_n}}{\partial x_1^{q_1}\,\partial x_2^{q_2}\dots\partial x_n^{q_n}}$;

$k_1,k_2 \dots k_n$ and $q_1,q_2 \dots q_n$ are arbitrary non negative integers.

The sequence $\{\varphi_n(x)\}$ is defined to converge to zero, when the sequences $\{|x^k\, D^q\,\varphi_n(x)|\}$ converge for all values of x_i uniformly to zero for any k and q.

The space of distributions, defined on S, is denoted by S'; these distributions are called tempered distributions.
In quantum-field theory one takes often for the space of testfunctions the class C(q,r,1), consisting of all functions $\varphi(x)$, which are defined in R_4 and which are continuous together with all their derivatives up to the q^{th} order inclusive; moreover all the products

$$x_1^{r_1}\, x_2^{r_2}\, x_3^{r_3}\, x_o^{r_o}\, \frac{\partial^p \varphi(x_1,x_2,x_3,x_o)}{\partial x_1^{p_1}\partial x_2^{p_2}\partial x_3^{p_3}\partial x_o^{p_o}} \tag{2.2}$$

are uniformly bounded for all values of x_1,x_2,x_3 and x_o; p_i and r_i are non negative integers with $\sum p_i = p \leqslant q$ and $\sum r_i \leqslant r$. x_1,x_2,x_3 are the space coordinates and x_o the time coordinate of a particle.
Dealing with n particles the class C(q,r,n) is used, which is defined in the same way as C(q,r,1) with the only difference that we have 4 n independent variables instead of four independent variables.
Convergence of a sequence of testfunctions to zero is defined analogously as in the space S: the sequence $\{\varphi_n(x)\}$ converges to zero when the sequences

$$\left\{\left| x_1^{r_1}\, x_2^{r_2}\, x_3^{r_3}\, x_o^{r_o}\, \frac{\partial^p \varphi_n(x_1,x_2,x_3,x_o)}{\partial x_1^{p_1}\partial x_2^{p_2}\partial x_3^{p_3}\partial x_o^{p_o}}\right|\right\}$$

converge uniformly for all values of x_1,x_2,x_3 and x_o to zero for all combinations (r_1,r_2,r_3,r_o) and (p_1,p_2,p_3,p_o) with $\sum r_i \leqslant r$ and $\sum p_i = p \leqslant q$.
The dual space of all the distributions defined on C(q,r,1) is denoted by C'(q,r,1).
It is clear that the cross-section of all spaces C(q,r,1) with arbitrary values of q and r is the space $S(R_4)$; hence the union of all spaces C'(q,r,1) coïncides with the space $S'(R_4)$.
Hence, when we have determined some distribution belonging to $S'(R_4)$, it belongs also to some C'(q,r,1); therefore, admitting sufficiently large values of q and r, we can without loss of generality restrict our considerations to distributions belonging to $S'(R_4)$.
An ordinary Lebesgue integrable function of n variables can be conceived as a distribution; the distribution is defined by the integral:

$$\langle f,\varphi\rangle = \int_{-\infty}^{+\infty} f(x)\ \varphi(x)\,dx \tag{2.3}$$

where the integration is performed over the whole n-dimensional space. When the distribution cannot be conceived as an integrable function, we retain nevertheless the notation of the right hand side of (2.3), allthough the integral notation has no meaning at all as integral in whatever sense; for instance:

$$\langle \delta(x),\ \varphi(x)\rangle = \int_{-\infty}^{+\infty} \delta(x)\ \varphi(x)\,dx = \varphi(0)$$

The operations on distributions are defined in such a way as to preserve their meaning when the distribution is an integrable function; e.g. the derivative of a distribution is defined by

$$\left\langle \frac{\partial f}{\partial x_i},\varphi\right\rangle = -\left\langle f,\frac{\partial \varphi}{\partial x_i}\right\rangle$$

When φ is a (C^{∞}) function, it follows that the distribution f is infinitely differentiable. A linear transformation T of the independent variables $x_1,\ldots,x_n$ is defined by:

$$\langle f(T\ x),\ \varphi(x)\rangle = \frac{1}{|T|}\ \langle f(x),\ \varphi(T^{-1}\ x)\rangle\ ,$$

where $|T|$ is the absolute value of the determinant of the matrix T.
A sequence $\{f_n(x)\}$ of distributions converges to the distribution f(x) in distributional sense, when

$$\lim_{n\to\infty}\ \langle f_n,\varphi\rangle = \langle f,\varphi\rangle$$

From this definition and from the definition of the derivative of a distribution it follows that the order of these operations may always be interchanged.
The Fourier transform of an absolutely integrable function is defined as

$$F\left[f(x)\right] = \mathcal{F}(k) = \int_{-\infty}^{+\infty} e^{i(k,x)}\ f(x)\,dx$$

with $(k,x)=k_1x_1+k_2x_2+\ldots+k_nx_n$.
The Fourier transform $\mathcal{F}(k)$ of a distribution f(x) is defined by Parseval's equality

$$\langle \tilde{f}(k), \tilde{\varphi}(k) \rangle = (2\pi)^n \langle f(x), \varphi(x) \rangle \qquad (2.4)$$

It is clear that the classical definition of the Fourier transform of an absolutely integrable function is preserved when the function is conceived as a distribution.

However, the definition (2.4) has only sense, when the space of the transforms of $\varphi(x)$, i.e. the space of the functions $\tilde{\varphi}(k)$, has such properties, that one can define on it linear continuous functionals. Moreover, the distributions $\tilde{f}(k)$, defined by (2.4), have to be continuous linear functionals on the space of test functions $\tilde{\varphi}(k)$. It is not difficult to prove that the Fourier transformation is a linear continuous 1-1 mapping of the space S onto itself and hence this is also true for the dual space S'.

For a complete account of the theory of distributions the reader is referred to lit.3 and 4.

In this paper we take for the space of testfunctions $\varphi(x)$ the space $S(R_4)$.

2.2 Distributions on surfaces

Since we shall use in the following pages often distributions which are concentrated on surfaces in R_4, we give in this section a special attention to this kind of distributions.

We consider distributions concentrated on a hypersurface in n-dimensional Euclidean space R_n. They are introduced according to the elegant method of lit.12 (compare also lit.13)

In the same way as the one dimensional distribution $\Theta(x)$ (unit step function) we define the distribution $\Theta(P)$ as:

$$\langle \Theta(P), \varphi(x) \rangle = \int_{P \geqslant 0} \varphi(x)\,dx \qquad (2.5)$$

where $P(x_1, x_2, \ldots, x_n) = 0$ is some surface in R_n and P is a (C^{∞}) function with $\nabla P = (\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, \ldots, \frac{\partial P}{\partial x_n})$ never zero on $\{P=0\}$.

As is well known we may define the one dimensional Dirac δ-function as the distributional limit:

$$\delta(x) = \lim_{c \to 0} \frac{1}{c}[\Theta(x+c) - \Theta(x)]$$

i.e. $\langle \delta(x), \varphi(x) \rangle = \lim_{c \to 0} \frac{1}{c} \langle \Theta(x+c) - \Theta(x), \varphi \rangle = \lim_{c \to 0} \frac{1}{c} \int_{-c}^{0} \varphi(x)\,dx = \varphi(0).$

In the same way one may define the generalization $\delta(P)$ as:

$$\langle \delta(P), \varphi(x) \rangle = \lim_{c \to 0} \frac{1}{c} \langle \Theta(P+c) - \Theta(P), \varphi(x) \rangle = \lim_{c \to 0} \frac{1}{c} \int_{-c \leq P < 0} \varphi(x_1, x_2, \ldots, x_n)\,dx_1 dx_2 \ldots dx_n \tag{2.6}$$

The existence of this limit is clear sence we may write for the latter integral (see fig.1):

$$\langle \delta(P), \varphi(x) \rangle = \lim_{c \to 0} \frac{1}{c} \int_{P=0} \varphi\, c \frac{d\sigma}{|\nabla P|} = \int_{P=0} \varphi(x_1, x_2, \ldots x_n) \frac{d\sigma}{|\nabla P|} \tag{2.7}$$

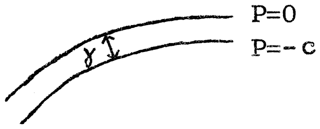

fig.1

where $d\sigma$ is the surface measure on $\{P=0\}$ and $|\nabla P| = \sqrt{(\nabla P, \nabla P)} \neq 0$; $dx_1 dx_2 \ldots dx_n = \gamma\, d\sigma = c \frac{d\sigma}{|\nabla P|}$

We apply this definition to the three dimensional distribution $\delta(x_o^2 - r^2)$ with $r^2 = x_1^2 + x_2^2 + x_3^2$ and where x_o is conceived as a parameter.
We introduce polar coordinates $x_i = r\,\omega_i$ (i=1,2,3); $d\sigma = r^2 d\Omega$, where $d\Omega$ denotes the surface element of the unit sphere Ω in R_3; $|\nabla P| = 2\,r$.
Substituting this into (2.7) we obtain:

$$\langle \delta(x_o^2 - r^2), \varphi \rangle = \tfrac{1}{2} \int_{r=|x_o|} \varphi(r\,\omega_1, r\,\omega_2, r\,\omega_3) \frac{r^2 d\Omega}{r} \quad \text{or}$$

$$\langle \delta(x_o^2 - r^2), \varphi \rangle = \tfrac{1}{2} |x_o| \int_{\Omega} \varphi(|x_o|\,\omega_1, |x_o|\,\omega_2, |x_o|\,\omega_3)\,d\Omega \tag{2.8}$$

A second example of a surface distribution, which we shall use frequently, is the four dimensional distribution $\delta(x^2 - m^2)$ with $x^2 = x_o^2 - x_1^2 - x_2^2 - x_3^2$; m is some constant.
Instead of x_1, x_2, x_3 we use again spherical coordinates, viz. $x_1 = r\,\omega_1, x_2 = r\,\omega_2, x_3 = r\,\omega_3$ with $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$; hence $dx_1\, dx_2\, dx_3 = r^2 dr\, d\Omega$, where $d\Omega$ is the surface element of the unit sphere Ω in (x_1, x_2, x_3) space.
Instead of x_o we take the new coordinate $P = x_o^2 - r^2 - m^2$ and so $x_o = \pm\sqrt{P + r^2 + m^2}$ and $dx_o = \pm \frac{1}{2}(P + r^2 + m^2)^{-\frac{1}{2}} dP$. Hence

$dx=dx_1 dx_2 dx_3 dx_o = \pm \frac{1}{2}(P+r^2+m^2)^{-\frac{1}{2}} r^2 dr\ dP\ d\Omega$ and

$$\langle \Theta(P),\ \varphi(x)\rangle = \frac{1}{2} \int_{P \geq 0} (P+r^2+m^2)^{-\frac{1}{2}} r^2 \varphi \, dr\ dP\ d\Omega$$

From this it follows immediately:

$$\frac{1}{c}\langle \Theta(P+c)-\Theta(P),\ \varphi(x)\rangle = \frac{1}{2c} \int_{-c \leq P<0} (P+r^2+m^2)^{-\frac{1}{2}}\ r^2 \varphi \, dr\ dP\ d\Omega$$

Taking the limit for $c \to 0$ we obtain finally:

$$\langle \delta(x^2-m^2),\ \varphi(x)\rangle = \frac{1}{2} \int_{x^2-m^2=0} (r^2+m^2)^{-\frac{1}{2}}\ r^2 \varphi \, dr\ d\Omega =$$

$$\frac{1}{2} \int_0^\infty\int_\Omega (r^2+m^2)^{-\frac{1}{2}}\ r^2\ \varphi(r\omega_1, r\omega_2, r\omega_3, +\sqrt{r^2+m^2})\, dr\ d\Omega\ +$$

$$\frac{1}{2} \int_0^\infty\int_\Omega (r^2+m^2)^{-\frac{1}{2}}\ r^2\ \varphi(r\omega_1, r\omega_2, r\omega_3, -\sqrt{r^2+m^2})\, dr\ d\Omega \qquad (2.9)$$

Performing the integration with respect to $d\Omega$ we obtain:

$$\langle \delta(x^2-m^2),\ \varphi(x)\rangle = \frac{1}{2} \int_0^\infty (r^2+m^2)^{-\frac{1}{2}}\ r^2\ \bar{\varphi}(r, +\sqrt{r^2+m^2})\, dr\ +$$

$$+\ \frac{1}{2} \int_0^\infty (r^2+m^2)^{-\frac{1}{2}}\ r^2\ \bar{\varphi}(r, -\sqrt{r^2+m^2})\, dr \qquad (2.10)$$

with

$$\bar{\varphi}(r, x_o) = \int_\Omega \varphi(x_1, x_2, x_3, x_o)\, d\Omega \qquad (2.11)$$

$\bar{\varphi}$ is, apart from a constant, the mean value of φ on a sphere with radius r in (x_1, x_2, x_3) space.

For m=0 we get a distribution concentrated on the lightcone $x_o^2=r^2$, viz:

$$\langle \delta(x^2),\varphi(x)\rangle = \frac{1}{2} \int_0^\infty r\ \bar{\varphi}(r,+r)\, dr + \frac{1}{2} \int_0^\infty r\ \bar{\varphi}(r,-r)\, dr \qquad (2.12)$$

The circumstance that the origin is a singular point of the cone surface $x^2=0$ is not essential since the integrals on the right hand side of (2.12) converge for r=0.

The distribution $\delta_{+}(x^2-m^2)$ which is concentrated only on the upper sheet of the hyperboloid $x^2-m^2=0$, is given by:

$$\langle \delta_{+}(x^2-m^2), \varphi(x) \rangle = \tfrac{1}{2} \int_0^{\infty}\int_{\Omega} (r^2+m^2)^{-\frac{1}{2}} \, r^2 \, \varphi(r\,\omega_1, r\,\omega_2, r\,\omega_3, +\sqrt{r^2+m^2}) \, dr \, d\Omega =$$

$$= \tfrac{1}{2} \int_0^{\infty} (r^2+m^2)^{-\frac{1}{2}} \, r^2 \, \bar{\varphi}(r, +\sqrt{r^2+m^2}) \, dr \qquad (2.13)$$

For the distribution $\delta_{-}(x^2-m^2)$ which is concentrated on the lower sheet of $x^2-m^2=0$, we may write analogously:

$$\langle \delta_{-}(x^2-m^2), \varphi(x) \rangle = \tfrac{1}{2} \int_0^{\infty}\int_{\Omega} (r^2+m^2)^{-\frac{1}{2}} \, r^2 \, \varphi(r\,\omega_1, r\,\omega_2, r\,\omega_3, -\sqrt{r^2+m^2}) \, dr \, d\Omega =$$

$$= \tfrac{1}{2} \int_0^{\infty} (r^2+m^2)^{-\frac{1}{2}} \, r^2 \, \bar{\varphi}(r, -\sqrt{r^2+m^2}) \, dr \qquad (2.14)$$

Taking again m=0, we have for the distribution, concentrated on the forward lightcone, the formula:

$$\langle \delta_{+}(x^2), \varphi(x) \rangle = \tfrac{1}{2} \int_0^{\infty} r \, \bar{\varphi}(r, +r) \, dr \qquad (2.15)$$

and for the distribution, concentrated on the backward lightcone, the formula:

$$\langle \delta_{-}(x^2), \varphi(x) \rangle = \tfrac{1}{2} \int_0^{\infty} r \, \bar{\varphi}(r, -r) \, dr \qquad (2.16)$$

It may be remarked that the distributions $\delta(P)$, $\delta_{+}(P)$ and $\delta_{-}(P)$ are even in P (compare eq. (2.7)) and hence $\delta(P) = \delta(-P)$ and $\delta_{\pm}(P) = \delta_{\pm}(-P)$.
The distributions $\delta(P)$, concentrated on the surface $P(x_1, x_2, \dots, x_n)=0$, can be differentiated to P; this derivative is defined by

$$\delta^{(k+1)}(P) = \lim_{c \to 0} \frac{1}{c} \left[\delta^{(k)}(P+c) - \delta^{(k)}(P) \right] \quad k=0,1,2,\dots \qquad (2.17)$$

As an example we determine the first derivative of the distribution $\delta(x_o^2-r^2)$ with respect to $(x_o^2-r^2)$, where x_o is conceived as a parameter. According to (2.8) and the definition (2.17) we may write:

$$\left\langle \delta^{(1)}(x_o^2-r^2),\varphi\right\rangle = \lim_{c\to 0} \tfrac{1}{2}\int_{\Omega} \frac{1}{c}\left[\varphi\left(\sqrt{x_o^2+c}\,\omega_1,\sqrt{x_o^2+c}\,\omega_2,\sqrt{x_o^2+c}\,\omega_3\right)\sqrt{x_o^2+c}\ -\right.$$

$$\left. -\ \varphi(|x_o|\,\omega_1,|x_o|\,\omega_2,|x_o|\,\omega_3)\,|x_o|\right]\,d\Omega =$$

$$\tfrac{1}{2}\int_{\Omega}\left[\frac{1}{2r}\ \frac{\partial}{\partial r}\left\{\varphi(r\,\omega_1,r\,\omega_2,r\,\omega_3)\,r\right\}\right]_{r=|x_o|} d\Omega \qquad (2.18)$$

The derivative of $\delta^{(k)}$ $(P(x_1,x_2,\ldots x_n))$ to one of the variables x_j is given by the chain rule:

$$\frac{\partial}{\partial x_j}\ \delta^{(k)}(P) = \delta^{(k+1)}(P)\ \frac{\partial P}{\partial x_j} \qquad (2.19)$$

When P depends on a parameter t, $P=P(x_1,x_2,\ldots,x_n;t)$, the derivative of $\delta^{(k)}(P)$ with respect to the parameter is also given by the chain rule:

$$\frac{\partial}{\partial t}\ \delta^{(k)}(P) = \delta^{(k+1)}(P)\ \frac{\partial P}{\partial t} \qquad (2.20)$$

For the proof of these chain-rules the reader is referred to lit.12. Finally, we mention the useful formula:

$$P\ \delta^{(k)}(P) = -k\ \delta^{(k-1)}(P) \qquad (2.21)$$

It follows from (2.7) that $P\,\delta(P)=0$ and by repeated differentiation of this result to P one obtains equation (2.21). By aid of the chain-rule and formula (2.21) one proves easily that $\delta(x^2)=\delta(x_o^2-x_1^2-x_2^2-x_3^2)$ satisfies the wave equation $\Box f=0$.

3. The Lorentz-invariant solutions of the Klein-Gordon equation.

3.1 The Fourier transformation

We introduce the modified Fourier transformation F^* which is defined for integrable functions, of four independent variables, by:

$$F^* [f(x)] = \hat{f}(k) = \int_{-\infty}^{+\infty} e^{ik.x} f(x)\,dx \qquad (3.1)$$

with $k.x = k_o x_o - k_1 x_1 - k_2 x_2 - k_3 x_3$ and $dx = dx_1\, dx_2\, dx_3\, dx_o$.
The integration hasto be performed over the whole four-dimensional space R_4.
The inverse transformation F^{*-1} is, in case it exists, given by

$$f(x) = F^{*-1} [\hat{f}(k)] = \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} e^{-i\,k.x} \hat{f}(k)\,dk \qquad (3.2)$$

with $dk = dk_1\, dk_2\, dk_3\, dk_o$.
We have adopted here the Lorentz metric

$$x^2 = x.x = x_o^2 - x_1^2 - x_2^2 - x_3^2 \qquad (3.3)$$

A Lorentz transformation is a linear transformation of the coordinates which leaves x^2 invariant and which does not reverse the direction of time. We consider only the so called proper Lorentz-transformations, which means that we do not consider reflections and so the determinant of the transformations equals always +1.
By using the Lorentz metric in the definition of the Fourier transformation F^* we have achieved that a Lorentz-invariant function has also a Lorentz-invariant transform and vice-versa.
A distribution is invariant under a Lorentz transformation Λ, when

$$\langle f(\Lambda x), \varphi(x) \rangle = \langle f(x), \varphi(\Lambda^{-1} x) \rangle = \langle f(x), \varphi(x) \rangle \qquad (3.4)$$

for any testfunction $\varphi(x)$.
The Fourier transform F^* of a distribution $f(x)$ is defined by the relation:

$$\langle F^*[f], F^*[\varphi] = \langle \hat{f}(k), \hat{\varphi}(k) \rangle = (2\pi)^4 \langle f(x), \varphi(x) \rangle \qquad (3.5)$$

By aid of (3.1) one can easily check the relation:

$$F^* [\varphi(\Lambda x)] = \hat{\varphi}(\Lambda k) \qquad (3.6)$$

Since according to the definition of transformation of the independent variables

$$\langle f(\Lambda x), \varphi(x)\rangle = \langle f(x), \varphi(\Lambda^{-1}x)\rangle$$

we have

$$\langle F^{*}[f(\Lambda x)], F^{*}[\varphi(x)]\rangle = (2\pi)^4 \langle f(\Lambda x), \varphi(x)\rangle =$$

$$(2\pi)^4 \langle f(x), \varphi(\Lambda^{-1}x)\rangle = \langle \hat{f}(k), \hat{\varphi}(\Lambda^{-1}k)\rangle = \langle \hat{f}(\Lambda k), \hat{\varphi}(k)\rangle \quad .$$

Hence we have also for any distribution:

$$F^{*}[f(\Lambda x)] = \hat{f}(\Lambda k) \quad . \tag{3.7}$$

When the distribution f(x) is Lorentz invariant, the relation (3.4) is valid and hence:

$$\langle \hat{f}(\Lambda k), \hat{\varphi}(k)\rangle = \langle \hat{f}(k), \hat{\varphi}(k)\rangle$$

Thus a Lorentz-invariant distribution f(x) has also a Lorentz-invariant modified Fourier transform $\hat{f}(k)$ and vice versa; the proof in the opposite direction is similar. It may be remarked, that for any test-function $\varphi(x)$

$$(2\pi)^4 \varphi(x_1, x_2, x_3, -x_o) = \int_{-\infty}^{+\infty} e^{i(k,x)} \hat{\varphi}(k)\,dk = F[\hat{\varphi}(k)] \tag{3.8}$$

with $(k,x) = k_1x_1 + k_2x_2 + k_3x_3 + k_ox_o$.

When the Fourier transformation F, in the usual sense, is applied to $\hat{\varphi}(k)$, we obtain $(2\pi)^4 \varphi(x_1, x_2, x_3, -x_o)$.

According to Parseval's equality we have also for the Fourier transformation F the relation:

$$\langle F[\hat{f}(k)], F[\hat{\varphi}(k)] = (2\pi)^4 \langle \hat{f}(k), \hat{\varphi}(k)\rangle \qquad \text{or}$$

$$\langle F[\hat{f}(k)], \varphi(x_1, x_2, x_3, -x_o)\rangle = \langle \hat{f}(k), \hat{\varphi}(k)\rangle = (2\pi)^4 \langle f(x), \varphi(x)\rangle$$

and so we have:

$$F[\hat{f}(k)] = (2\pi)^4 \; f(x_1, x_2, x_3, -x_o) \tag{3.9}$$

By aid of (3.1) we obtain

$$F\,F^{*}[f(x_1, x_2, x_3, x_o)] = (2\pi)^4 f(x_1, x_2, x_3, -x_o) \tag{3.10}$$

3.2 Lorentz-invariant solutions of the homogeneous Klein-Gordon equation

Applying the modified Fourier transformation to the homogeneous Klein-Gordon equation:

$$(\Box - m^2)\ f(x)=0 \tag{1.1}$$

we obtain $(k^2-m^2)\ \hat{f}(k)=0$ (3.11)

with

$$k^2=k.k=k_o^2-k_1^2-k_2^2-k_3^2 \tag{3.12}$$

It is clear that the solution of (3.11) is a distribution concentrated in the two sheets of the hyperboloid

$$k^2-m^2=0 \tag{3.13}$$

Outside the two sheets of this hyperboloid $\hat{f}(k) \equiv 0$.

R.T. Seeley has shown in a private communication that the solutions of (3.11), invariant under the group of proper Lorentz-transformations, can be written as:

$$\hat{f}(k) = c_+\ \delta_+(k^2-m^2) + c_-\ \delta_-(k^2-m^2) \tag{3.14}$$

where c_+ and c_- are arbitrary constants; $\delta_+(k^2-m^2)$ and $\delta_-(k^2-m^2)$ are distributions concentrated on the upper respectively lower sheet of the hyperboloid $(k^2-m^2)=0$; they are defined by the equations (2.13) and (2.14), viz.:

$$\langle \delta_+(k^2-m^2),\ \hat{\varphi}(k)\rangle = \tfrac{1}{2} \int_0^\infty \int_\Omega (x^2+m^2)^{-\frac{1}{2}}\ x^2\ \hat{\varphi}(x\,\omega_1, x\,\omega_2, x\omega_3, +\sqrt{x^2+m^2})\,dx\ d\Omega$$

$$= \tfrac{1}{2} \int_0^\infty (x^2+m^2)^{-\frac{1}{2}}\ x^2\ \bar{\hat{\varphi}}(x,+\sqrt{x^2+m^2})\,dx \tag{3.15}$$

and

$$\langle \delta_-(k^2-m^2),\ \hat{\varphi}(k)\rangle = \tfrac{1}{2} \int_0^\infty \int_\Omega (x^2+m^2)^{-\frac{1}{2}}\ x^2\ \hat{\varphi}(x\omega_1, x\omega_2, x\omega_3, -\sqrt{x^2+m^2})\,dx\ d\Omega$$

$$= \tfrac{1}{2} \int_0^\infty (x^2+m^2)^{-\frac{1}{2}}\ x^2\ \bar{\hat{\varphi}}(x,-\sqrt{x^2+m^2})\,dx \tag{3.16}$$

with $x^2=k_1^2+k_2^2+k_3^2$; Ω denotes now the unit sphere in (k_1, k_2, k_3) space and $d\Omega$ the surface measure on it.

$\bar{\hat{\varphi}}$ is defined analogously to (2.11)

$$\bar{\hat{\varphi}}(x,k_o)= \int_\Omega \hat{\varphi}(k_1,k_2,k_3,k_o)d\Omega=\int_\Omega \hat{\varphi}(x\omega_1,x\omega_2,x\omega_3,k_o)\,d\Omega \tag{3.17}$$

For $c_+=c_-=1$ we obtain $\hat{f}(k)=\delta(k^2-m^2)$ as a special solution of (3.11). For the sake of completeness we repeat in the next section the proof of Mr.Seeley.

The Lorentz-invariant solutions of (1.1) are according to the foregoing section 3.1 determined by the Fourier-transform F of the distribution $\delta_+(k^2-m^2)$ and $\delta_-(k^2-m^2)$.

Hence to obtain the general Lorentz-invariant solution of the homogeneous Klein-Gordon equation all we have to do is to determine the Fourier transform of these two distributions. This will be carried out in chapter 6.

3.3 The Lorentz-invariant solutions of $(k^2-m^2)\ \hat{f}(k)=0$.

Any solution of $(k^2-m^2)\ \hat{f}(k)=0$ invariant under the group of proper Lorentz-transformations can be written as:

$$\hat{f}(k)= c_+\ \delta_+(k^2-m^2)+ c_-\ \delta_-(k^2-m^2) \tag{3.14}$$

where c_+ and c_- are arbitrary constants and δ_+ and δ_- are defined by the equations (3.15) and (3.16).

Proof

We define two continuous maps, T_+ and T_-, from the space of testfunctions $S(R_4)$ to the space of testfunctions $S(R_3)$, by:

$$T_{\pm}\ \hat{\varphi}(k_1,k_2,k_3,k_o)= \frac{\hat{\varphi}(k_1,k_2,k_3,\pm\sqrt{x^2+m^2})}{2\ \sqrt{x^2+m^2}} \tag{3.18}$$

Any distribution of $S'(R_4)$, satisfying $(k^2-m^2)\ \hat{f}(k)=0$ has the form:

$$\langle\hat{f}(k),\ \hat{\varphi}(k)\rangle = \langle\hat{g}_+(k_1,k_2,k_3),\ T_+\ \hat{\varphi}(k)\rangle\ +$$

$$+\langle\ \hat{g}_-(k_1,k_2,k_3),\ T_-\ \hat{\varphi}(k)\rangle \tag{3.19}$$

Since $\langle(k^2-m^2)\ \hat{f}(k),\ \hat{\varphi}(k)\rangle = \langle\hat{f}(k),(k^2-m^2)\ \hat{\varphi}(k)\rangle$ and $T_{\pm}\ (k^2-m^2)\ \hat{\varphi}(k)\equiv 0$, it is clear that any distribution of the form (3.19) satisfies the relation $(k^2-m^2)\ \hat{f}(k)=0$.

Suppose now $(k^2-m^2)\ \hat{f}(k)=0$ and we have to prove the relation (3.19).
We introduce the (C^{∞}) function $\chi(t)$, such that $\chi(t)\equiv 1$ for $|t|\leq\frac{m^2}{4}$ and $\chi(t)\equiv 0$ for $|t|\geq\frac{m^2}{2}$.
The function $\chi(k^2-m^2)$ is a (C^{∞}) function which has its support in a neighbourhood of the upper and lower sheet of the hyperboloid $k^2-m^2=0$; the neighbourhood of the upper sheet lies within the region $k_o > \frac{1}{2}\, m\sqrt{2}$ and that of the lower sheet in the region $k_o < -\frac{1}{2}\, m\sqrt{2}$.
Hence $\Theta(\pm\, k_o)\ \chi(k^2-m^2)$ is also in (C^{∞}) and we have also $\hat{f}=\chi(k^2-m^2)\ \hat{f}$, since $\hat{f}$ is concentrated on the hyperboloid.
For any testfunction $\hat{\psi}(k_1,k_2,k_3)$, belonging to $S(R_3)$, we define the distributions:

$$\langle \hat{g}_{\pm}, \hat{\psi}\rangle = \langle \hat{f}, 2\sqrt{\boldsymbol{x}^2+m^2}\ \Theta(\pm\, k_o)\ \chi(k^2-m^2)\ \hat{\psi}\rangle \tag{3.20}$$

It is clear that $2\sqrt{\boldsymbol{x}^2+m^2}\ \Theta(\pm\, k_o)\ \chi(k^2-m^2)\ \hat{\psi}$ belongs to $S(R_4)$.
Consider a testfunction $\hat{\varphi}(k)\in S(R_4)$ which vanishes for $k_o < -\frac{m}{2}$. From (3.18) it follows immediately $T_-\ \hat{\varphi}\equiv 0$ and

$$\langle \hat{f},\hat{\varphi}\rangle - \langle \hat{g}_+,T_+\,\hat{\varphi}\rangle - \langle \hat{g}_-,T_-\,\hat{\varphi}\rangle = \langle \hat{f},\hat{\varphi}\rangle - \langle \hat{g}_+,T_+\,\hat{\varphi}\rangle =$$

$$\langle \hat{f},\ \chi(k^2-m^2)\hat{\varphi}\rangle - \langle \hat{g}_+,T_+\ \hat{\varphi}\rangle\ .$$

Since $\chi(k^2-m^2)\,\hat{\varphi}\ (k)$ vanishes for $k_o < +\frac{1}{2}\, m\sqrt{2}$, we may write:

$$\langle \hat{f},\hat{\varphi}\rangle - \langle \hat{g}_+,T_+\hat{\varphi}\rangle = \langle \hat{f},\Theta(k_o)\ \chi(k^2-m^2)\hat{\varphi}\rangle - \langle \hat{g}_+,T_+\ \hat{\varphi}\rangle.$$

Using now (3.20) and (3.18) we obtain:

$$\langle \hat{f},\hat{\varphi}\rangle - \langle \hat{g}_+,T_+\hat{\varphi}\rangle = \langle (k^2-m^2)\hat{f},\Theta(k_o)\ \chi(k^2-m^2)\ \frac{\hat{\varphi}(k_1,k_2,k_3,k_o)-\hat{\varphi}(k_1,k_2,k_3,+\sqrt{\boldsymbol{x}^2+m^2})}{(k^2-m^2)}\rangle \tag{3.21}$$

The testfunction, appearing in the right hand side of (3.21) belongs certainly to $S(R_4)$, since $\hat{\varphi}$ is in (C^{∞}) and $k_o\longrightarrow\sqrt{\boldsymbol{x}^2+m^2}$ when $(k^2-m^2)\longrightarrow 0$.
Due to the supposition $(k^2-m^2)\hat{f}=0$ we get finally $\langle \hat{f},\hat{\varphi}\rangle = \langle \hat{g}_+,T_+\,\hat{\varphi}\rangle$ and the relation (3.19) has been proved for testfunctions vanishing for $k_o < -\frac{m}{2}$.
In the same way $\langle \hat{f},\hat{\varphi}\rangle = \langle \hat{g}_-,T_-\,\hat{\varphi}\rangle$ for testfunctions $\hat{\varphi}$ vanishing for $k_o > \frac{m}{2}$. As any testfunction can be written as the sum of two testfunctions

vanishing respectively for $k_o < -\frac{m}{2}$ and $k_o > +\frac{m}{2}$, the relation (3.19) has been established.

To complete the proof of the theorem we need the following lemma:

A distribution, invariant under the group of proper Lorentz-transformations, satisfies the relations

$$k_j \frac{\partial \hat{f}}{\partial k_o} = -k_o \frac{\partial \hat{f}}{\partial k_j} \qquad j=1,2,3. \qquad (3.22)$$

Proof

Three special transformations of this group are the transformations $\Lambda_1(\theta)$, $\Lambda_2(\theta)$ and $\Lambda_3(\theta)$ which mix respectively k_o and k_1, k_o and k_2, and k_o and k_3. They are given by matrices such as e.g.:

$$\Lambda_1(\theta) = \begin{pmatrix} \cosh\theta & 0 & 0 & \sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\theta & 0 & 0 & \cosh\theta \end{pmatrix} \qquad (3.23)$$

To obtain the new coordinates (k_1', k_2', k_3', k_o') the matrix has to be multiplied by the vector (k_1, k_2, k_3, k_o). When $\hat{f}(k)$ is invariant under Λ_1 we have:

$$\langle \hat{f}(k), \hat{\varphi}(k)\rangle = \langle \hat{f}(\Lambda_1 k), \hat{\varphi}(k)\rangle = \langle \hat{f}(k), \hat{\varphi}(\Lambda_1^{-1} k)\rangle$$

or
$$\left\langle \hat{f}(k), \frac{\hat{\varphi}(k) - \hat{\varphi}(\Lambda_1^{-1} k)}{\theta} \right\rangle = 0 \qquad (3.24)$$

By aid of (3.23) we may write:

$$\frac{\hat{\varphi}(k) - \hat{\varphi}(\Lambda_1^{-1} k)}{\theta} = \frac{\hat{\varphi}(k_1, k_2, k_3, k_o) - \hat{\varphi}(\cosh\theta\, k_1 - \sinh\theta\, k_o, k_2, k_3, -\sinh\theta\, k_1 + \cosh\theta\, k_o)}{\theta}$$

and hence

$$\lim_{\theta \to 0} \frac{\hat{\varphi}(k) - \hat{\varphi}(\Lambda_1^{-1} k)}{\theta} = k_o \frac{\partial \hat{\varphi}}{\partial k_1} + k_1 \frac{\partial \hat{\varphi}}{\partial k_o} \qquad (3.25)$$

Substituting (3.25) into (3.24) we obtain

$$\left\langle \hat{f}, k_o \frac{\partial \hat{\varphi}}{\partial k_1} + k_1 \frac{\partial \hat{\varphi}}{\partial k_o} \right\rangle = -\left\langle k_o \frac{\partial \hat{f}}{\partial k_1} + k_1 \frac{\partial \hat{f}}{\partial k_o}, \hat{\varphi} \right\rangle = 0$$

and hence

$$k_1 \frac{\partial \hat{f}}{\partial k_o} = -k_o \frac{\partial \hat{f}}{\partial k_1} .$$

One proves in the some way:

$$k_j \frac{\partial \hat{f}}{\partial k_o} = - k_o \frac{\partial \hat{f}}{\partial k_j} \qquad j=2,3. \tag{3.22}$$

It has been proved above that any distribution $\hat{f}(k)$ satisfying $(k^2-m^2)\hat{f}=0$ can be written in the form (3.19), where $\hat{g}_+$ and $\hat{g}_-$ are defined by (3.20). We use now the conditions that $\hat{f}$ has to be invariant under the group of proper Lorentz-transformations.

Given $\hat{\varphi}(k_1,k_2,k_3)$ in $S(R_3)$, we take $\hat{\varphi}_{\pm}(k_1,k_2,k_3,k_o)=\Theta(\pm k_o)\ \chi(k^2-m^2)\hat{\varphi}(k_1,k_2,k_3)$. In a neighbourhood of $(k^2-m^2)=0$ one has:

$$\frac{\partial \hat{\varphi}_{\pm}}{\partial k_o} = 0 \quad \text{and} \quad \frac{\partial \hat{\varphi}_{\pm}}{\partial k_1} = \Theta(\pm k_o)\ \chi(k^2-m^2)\frac{\partial \hat{\varphi}}{\partial k_1}$$

Applying now the lemma we obtain:

$$\left\langle k_o \frac{\partial \hat{f}}{\partial k_1} + k_1 \frac{\partial \hat{f}}{\partial k_o}, \hat{\varphi}_{\pm} \right\rangle = -\left\langle \hat{f}, k_o\ \Theta(\pm k_o)\ \chi(k^2-m^2)\ \frac{\partial \hat{\varphi}}{\partial k_1} \right\rangle = 0.$$

Since $\hat{f}$ is concentrated on the hyperboloid $k^2-m^2=0$, one may substitute $k_o=\sqrt{x^2+m^2}$; inserting (3.20) yields finally

$$-\tfrac{1}{2}\left\langle \hat{g}_{\pm}, \frac{\partial \hat{\varphi}}{\partial k_1} \right\rangle = +\tfrac{1}{2}\left\langle \frac{\partial \hat{g}_{\pm}}{\partial k_1}, \hat{\varphi} \right\rangle = 0$$

and hence $\hat{g}_{\pm}$ is independent of k_1. The independence of $\hat{g}_{\pm}$ from k_2 and k_3 is proved in the same way and so $\hat{g}_{\pm}$ is constant. We put $\hat{g}_+=c_+$ and $\hat{g}_-=c_-$. Collecting the results gives:

$$\left\langle \hat{f}(k), \hat{\varphi}(k) \right\rangle = \left\langle c_+, T_+\ \hat{\varphi}(k) \right\rangle + \left\langle c_-, T_-\ \hat{\varphi}(k) \right\rangle =$$

$$\tfrac{1}{2}\ c_+ \iiint_{-\infty}^{+\infty} \frac{\hat{\varphi}(k_1,k_2,k_3,+\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}}\ dk_1\ dk_2\ dk_3 +$$

$$\tfrac{1}{2}\ c_- \iiint_{-\infty}^{+\infty} \frac{\hat{\varphi}(k_1,k_2,k_3,-\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}}\ dk_1\ dk_2\ dk_3 =$$

$$c_+ \left\langle \delta_+(k^2-m^2), \hat{\varphi}(k) \right\rangle + c_- \left\langle \delta_-(k^2-m^2), \hat{\varphi}(k) \right\rangle \tag{3.14}$$

q.e.d.

3.4. Lorentz-invariant solutions of the inhomogeneous Klein-Gordon equation.

The Lorentz invariant solutions of the inhomogeneous Klein-Gordon equation:

$$(\square - m^2) g(x) = -\delta(x) \tag{1.3}$$

are obtained by the Lorentz invariant solutions of the homogeneous equation (1.1) and a particular Lorentz-invariant solution of (1.3). The latter can be obtained by applying again the modified Fourier transformation F^* to (1.3). Transformation of (1.3) yields:

$$(k^2 - m^2) \hat{g}(k) = -1 \tag{3.26}$$

A Lorentz invariant solution of (3.26) is

$$\hat{g}(k) = \frac{1}{m^2 - k^2} \tag{3.27}$$

The distribution $\left\langle \frac{1}{m^2 - k^2}, \hat{\varphi}(k) \right\rangle$ is defined as the Cauchy principal value:

$$P \int \frac{\hat{\varphi}(k)}{m^2 - k^2} dk = \lim_{\varepsilon \to 0} \int_{|m^2 - k^2| > \varepsilon} \frac{\hat{\varphi}(k)}{m^2 - k^2} dk \tag{3.28}$$

The integration is performed over the whole space R_4.
The existence of this principal value is shown in the next chapter.
By aid of the Fourier transform of the distribution $\frac{1}{m^2 - k^2}$ we get a Lorentz-invariant particular solution of (1.3).
In order to obtain the solutions of (1.3), invariant under the group of proper Lorentz transformations, we have only to determine the Fourier transforms of the distributions $\delta_+(k^2 - m^2)$, $\delta_-(k^2 - m^2)$ and $\frac{1}{m^2 - k^2}$.
This will be done in chapters 5 and 6.

4. The distribution $(m^2 + P \pm i\,0)^{-1}$

We consider for the moment the one-dimensional distribution $\ln(k + i0)$.
It is defined by:

$$\ln(k + i0) = \lim_{\varepsilon \to +0} \ln(k + i\varepsilon) = \ln|k| + i\pi\Theta(-k).$$

Differentiation of this distribution to k yields:

$$\frac{1}{k+i0} = \lim_{\varepsilon \to +0} \frac{1}{k+i\varepsilon} = \frac{d}{dk} \ln|k| - i\pi\delta(k) \quad (4.1)$$

The functional $\frac{d}{dk}\ln(k)$ can be reduced as follows:

$$\left\langle \frac{d}{dk}\ln|k|, \hat{\varphi}(k)\right\rangle = -\left\langle \ln|k|, \hat{\varphi}'(k)\right\rangle = -\int_{-\infty}^{+\infty} \ln|k| \; \hat{\varphi}'(k)\,dk =$$

$$= -\lim_{\varepsilon \to +0} \int_{|k|>\varepsilon} \ln|k| \; \hat{\varphi}'(k)\,dk = -\lim_{\varepsilon \to +0} \left[\ln|k| \; \hat{\varphi}(k) \Big|_{-\infty}^{-\varepsilon} + \ln|k| \; \hat{\varphi}(k) \Big|_{+\varepsilon}^{\infty} \right.$$

$$\left. - \int_{|k|>\varepsilon} \frac{\hat{\varphi}(k)}{k}\,dk \right] = \lim_{\varepsilon \to 0} \int_{|k|>\varepsilon} \frac{\hat{\varphi}(k)}{k}\,dk \; .$$

Hence

$$\frac{1}{k+i0} = \lim_{\varepsilon \to +0} \frac{1}{k+i\varepsilon} = \frac{1}{k} - \pi i\delta(k) \quad (4.2)$$

and in the same way

$$\frac{1}{k-i0} = \lim_{\varepsilon \to +0} \frac{1}{k-i\varepsilon} = \frac{1}{k} + \pi i\delta(k) \quad (4.3)$$

where the distribution $\left\langle \frac{1}{k}, \hat{\varphi}(k)\right\rangle$ has to be conceived as a principal value of Cauchy.

This well known result can be generalized for the distribution $(m^2+P(k) \pm i0)^{-1}$, where $P(k)$ is some real quadratic form in $k_1, k_2, \dots, k_n$. Since we need it here only for $P(k)=k_1^2+k_2^2+k_3^2-k_o^2=\varkappa^2-k_o^2$, we shall confine our considerations to this case. We define the distribution $(m^2+P \pm i0)^\lambda$ for $\mathrm{Re}\,\lambda > -1$ by:

$$(m^2+P \pm i0)^\lambda = (m^2+P)_+^\lambda + e^{\pm \pi\lambda i} (m^2+P)_-^\lambda \quad (4.4)$$

with

$$(m^2+P)_+^\lambda = \begin{cases} (m^2+P)^\lambda & \text{for } m^2+P \geq 0 \\ 0 & \text{for } m^2+P < 0 \end{cases} \quad (4.5)$$

and

$$(m^2+P)_-^\lambda = \begin{cases} 0 & \text{for } m^2+P > 0 \\ (-m^2-P)^\lambda & \text{for } (m^2+P) \leq 0. \end{cases} \quad (4.6)$$

It is clear that the functional $\langle (m^2+P\pm i0)^\lambda , \hat{\varphi}(k)\rangle$ is an analytic function of λ for $\mathrm{Re}\,\lambda > -1$.
The distribution $(m^2+P\pm i0)^\lambda$ with $\mathrm{Re}\,\lambda \leqslant -1$ is defined by the analytic continuation of $\langle (m^2+P\pm i0)^\lambda , \hat{\varphi}(k)\rangle$ as function of the complex variable λ.
This analytic continuation will be obtained by the analytic continuation of $\langle (m^2+P)_+^\lambda , \hat{\varphi}\rangle$ and $\langle (m^2+P)_-^\lambda , \hat{\varphi}\rangle$.
For $\mathrm{Re}\,\lambda > -1$ we may write:

$$\langle (m^2+P)_+^\lambda , \hat{\varphi}\rangle = \int_{m^2+P \geqslant 0} (m^2+P)^\lambda \, \hat{\varphi}(k)\, dk \qquad (4.7)$$

It is also clear that $\langle (m^2+P)_+^\lambda , \hat{\varphi}\rangle$ is an analytic function of λ for $\mathrm{Re}\,\lambda > -1$. We introduce again polar coordinates for k_1, k_2 and k_3: $k_i = x\omega_i$ $i=1,2,3$, $x = \sqrt{k_1^2+k_2^2+k_3^2}$ and instead of k_o we take

$$Q = x^2 - k_o^2 + m^2 \quad \text{or} \quad k_o = \pm \sqrt{x^2+m^2-Q} \;;$$

hence $dk_1\, dk_2\, dk_3\, dk_o = \mp \frac{1}{2}(x^2+m^2-Q)^{-\frac{1}{2}}\, x^2\, dx\, dQ\, d\Omega$, where $d\Omega$ is the surface element on the unit sphere in (k_1,k_2,k_3)-space.
Substitution into (4.7) yields:

$$\langle (m^2+P)_+^\lambda , \hat{\varphi}\rangle = \frac{1}{2} \int_0^\infty \left\{ \int_0^{x^2+m^2} Q^\lambda \, \frac{\bar{\hat{\varphi}}(x, +\sqrt{x^2+m^2-Q})}{\sqrt{x^2+m^2-Q}} \, dQ \right\} x^2\, dx \; +$$

$$+ \frac{1}{2} \int_0^\infty \left\{ \int_0^{x^2+m^2} Q^\lambda \, \frac{\bar{\hat{\varphi}}(x, -\sqrt{x^2+m^2-Q})}{\sqrt{x^2+m^2-Q}} \, dQ \right\} x^2\, dx \qquad (4.8)$$

where the integration to $d\Omega$ has been carried out and $\bar{\hat{\varphi}}$ is defined by (3.17)
The right hand side of (4.8) may be written for $\mathrm{Re}\,\lambda > -1$ as:

$$\left\langle (m^2+P)_+^{\lambda}, \hat{\varphi} \right\rangle =$$

$$\tfrac{1}{2} \int_0^{\infty} \Big[\int_0^{x^2+m^2} Q^{\lambda} \Big\{ \frac{\bar{\hat{\varphi}}(x,+\sqrt{x^2+m^2-Q})}{\sqrt{x^2+m^2-Q}} - \frac{\bar{\hat{\varphi}}(x,+\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}} \Big\} dQ \Big] x^2 \, dx +$$

$$+ \tfrac{1}{2} \int_0^{\infty} \Big[\int_0^{x^2+m^2} Q^{\lambda} \Big\{ \frac{\bar{\hat{\varphi}}(x,-\sqrt{x^2+m^2-Q})}{\sqrt{x^2+m^2-Q}} - \frac{\bar{\hat{\varphi}}(x,-\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}} \Big\} dQ \Big] x^2 \, dx +$$

$$+ \tfrac{1}{2} \frac{1}{\lambda+1} \int_0^{\infty} \frac{(x^2+m^2)^{\lambda+1}}{\sqrt{x^2+m^2}} \Big\{ \bar{\hat{\varphi}}(x,+\sqrt{x^2+m^2}) + \bar{\hat{\varphi}}(x,-\sqrt{x^2+m^2}) \Big\} x^2 \, dx \qquad (4.9)$$

The first two terms are analytic functions of λ for $\mathrm{Re}\,\lambda > -2$, while the third term is an analytic function of λ for all values of λ except $\lambda = -1$, where it has a simple pole.

Hence the distribution $\left\langle (m^2+P)_+^{\lambda}, \hat{\varphi} \right\rangle$ is defined for $\mathrm{Re}\,\lambda > -2$ by (4.9) with the exception of $\lambda = -1$; the distribution $(m^2+P)_+^{\lambda}$ does not exist for $\lambda = -1$, since $\left\langle (m^2+P)_+^{\lambda}, \hat{\varphi} \right\rangle$ has a pole in $\lambda = -1$. According to (3.15) and (3.16) the residue in this pole equals $\left\langle \delta(m^2+P), \hat{\varphi} \right\rangle$.

The Laurent expansion of $\left\langle (m^2+P)_+^{\lambda}, \hat{\varphi} \right\rangle$ in the neighbourhood of the point $\lambda = -1$ is:

$$\left\langle (m^2+P)_+^{\lambda}, \hat{\varphi} \right\rangle = \frac{1}{\lambda+1} \left\langle \delta(m^2+P), \hat{\varphi} \right\rangle + \left\langle (m^2+P)_+^{-1}, \hat{\varphi} \right\rangle + \Theta(\lambda+1) \qquad (4.10)$$

where the distribution $\left\langle (m^2+P)_+^{-1}, \hat{\varphi} \right\rangle$ is defined by:

$$\left\langle (m^2+P)_+^{-1}, \hat{\varphi} \right\rangle = \tfrac{1}{2} \int_0^{\infty} \frac{\log(x^2+m^2)}{\sqrt{x^2+m^2}} \Big\{ \bar{\hat{\varphi}}(x,+\sqrt{x^2+m^2}) + \bar{\hat{\varphi}}(x,-\sqrt{x^2+m^2}) \Big\} x^2 dx$$

$$+ \tfrac{1}{2} \int_0^{\infty} \Big[\int_{-m^2}^{x^2} (m^2+P)^{-1} \Big\{ \frac{\bar{\hat{\varphi}}(x,+\sqrt{x^2-P})}{\sqrt{x^2-P}} - \frac{\bar{\hat{\varphi}}(x,+\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}} \Big\} dP \Big] x^2 dx \, .$$

$$+ \tfrac{1}{2} \int_0^{\infty} \Big[\int_{-m^2}^{x^2} (m^2+P)^{-1} \Big\{ \frac{\bar{\hat{\varphi}}(x,-\sqrt{x^2-P})}{\sqrt{x^2-P}} - \frac{\bar{\hat{\varphi}}(x,-\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}} \Big\} dP \Big] x^2 dx \qquad (4.11)$$

Needless to remark that the distribution $(m^2+P)_+^{-1}$ is not the distribution

$(m^2+P)_+^\lambda$ for $\lambda=-1$, but $\langle(m^2+P)_+^{-1},\hat{\varphi}\rangle$ is the value of the regular part of the Laurent-expansion of $\langle(m^2+P)_+^\lambda,\hat{\varphi}\rangle$ in the point $\lambda=-1$.
The Laurent expansion of $\langle(m^2+P)_-^\lambda,\hat{\varphi}\rangle$ for λ in the neighbourhood of the point $\lambda=-1$ can be found in the same way.
For Re $\lambda>-1$ we may write:

$$\langle(m^2+P)_-^\lambda,\hat{\varphi}\rangle=\int_{m^2+P\leqq 0}(-m^2-P)^\lambda\,\hat{\varphi}(k)\,dk=\int_{Q\geqslant 0}Q^\lambda\,\hat{\varphi}(k)\,dk$$

with $Q=-m^2-P=k_0^2-x^2-m^2$.
We introduce again the new coordinates $k_i=x\,\omega_i$, $i=1,2,3$ with $dk_1dk_2dk_3=x^2dx\,d\Omega$ integrating with respect to $d\Omega$ yields:

$$\langle(m^2+P)_-^\lambda,\hat{\varphi}\rangle=\int_{Q\geqslant 0}Q^\lambda\,\bar{\hat{\varphi}}(x,k_0)\,x^2dx\,dk_0$$

Instead of x we use now the coordinate Q with $x=\sqrt{k_0^2-Q-m^2}$ and $dx=-\frac{1}{2}\,dQ/\sqrt{k_0^2-Q-m^2}$; hence we obtain:

$$\langle(m^2+P)_-^\lambda,\hat{\varphi}\rangle=\tfrac{1}{2}\left\{\int_{-\infty}^{-m}+\int_{+m}^{+\infty}\right\}dk_0\left[\int_0^{k_0^2-m^2}Q^\lambda\,\bar{\hat{\varphi}}(\sqrt{k_0^2-Q-m^2},k_0)\sqrt{k_0^2-Q-m^2}\,dQ\right] \tag{4.12}$$

The right hand side of (4.12) may be written for Re $\lambda>-1$ as:

$$\langle(m^2+P)_-^\lambda,\hat{\varphi}\rangle=$$

$$\tfrac{1}{2}\left\{\int_{-\infty}^{-m}+\int_{+m}^{+\infty}\right\}dk_0\left[\int_0^{k_0^2-m^2}Q^\lambda\left\{\bar{\hat{\varphi}}(\sqrt{k_0^2-Q-m^2},k_0)\sqrt{k_0^2-Q-m^2}-\bar{\hat{\varphi}}(\sqrt{k_0^2-m^2},k_0)\sqrt{k_0^2-m^2}\right\}dQ\right]+$$

$$+\tfrac{1}{2}\,\frac{1}{\lambda+1}\left\{\int_{-\infty}^{-m}+\int_{+m}^{+\infty}\right\}(k_0^2-m^2)^{\lambda+1}\sqrt{k_0^2-m^2}\;\bar{\hat{\varphi}}(\sqrt{k_0^2-m^2},k_0)\,dk_0$$

After substitution of $k_0=\pm\sqrt{x^2+m^2}$ the second term is reduced to

$$\tfrac{1}{2}\,\frac{1}{\lambda+1}\int_0^\infty x^{2(\lambda+1)}\,\bar{\hat{\varphi}}(x,+\sqrt{x^2+m^2})\,\frac{x^2\,dx}{\sqrt{x^2+m^2}}+\tfrac{1}{2}\,\frac{1}{\lambda+1}\int_0^\infty x^{2(\lambda+1)}\,\bar{\hat{\varphi}}(x,-\sqrt{x^2+m^2})\,\frac{x^2\,dx}{\sqrt{x^2+m^2}}$$

and we obtain for Re $\lambda>-1$

$$\left\langle (m^2+P)_-^{\lambda}, \hat{\varphi} \right\rangle =$$

$$\tfrac{1}{2}\left\{ \int_{-\infty}^{-m} + \int_{+m}^{+\infty} \right\} dk_o \left[\int_0^{k_o^2-m^2} Q^{\lambda} \left\{ \bar{\hat{\varphi}}(\sqrt{k_o^2-Q-m^2}, k_o)\sqrt{k_o^2-Q-m^2} - \bar{\hat{\varphi}}(\sqrt{k_o^2-m^2}, k_o)\sqrt{k_o^2-m^2} \right\} dQ \right] \tag{4.13}$$

$$+ \frac{1}{\lambda+1}\left\{ \tfrac{1}{2}\int_0^{\infty} x^{2(\lambda+1)} \bar{\hat{\varphi}}(x, +\sqrt{x^2+m^2}) \frac{x^2 dx}{\sqrt{x^2+m^2}} + \tfrac{1}{2}\int_0^{\infty} x^{2(\lambda+1)} \bar{\hat{\varphi}}(x, -\sqrt{x^2+m^2}) \frac{x^2 dx}{\sqrt{x^2+m^2}} \right\}$$

The right hand side of (4.13) is an analytic function of λ for $\mathrm{Re}\,\lambda > -2$ with the exception of the point $\lambda=-1$, where it has a simple pole. According to (3.15) and (3.16) the residu in this pole equals $\left\langle \delta(m^2+P), \hat{\varphi}(k) \right\rangle$.
The right hand side of (4.13) gives the analytic continuation of $\left\langle (m^2+P)_-^{\lambda}, \hat{\varphi} \right\rangle$ in the region of $\mathrm{Re}\,\lambda > -2$.
The Laurent expansion of $\left\langle (m^2+P)_-^{\lambda}, \hat{\varphi} \right\rangle$ in the neighbourhood of the point $\lambda=-1$ runs as follows:

$$\left\langle (m^2+P)_-^{\lambda}, \hat{\varphi} \right\rangle = \frac{1}{(\lambda+1)} \left\langle \delta(m^2+P), \hat{\varphi} \right\rangle + \left\langle (m^2+P)_-^{-1}, \hat{\varphi} \right\rangle + O(\lambda+1) \tag{4.14}$$

where $\left\langle (m^2+P)_-^{-1}, \hat{\varphi} \right\rangle$ is again the value of the regular part of $\left\langle (m^2+P)_-^{\lambda}, \hat{\varphi} \right\rangle$ with $\lambda=-1$. The distribution $\left\langle (m^2+P)_-^{-1}, \hat{\varphi} \right\rangle$ is defined by:

$$\left\langle (m^2+P)_-^{-1}, \hat{\varphi} \right\rangle =$$

$$\tfrac{1}{2}\left\{ \int_{-\infty}^{-m} + \int_{+m}^{+\infty} \right\} \bar{\hat{\varphi}}(\sqrt{k_o^2-m^2}, k_o) \sqrt{k_o^2-m^2}\,.\ln(k_o^2-m^2)\, dk_o -$$

$$-\tfrac{1}{2}\left\{ \int_{-\infty}^{-m} + \int_{+m}^{+\infty} \right\} dk_o \left[\int_{-k_o^2}^{-m^2} (m^2+P)^{-1} \left\{ \bar{\hat{\varphi}}(\sqrt{k_o^2+P}, k_o)\sqrt{k_o^2+P} - \bar{\hat{\varphi}}(\sqrt{k_o^2-m^2}, k_o)\sqrt{k_o^2-m^2} \right\} dP \right] \tag{4.15}$$

The Laurent expansion of $\left\langle (m^2+P+i0)^{\lambda}, \hat{\varphi} \right\rangle = \left\langle (m^2+P)_+^{\lambda}, \hat{\varphi} \right\rangle + e^{\pi\lambda i} \left\langle (m^2+P)_-^{\lambda}, \hat{\varphi} \right\rangle$ in the neighbourhood of $\lambda=-1$ can now easily be obtained by aid of (4.10) and (4.14).
The poles of $\left\langle (m^2+P)_{\pm}^{\lambda}, \hat{\varphi} \right\rangle$ cancel each other and the result is:

$$\left\langle (m^2+P+i0)^{\lambda}, \hat{\varphi} \right\rangle = \left\langle (m^2+P)_+^{-1} - (m^2+P)_-^{-1}, \hat{\varphi} \right\rangle - \pi i \left\langle \delta(m^2+P), \hat{\varphi} \right\rangle + O(\lambda+1).$$

Thus $\langle (m^2+P+i0)^\lambda, \hat{\varphi}\rangle$ is an analytic function of λ in the neighbourhood of $\lambda=-1$ and for $\lambda=-1$ we get the result:

$$\langle (m^2+P+i0)^{-1}, \hat{\varphi}\rangle = \langle (m^2+P)_+^{-1} - (m^2+P)_-^{-1}, \hat{\varphi}\rangle - \pi i \langle \delta(m^2+P), \hat{\varphi}\rangle \qquad (4.16)$$

where the distributions $\langle (m^2+P)_\pm^{-1}, \hat{\varphi}\rangle$ are defined by (4.11) and (4.15). In order to reduce the right hand side of (4.16) further we consider once again the distributions $\langle (m^2+P)_\pm^{-1}, \hat{\varphi}\rangle$.

The right hand side of (4.11) can be reduced as follows:

$$\langle (m^2+P)_+^{-1}, \hat{\varphi}\rangle = \tfrac{1}{2} \int_0^\infty \frac{\log(x^2+m^2)}{\sqrt{x^2+m^2}} \left\{ \bar{\hat{\varphi}}(x, +\sqrt{x^2+m^2}) + \bar{\hat{\varphi}}(x, -\sqrt{x^2+m^2}) \right\} x^2 dx$$

$$+\tfrac{1}{2} \lim_{\varepsilon\to+0} \left[\int_0^\infty x^2 dx \left\{ \int_{-m^2+\varepsilon}^{x^2} (m^2+P)^{-1} \frac{\bar{\hat{\varphi}}(x, +\sqrt{x^2-P})}{\sqrt{x^2-P}} dP + \frac{\bar{\hat{\varphi}}(x, +\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}} \log\varepsilon \right\} \right]$$

$$-\tfrac{1}{2} \int_0^\infty \log(x^2+m^2) \frac{\bar{\hat{\varphi}}(x, +\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}} x^2 dx +$$

$$+\tfrac{1}{2} \lim_{\varepsilon\to+0} \left[\int_0^\infty x^2 dx \left\{ \int_{-m^2+\varepsilon}^{x^2} (m^2+P)^{-1} \frac{\bar{\hat{\varphi}}(x, -\sqrt{x^2-P})}{\sqrt{x^2-P}} dP + \frac{\bar{\hat{\varphi}}(x, -\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}} \log\varepsilon \right\} \right]$$

$$-\tfrac{1}{2} \int_0^\infty \log(x^2+m^2) \frac{\bar{\hat{\varphi}}(x, -\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}} x^2 dx =$$

$$\tfrac{1}{2} \lim_{\varepsilon\to+0} \left\{ \left[\int_0^\infty x^2 dx \left[\int_{-m^2+\varepsilon}^{x^2} (m^2+P)^{-1} \left\{ \frac{\bar{\hat{\varphi}}(x, +\sqrt{x^2-P}) + \bar{\hat{\varphi}}(x, -\sqrt{x^2-P})}{\sqrt{x^2-P}} \right\} dP + \right.\right.\right.$$

$$\left.\left.\left. + \log\varepsilon \left\{ \frac{\bar{\hat{\varphi}}(x, +\sqrt{x^2+m^2}) + \bar{\hat{\varphi}}(x, -\sqrt{x^2+m^2})}{\sqrt{x^2+m^2}} \right\} \right] \right] \right\}$$

By virtue of (3.15) and (3.16) we get finally:

$$\left\langle (m^2+P)_+^{-1}, \hat{\varphi} \right\rangle = \lim_{\varepsilon\to+0} \left[\frac{1}{2} \int_0^\infty x^2 dx \int_{-m^2+\varepsilon}^{x^2} (m^2+P)^{-1} \left\{ \frac{\bar{\hat{\varphi}}(x,+\sqrt{x^2-P})+\bar{\hat{\varphi}}(x,-\sqrt{x^2-P})}{\sqrt{x^2-P}} \right\} dP + \right.$$

$$\left. + \log\varepsilon \left\langle \delta(m^2+P), \hat{\varphi} \right\rangle \right] \qquad (4.17)$$

We can reduce in the same way equation (4.15).

$$\left\langle (m^2+P)_-^{-1}, \hat{\varphi} \right\rangle = \tfrac{1}{2}\left\{ \left(\int_{-\infty}^{-m} + \int_{+m}^{+\infty} \right) \bar{\hat{\varphi}}(\sqrt{k_o^2-m^2}, k_o)\sqrt{k_o^2-m^2}\, \ln(k_o^2-m^2)\, dk_o \right\} -$$

$$-\tfrac{1}{2} \lim_{\varepsilon\to+0} \left[\left(\int_{-\infty}^{-\sqrt{m^2+\varepsilon}} + \int_{+\sqrt{m^2+\varepsilon}}^{\infty} \right) dk_o \left[\int_{-k_o^2}^{-m^2-\varepsilon} (m^2+P)^{-1}\, \bar{\hat{\varphi}}(\sqrt{k_o^2+P}, k_o)\sqrt{k_o^2+P}\, dP - \right.\right.$$

$$\left.\left. -\log\varepsilon \,.\, \bar{\hat{\varphi}}(\sqrt{k_o^2-m^2}, k_o)\sqrt{k_o^2-m^2} \right] \right] -$$

$$-\tfrac{1}{2}\left\{ \left(\int_{-\infty}^{-m} + \int_{+m}^{+\infty} \right) \bar{\hat{\varphi}}(\sqrt{k_o^2-m^2}, k_o)\sqrt{k_o^2-m^2}\, \ln(k_o^2-m^2)\, dk_o \right\}$$

and hence

$$\left\langle (m^2+P)_-^{-1}, \hat{\varphi} \right\rangle = -\lim_{\varepsilon\to+0} \left[\frac{1}{2} \left(\int_{-\infty}^{-\sqrt{m^2+\varepsilon}} + \int_{+\sqrt{m^2+\varepsilon}}^{\infty} \right) dk_o \left\{ \int_{-k_o^2}^{-m^2-\varepsilon} (m^2+P)^{-1}\, \bar{\hat{\varphi}}(\sqrt{k_o^2+P}, k_o)\sqrt{k_o^2+P}\, dP \right\} \right.$$

$$\left. - \log\varepsilon \,.\, \left\langle \delta(m^2+P), \hat{\varphi} \right\rangle \right] \qquad (4.18)$$

Combining (4.17) and (4.18) we obtain finally:

$$\left\langle (m^2+P)_+^{-1} - (m^2+P)_-^{-1}, \hat{\varphi} \right\rangle =$$

$$\lim_{\varepsilon\to+0} \left[\frac{1}{2} \int_0^\infty x^2 dx \int_{-m^2+\varepsilon}^{x^2} (m^2+P)^{-1} \left\{ \frac{\bar{\hat{\varphi}}(x,+\sqrt{x^2-P})+\bar{\hat{\varphi}}(x,-\sqrt{x^2-P})}{\sqrt{x^2-P}} \right\} dP + \right.$$

$$\left. +\frac{1}{2} \left(\int_{-\infty}^{-\sqrt{m^2+\varepsilon}} + \int_{+\sqrt{m^2+\varepsilon}}^{\infty} \right) dk_o \left\{ \int_{-k_o^2}^{-m^2-\varepsilon} (m^2+P)^{-1} \bar{\hat{\varphi}}(\sqrt{k_o^2+P}, k_o)\sqrt{k_o^2+P}\, dP \right\} \right] \qquad (4.19)$$

However, the right hand side of (4.19) is nothing else than

$$\lim_{\varepsilon\to+0} \int_{|m^2+P|>\varepsilon} (m^2+P)^{-1}\, \hat{\varphi}(k)\, dk$$

and hence:

$$\left\langle (m^2+P)_+^{-1} - (m^2+P)_-^{-1}, \hat{\varphi}(k) \right\rangle = \left\langle \frac{1}{m^2+P}, \hat{\varphi}(k) \right\rangle \tag{4.20}$$

where the distribution $\left\langle \frac{1}{m^2+P}, \hat{\varphi}(k) \right\rangle$ has to be conceived as the Cauchy principal value of

$$\int \frac{\hat{\varphi}(k)}{m^2+P}\, dk.$$

Combining equations (4.16) and (4.20) we obtain:

$$\left\langle (m^2+P+i0)^{-1}, \hat{\varphi}(k) \right\rangle = \left\langle \frac{1}{m^2+P}, \hat{\varphi}(k) \right\rangle - \pi i \left\langle \delta(m^2+P), \hat{\varphi}(k) \right\rangle \tag{4.21}$$

We have, of course, in the same way the relation:

$$\left\langle (m^2+P-i0)^{-1}, \hat{\varphi}(k) \right\rangle = \left\langle \frac{1}{m^2+P}, \hat{\varphi}(k) \right\rangle + \pi i \left\langle \delta(m^2+P), \hat{\varphi}(k) \right\rangle \tag{4.22}$$

The formulae (4.21) and (4.22) are generalizations of the one-dimensional formulae:

$$\left\langle (k\pm i0)^{-1}, \hat{\varphi}(k) \right\rangle = \left\langle \frac{1}{k}, \hat{\varphi}(k) \right\rangle \mp \pi i \left\langle \delta(k), \hat{\varphi}(k) \right\rangle \qquad (4.2)(4.3)$$

From (4.21) and (4.22) it follows that

$$\left\langle \frac{1}{m^2+P}, \hat{\varphi}(k) \right\rangle = \tfrac{1}{2} \left\langle (m^2+P+i0)^{-1}, \hat{\varphi}(k) \right\rangle + \tfrac{1}{2} \left\langle (m^2+P-i0)^{-1}, \hat{\varphi}(k) \right\rangle \tag{4.23}$$

and

$$\left\langle \delta(m^2+P), \hat{\varphi}(k) \right\rangle = \frac{1}{2\pi i} \left\langle (m^2+P-i0)^{-1}, \hat{\varphi}(k) \right\rangle - \frac{1}{2\pi i} \left\langle (m^2+P+i0)^{-1}, \hat{\varphi}(k) \right\rangle \tag{4.24}$$

To obtain Lorentz invariant solutions of the homogeneous and inhomogeneous Klein-Gordon equation we have to determine a.o. the Fourier transforms of $\frac{1}{m^2+P}$ and $\delta(m^2+P)$. This can be done by aid of the Fourier transform of $(m^2+P\pm i0)^{-1}$; therefore we shall turn now our attention to the determination of the Fourier transform of the distributions $(m^2+P\pm i0)^{-1}$.

5. The Fourier transform of $(m^2+P\pm i0)^{-1}$

5.1. The Fourier transform of $(m^2+\mathcal{D})^{\lambda}$

Let D be a positive definite quadratic form in k. The Fourier transform of $(m^2+D)^{\lambda}$ with $\mathrm{Re}\,\lambda<-2$ is

$$F\left[(m^2+D)^{\lambda}\right] = \int_{-\infty}^{+\infty} (m^2+D)^{\lambda} \; e^{i(x,k)} \, dk \qquad (5.1)$$

where $(x,k)= x_1k_1+x_2k_2+x_3k_3+x_ok_o$ (5.2)

We write D in the form:

$$D=(k,Gk) = \sum_{r,s=0}^{3} g_{rs} \, k_r \, k_s \qquad (5.3)$$

where G denotes the matrix of the coefficients g_{rs}.
Making a transformation k=Tk' to principal axes, the quadratic form D can be written as:

$$D=(k,Gk)=(k',T^{*}G\,T\,k')=(k',k')$$

where the matrix T^{*} is the adjoint of the matrix T.
$T^{*}\,G\,T=1$ and $|T|^2=\frac{1}{|G|}$, where $|T|$ and $|G|$ are the determinants of the matrices T resp. G.
Hence

$$F\left[(m^2+D)^{\lambda}\right] = \frac{1}{\sqrt{|G|}} \int_{-\infty}^{+\infty} \left\{m^2+(k',k')\right\}^{\lambda} \; e^{i(T^{*}x,k')} dk'$$

$$= \frac{1}{\sqrt{|G|}} \int_{-\infty}^{+\infty} \left\{m^2+(k',k')\right\}^{\lambda} \; e^{i(x',k')} dk'$$

with $x'=T^{*}x$.
It is clear, that $F\left[(m^2+D)^{\lambda}\right]$ depends only on the length $|x'|$ of the four-vector x'. Taking $x'=(|x'|,0,0,0)$, we obtain:

$$F\left[(m^2+D)^{\lambda}\right] = \frac{1}{\sqrt{|G|}} \int_{-\infty}^{+\infty} \left\{m^2+(k',k')\right\}^{\lambda} \; e^{i|x'|\,k_1'} \, dk' \qquad (5.4)$$

To reduce further the right hand side of (5.4) we introduce spherical coordinates:

$$k_1'= \varkappa \cos\varphi_1, \quad k_2'= \varkappa \sin\varphi_1 \cos\varphi_2, \quad k_3'= \varkappa \sin\varphi_1 \sin\varphi_2 \cos\varphi_3$$

and $k_o'= \varkappa \sin\varphi_1 \sin\varphi_2 \sin\varphi_3$.

$\varkappa^2=(k',k')=k_1'^2+k_2'^2+k_3'^2+k_o'^2$ and $dk'=\varkappa^3 \sin^2\varphi_1 \, d\varkappa \, d\varphi_1 \, d\Omega$, where $d\Omega$ is the surface element of the unity sphere in R_3.
Performing the integration to Ω we obtain:

$$F[(m^2+D)^\lambda] = \frac{4\pi}{\sqrt{|G|}} \int_0^\infty \int_0^\pi (m^2+\varkappa^2)^\lambda e^{i|x'|\varkappa\cos\varphi_1} \varkappa^3 \sin^2\varphi_1 \, d\varphi_1 \, d\varkappa$$

Using the following integral relations for Bessel function:

$$\int_0^\pi e^{i|x'|\varkappa\cos\varphi_1} \sin^2\varphi_1 \, d\varphi_1 = \frac{\pi}{\varkappa|x'|} J_1(\varkappa|x'|)$$

and

$$\int_0^\infty \varkappa^2 J_1(\varkappa|x'|)(m^2+\varkappa^2)^\lambda \, d\varkappa = \left(\frac{2}{|x'|}\right)^{\lambda+1} \frac{m^{2+\lambda}}{\Gamma(-\lambda)} K_{2+\lambda}(m|x'|)$$

where J_1 is the Bessel function of the first kind and $K_{2+\lambda}$ the modified Bessel-function (see lit.14), we get the result:

$$F[(m^2+D)^\lambda] = \frac{2\pi^2}{\sqrt{|G|}} \left(\frac{2m}{|x'|}\right)^{\lambda+2} \frac{1}{\Gamma(-\lambda)} K_{\lambda+2}(m|x'|) \qquad (5.5)$$

Since $x'=T^*x$, we have

$$|x'|^2=(x',x')=(x,T\,T^*x)=(x,G^{-1}x)=\sum_{r,s=0}^{3} g^{rs} x_r x_s$$

Hence $|x'|^2$ is a positive definite quadratic form, whose coefficients form a matrix which is the inverse of the matrix of the coefficients of the positive quadratic form D.
Putting

$$|x'|^2 = \sum_{r,s=0}^{3} g^{rs} x_r x_s = E \qquad (5.6)$$

we obtain finally

$$F[(m^2+D)^\lambda] = \frac{2\pi^2}{\sqrt{|G|}} \left(\frac{2m}{E^{\frac{1}{2}}}\right)^{\lambda+2} \frac{1}{\Gamma(-\lambda)} K_{\lambda+2}(m\,E^{\frac{1}{2}}) \qquad (5.7)$$

By aid of the principle of analytic continuation one proves easily that this result, obtained for $\mathrm{Re}\,\lambda<-2$, is also valid for all other complex

values of λ.
We have derived formula (5.7) with the assumption that D is a positive definite quadratic form.
The left and right hand side of (5.7) is also an analytic function of the coefficients g_{rs}, belonging to the quadratic form D, and this is true for those ranges of g_{rs}, where D is positive definite.
We continue now the left and right hand side of (5.7) analytically with respect to the coefficients g_{rs} into those ranges of complex values of g_{rs}, where the so obtained new quadratic form $\mathcal{D}$ has either a positive definite or a negative definite imaginary part.
The function $\mathcal{D}^{\lambda}$ is defined as

$$\mathcal{D}^{\lambda} = |\mathcal{D}|^{\lambda} \exp(\lambda i \arg\mathcal{D}) \text{ with } 0 \leqslant \arg\mathcal{D} < \pi \text{ or } -\pi < \arg\mathcal{D} \leqslant 0 \qquad (5.8)$$

The function $\mathcal{D}^{\lambda}$ has a cut along the negative real axis of the complex $\mathcal{D}$-plane.
Using again the principle of analytic continuation, we have for the complex quadratic forms $\mathcal{D}$ and all values of λ the formula:

$$F[(m^2+\mathcal{D})^{\lambda}] = \frac{2\pi^2}{\sqrt{|g|}} \left(\frac{2\, m}{\mathcal{E}^{\frac{1}{2}}}\right)^{\lambda+2} \frac{1}{\Gamma(-\lambda)} K_{\lambda+2}(m\, \mathcal{E}^{\frac{1}{2}}) \qquad (5.9)$$

where $|g|$ is the discriminant of the quadratic form $\mathcal{D}$.
$\mathcal{E}$ is a quadratic form in x, whose coefficients form a matrix which is the inverse of the matrix of the coefficients of $\mathcal{D}$.
When $\mathcal{D}$ has a positive or negative definite imaginary part, $\mathcal{E}$ has a negative respectively positive definite imaginary part; when $\mathcal{D}$ itself is positive or negative definite, $\mathcal{E}$ is also positive respectively negative definite.
Hence the function $\mathcal{E}^{\frac{1}{2}}$, appearing in formula (5.9) has a cut along the negative real axis of the complex $\mathcal{E}$-plane.

5.2 The Fourier transforms of $(m^2+P\pm i0)^{-1}$, $(m^2+P)^{-1}$ and $\delta(m^2+P)$

In section 4 we have defined the distribution $(m^2+P\pm i0)^{\lambda}$.
For Re $\lambda > -1$ we have:

$$(m^2+P\pm i0)^{\lambda} = (m^2+P)_{+}^{\lambda} + e^{\pm\pi\lambda i}(m^2+P)_{-}^{\lambda} \qquad (4.4)$$

For other values of λ the distribution $(m^2+P\pm i0)^\lambda$ is defined by the analytical continuation of $\langle (m^2+P\pm i0)^\lambda , \hat{\varphi}(k)\rangle$ with respect to λ .
Another method to define the distribution $(m^2+P\pm i0)^\lambda$ is as follows; for $\mathrm{Re}\,\lambda \geqslant 0$ we define $(m^2+P\pm i0)^\lambda$ as

$$(m^2+P\pm i0)^\lambda = \lim_{\varepsilon \to +0} (m^2+P\pm i\,\varepsilon\, P_1)^\lambda \tag{5.10}$$

where P_1 is the positive definite quadratic form $k_1^2+k_2^2+k_3^2+k_o^2$ and where $(m^2+P\pm i\,\varepsilon\, P_1)^\lambda$ is defined as $\left|m^2+P\pm i\,\varepsilon\, P_1\right|^\lambda \exp\left\{\lambda i \arg(m^2+P\pm i\,\varepsilon\, P_1)\right\}$ with $0 \leqslant \arg(m^2+P+i\,\varepsilon\, P_1) < \pi$ and $-\pi < \arg(m^2+P-i\,\varepsilon\, P_1) \leqslant 0$.
For other values of λ the distribution $(m^2+P\pm i0)^\lambda$ is defined as the analytical continuation of

$$\lim_{\varepsilon \to +0} \langle (m^2+P\pm i\,\varepsilon\, P_1)^\lambda , \hat{\varphi}(k)\rangle \quad \text{with respect to } \lambda .$$

The right hand sides of the formulae (4.4) and (5.10) are equal to each other for $\mathrm{Re}\,\lambda \geqslant 0$ and hence due to the principle of analytic continuation both definitions of the distribution $(m^2+P\pm i0)^\lambda$ are equivalent. .
For the complex quadratic form $\mathcal{D}$ we take $\mathcal{D}=D+i\,\varepsilon\, D_1$, where D is an arbitrary non-degenerate real quadratic form and D_1 the positive definite quadratic form $k_1^2+k_2^2+k_3^2+k_o^2$.
Substituting this into (5.9) and letting consecutively ε go to $\pm$ 0, we obtain due to the continuity of the Fourier transformation:

$$F\left[(m^2+D\pm i0)^\lambda\right] = \frac{4\pi^2}{\sqrt{|\mathcal{g}_\pm|}}\ \frac{2^{\lambda+1}}{\Gamma(-\lambda)}\ m^{\lambda+2}\ \frac{K_{\lambda+2}\left\{m(E\mp i0)^{\frac{1}{2}}\right\}}{(E\mp i0)^{\frac{1}{2}(\lambda+2)}} \tag{5.11}$$

The matrix of the coefficients of E is of course again the inverse of the matrix of the coefficients of D.
In particular we take now $\lambda=-1$ and $D=k_1^2+k_2^2+k_3^2-k_o^2=P$.
Hence

$$E=x_1^2+x_2^2+x_3^2-x_o^2= Q \tag{5.12}$$

and

$$|\mathcal{g}_\pm| = (1\pm i0)^3\ (-1\pm i0) = e^{\pm \pi i} \tag{5.13}$$

Putting (5.12) and (5.13) into (5.11) we get finally the result:

$$F\left[(m^2+P\pm i0)^{-1}\right] = \mp 4\pi^2 \, i \, m \, \frac{K_1\left\{ m(Q\mp i0)^{\frac{1}{2}}\right\}}{(Q\mp i0)^{\frac{1}{2}}} \tag{5.14}$$

By aid of (4.23) and (4.24) we obtain:

$$F\left[\delta(m^2+P)\right] = 2\pi m\left[\frac{K_1\left\{ m(Q+i0)^{\frac{1}{2}}\right\}}{(Q+i0)^{\frac{1}{2}}} + \frac{K_1\left\{ m(Q-i0)^{\frac{1}{2}}\right\}}{(Q-i0)^{\frac{1}{2}}}\right] \tag{5.15}$$

$$F\left[\frac{1}{m^2+P}\right] = 2\pi^2 \, i \, m\left[\frac{K_1\left\{ m(Q+i0)^{\frac{1}{2}}\right\}}{(Q+i0)^{\frac{1}{2}}} - \frac{K_1\left\{ m(Q-i0)^{\frac{1}{2}}\right\}}{(Q-i0)^{\frac{1}{2}}}\right] \tag{5.16}$$

In the special case that m=0, we find by taking the limit for m going to zero:

$$F\left[\delta(P)\right] = 2\pi\left[\frac{1}{Q+i0} + \frac{1}{Q-i0}\right] = \frac{4\pi}{Q} \tag{5.17}$$

$$F\left[\frac{1}{P}\right] = 2\pi^2 i\left[\frac{1}{Q+i0} - \frac{1}{Q-i0}\right] = 4\pi^3\delta(Q) \tag{5.18}$$

Since $(Q\pm i0)^{\frac{1}{2}}$ has a cut along the negative axis of the complex Q-plane we have:

$(Q\pm i0)^{\frac{1}{2}} = Q^{\frac{1}{2}}$ for $Q > 0$ and $(Q\pm i0)^{\frac{1}{2}} = e^{\pm\frac{\pi}{2}i}(-Q)^{\frac{1}{2}}$ for $Q < 0$.

Using the relations:

$$K_1(z\,e^{i\frac{\pi}{2}}) = -\tfrac{1}{2}\pi\, H_1^{(2)}(z) = -\tfrac{1}{2}\pi\, J_1(z) + \tfrac{1}{2}\, i\pi\, Y_1(z)$$

$$K_1(z\,e^{-i\frac{\pi}{2}}) = -\tfrac{1}{2}\pi\, H_1^{(1)}(z) = -\tfrac{1}{2}\pi\, J_1(z) - \tfrac{1}{2}\, i\pi\, Y_1(z)$$

where $H_1^{(i)}$ is the Hankel- and Y_1 the Neumann-function, we can also write:

$$F\left[\delta(m^2+P)\right] = 4\pi m \, \frac{K_1(m\,Q^{\frac{1}{2}})}{Q^{\frac{1}{2}}} \quad \text{for } Q > 0\text{, i.e. outside the light-cone} \tag{5.19}$$

and

$$F\left[\delta(m^2+P)\right] = i\pi^2 m \, \frac{H_1^{(2)}\left\{m(-Q)^{\frac{1}{2}}\right\} - H_1^{(1)}\left\{m(-Q)^{\frac{1}{2}}\right\}}{(-Q)^{\frac{1}{2}}} \quad \text{for } Q < 0, \tag{5.20}$$

i.e. inside the light-cone

It follows from (5.15) that for small values of $|Q|$

$$F[\delta(m^2+P)] = 2\pi\left(\frac{1}{Q+i0}+\frac{1}{Q-i0}\right)+O(\ln|Q|)=\frac{4\pi}{Q}+O(\ln|Q|) \tag{5.21}$$

where the distribution $\langle \frac{1}{Q}, \varphi(x)\rangle$ is taken in the sense of a principal value.

Hence for any testfunction $\varphi(x)$ we may write:

$$\langle F[\delta(m^2+P)], \varphi(x)\rangle = \lim_{\varepsilon\to+0}\Big[4\pi m\int_{Q>\varepsilon}\frac{K_1(m\,Q^{\frac{1}{2}})}{Q^{\frac{1}{2}}}\varphi(x)\,dx +$$

$$+ i\pi^2 m\int_{Q<-\varepsilon}\frac{H_1^{(2)}\{m(-Q)^{\frac{1}{2}}\} - H_1^{(1)}\{m(-Q)^{\frac{1}{2}}\}}{(-Q)^{\frac{1}{2}}}\varphi(x)\,dx \tag{5.22}$$

In the same way we find

$$F\left[\frac{1}{m^2+P}\right] = 0 \quad \text{for } Q>0, \text{ i.e. outside the light cone} \tag{5.23}$$

$$F\left[\frac{1}{m^2+P}\right] = -\frac{2\pi^3 m}{(-Q)^{\frac{1}{2}}}J_1\{m(-Q)^{\frac{1}{2}}\} \quad \text{for } Q<0, \text{ i.e. inside the light cone} \tag{5.24}$$

For small values of $|Q|$ we have

$$F\left[\frac{1}{m^2+P}\right] = 2\pi^2 i\left[\frac{1}{Q+i0}-\frac{1}{Q-i0}\right]+O(\ln|Q|)=4\pi^3\,\delta(Q)+O(\ln|Q|) \tag{5.25}$$

Thus we may write:

$$F\left[\frac{1}{m^2+P}\right] = -2\pi^3\, m\,\Theta(-Q)\,\frac{J_1\{m(-Q)^{\frac{1}{2}}\}}{(-Q)^{\frac{1}{2}}} + 4\pi^3\,\delta(Q) \tag{5.26}$$

or

$$\left\langle F\left[\frac{1}{m^2+P}\right], \varphi(x)\right\rangle = -2\pi^3 m\int_{Q<0}\frac{J_1\{m(-Q)^{\frac{1}{2}}\}}{(-Q)^{\frac{1}{2}}}\varphi(x)\,dx + 4\pi^3\langle\delta(Q),\varphi(x)\rangle \tag{5.27}$$

$F\left[\frac{1}{m^2+P}\right]$ is an analytic function of x inside the light cone, it vanishes outside the light cone and on the light cone it is highly singular, where it must be considered as the distribution $4\pi^3\,\delta(Q)$.

Having obtained the Fourier transforms of the distributions $\delta(m^2+P)$ and $\frac{1}{m^2+P}$, we have still only to determine the Fourier transforms of the distributions $\delta_{\pm}(m^2+P)$; this will be done in the next chapter.

6. The Fourier transform of $\delta_{\pm}(m^2+P)$

6.1 Lemma

To determine the Fourier transforms of the distributions $\delta_{\pm}(m^2+P)$ we need a lemma, which is formulated as follows:

Let $g(u_1,u_2,\ldots,u_n,w_o)=g(u_1,u_2,\ldots,u_n,u_o+iv_o)$ be holomorphic in the upper halfplane $v_o>0$ or in the lower halfplane $v_o<0$ for any set of real values $(u_1,u_2,\ldots,u_n)$.

$g(u_1,u_2,\ldots,u_n,u_o+iv_o)$ is assumed for any $v_o\neq 0$ to be absolutely integrable over any finite region of the space R_{n+1} of the variables $u_1,u_2,\ldots,u_n,u_o$. There exist a positive constant C and positive integers $p_1,p_2,\ldots,p_n,p_o$, the latter independent of v_o, such that for sufficient large values of u_i $(i=1,2,\ldots,n,0)$ the function $g(u_1,u_2,\ldots,u_n,u_o+iv_o)$ is majorated as:

$$|g(u_1,u_2,\ldots u_n,u_o+iv_o)| < C\left\{\prod_{i=1}^{n}(1+u_i^2)^{p_i}\right\}|u_o+iv_o|^{p_o}$$

When it is further assumed that $\lim_{v_o\to\pm 0} g(u_1,u_2,\ldots u_n,u_o+iv_o)$ exists in the distributional sense, then $\lim_{v_o\to\pm 0} g(u_1,u_2,\ldots u_n,u_o+iv_o)$ is the (n+1)-dimensional Fourier transform of a distribution $f_{\pm}(x_1,x_2,\ldots,x_n,x_o)$ which vanishes for $x_o<0$ respectively $x_o>0$.

Proof

Assume $g(u_1,u_2,\ldots u_n,w_o)$ holomorphic in the upper halfplane $v_o>0$. According to the assumptions there exist positive integers $p_1,p_2,\ldots p_n,p_o$, independent of v_o, such that the function

$$h(u_1,u_2,\ldots,u_n,u_o+iv_o)=\left\{\prod_{i=1}^{n}(1+u_i^2)^{-p_i-1}\right\}(u_o+iv_o)^{-p_o-2}\; g(u_1\ldots u_n,u_o+iv_o)$$

is absolutely integrable over the whole space R_{n+1} for any value of $v_o>0$. We consider now the integral:

$$\frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-i\{x_1u_1+..+x_nu_n+x_o(u_o+iv_o)\}} h(u_1,u_2,..,u_n,u_o+iv_o) \cdot du_1 du_2 \dots du_n du_o =$$

$$\frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-i(x_1u_1+\dots+x_nu_n)} du_1 \dots du_n \int_{-\infty}^{+\infty} e^{-ix_o(u_o+iv_o)} h(u_1,\dots,u_o+iv_o) du_o$$

Since the function h is holomorphic in the upper halfplane $v_o > 0$, this integral is independent of v_o and vanishes for $x_o < 0$. We denote this integral by $f_+^*(x_1,x_2,\dots,x_n,x_o)$ and hence

$$F\left[e^{-v_o x_o} f_+^*(x_1,x_2,..x_n,x_o) \right] = h(u_1,u_2,\dots u_n,u_o+iv_o)$$

Applying to $e^{-v_o x_o} f_+^*(x_1,x_2,\dots x_n,x_o)$ the operator

$$D=\left\{ \prod_{i=1}^{n} \left(1-\frac{\partial^2}{\partial x_i^2}\right)^{p_i+1} \right\}, \text{ we obtain}$$

$$F\left[D\, e^{-v_o x_o} f_+^*(x_1,x_2,\dots x_n,x_o)\right]=(u_o+iv_o)^{-p_o-2} g(u_1,u_2,\dots u_n,u_o+iv_o)$$

or

$$g(u_1,u_2,\dots u_n,u_o+iv_o)=F\left[\left(i\frac{\partial}{\partial x_o}+iv_o\right)^{p_o+2} e^{-v_o x_o} D\, f_+^*(x_1,x_2,\dots,x_n,x_o)\right]$$

Taking the limit for $v_o \longrightarrow +0$ we get due to the continuity of the Fourier transformation:

$$\lim_{v_o \to +0} g(u_1,u_2,\dots u_n,u_o+iv_o)=$$

$$F\left[\lim_{v_o \to 0} \left\{\left(i\frac{\partial}{\partial x_o}+iv_o\right)^{p_o+2} e^{-v_o x_o} D\, f_+^*(x_1,x_2,\dots,x_n,x_o)\right\}\right]$$

Since f_+^* vanishes for $x_o < 0$, $\lim_{v_o \to +0} g(u_1,\dots u_n,u_o+iv_o)$ is the Fourier transform of a distribution vanishing for $x_o < 0$.

One can prove in the same way that the distributional limit of a function, holomorphic in the lower halfplane, for $v_o \to -0$ is the Fourier transform of a distribution vanishing for $x_o > 0$.

It may be remarked that in the case of distributions defined on (C^∞) functions with compact support one can omit in the lemma the assumption of the existence of the distributional limit $\lim_{v_o \to \underline{+}0} g(u_1,u_2,\dots u_n,u_o+iv_o)$; the latter can then be proved and one has again the result that

$\lim_{v_o \to \pm 0} g(u_1,u_2,\ldots u_n,u_o+iv_o)$ is the Fourier transform of a distribution vanishing for $x_o < 0$ respectively $x_o > 0$.
Since we consider distributions defined on $S(R_4)$ and since we do not need the lemma in this more general form, we content ourselves here with the above proven simplified form of the lemma.

6.2 The Fourier transform of $\delta_{\pm}(m^2+P)$

The distributions $\delta_{\pm}(m^2+P)$ were defined in section 3, viz:

$$\langle \delta_{+}(m^2+P)\,,\ \hat{\varphi}(k)\rangle = \tfrac{1}{2}\int_0^{\infty} (\varkappa^2+m^2)^{-\frac{1}{2}}\,\varkappa^2\ \bar{\hat{\varphi}}(\varkappa,+\sqrt{\varkappa^2+m^2})\,d\varkappa \qquad (3.15)$$

and

$$\langle \delta_{-}(m^2+P)\,,\ \hat{\varphi}(k)\rangle = \tfrac{1}{2}\int_0^{\infty} (\varkappa^2+m^2)^{-\frac{1}{2}}\,\varkappa^2\ \bar{\hat{\varphi}}(\varkappa,-\sqrt{\varkappa^2+m^2})\,d\varkappa \qquad (3.16)$$

with

$$P=k_1^2+k_2^2+k_3^2-k_o^2 = \varkappa^2-k_o^2$$

Hence the distribution $\delta_{+}(m^2+P)$ is concentrated on the upper sheet of the hyperboloid $m^2+P=0$ and vanishes for $k_o < m$, whereas the distribution $\delta_{-}(m^2+P)$ is concentrated on the lower sheet of this hyperboloid and vanishes for $k_o > -m$.
The distribution $\delta(m^2+P)$ is the sum of these distributions and the latter are the result of the unique splitting of $\delta(m^2+P)$ into its positive and negative "frequency" parts.
We consider now the Fourier transform of $\delta(m^2+P)$ and we put
$F\,[\,\delta(m^2+P)\,] = X(x_1,x_2,x_3,x_o)$.

Let it be possible to make a Hilbert splitting of the distribution $X(x_1,x_2,x_3,x_o)$ as follows:

$$X(x_1,x_2,x_3,x_o)=\lim_{y_o\to+0} X_1(x_1,x_2,x_3,x_o+iy_o)+\lim_{y_o\to-0} X_2(x_1,x_2,x_3,x_o+iy_o)$$

where the limits are taken in the distributional sense and where $X_1(x_1,x_2,x_3,x_o+iy_o)$ is a holomorphic function in the upper halfplane of the x_o+iy_o plane and $X_2(x_1,x_2,x_3,x_o+iy_o)$ is a holomorphic function in the lower half of this plane.

Moreover, we assume that X_1 and X_2 satisfye the conditions of the lemma with n=3.

From the lemma it follows that $F^{-1}\left[\lim_{y_o\to+0} X_1\right]$ is a distribution vanishing for $k_o<0$ and $F^{-1}\left[\lim_{y_o\to-0} X_2\right]$ is a distribution vanishing for $k_o>0$.

Since $F^{-1}\left[\lim_{y_o\to+0} X_1+\lim_{y_o\to-0} X_2\right]=\delta(m^2+P)$, the distribution $F^{-1}\left[\lim_{y_o\to+0} X_1\right]$ is concentrated on the upper sheet of the hyperboloid and the plane $k_o=0$, whereas $F^{-1}\left[\lim_{y_o\to-0} X_2\right]$ is concentrated on the lower sheet of the hyperboloid and the plane $k_o=0$.

When we succeed in making a Hilbert splitting of $X(x_1,x_2,x_3,x_o)$, which satisfies the homogeneous Klein-Gordon equation, such that $\lim_{y_o\to+0} X_1$ and $\lim_{y_o\to-0} X_2$ satisfye also the homogeneous Klein-Gordon equation, the part of the distributions $F^{-1}\left[\lim_{y_o\to+0} X_1\right]$ and $F^{-1}\left[\lim_{y_o\to-0} X_2\right]$ which is concentrated on the plane $k_o=0$ does not occur, since a distribution concentrated on the plane $k_o=0$ cannot satisfye the equation $(k^2-m^2)\hat{f}=0$.

Any Hilbert splitting is unique apart from a polynomial and we wish to make such a Hilbert splitting that this polynomial does not occur.

Since the splitting of $\delta(m^2+P)$ in its positive and negative frequency parts is unique, we have obtained the results

$$\delta_+(m^2+P)=F^{-1}\left[\lim_{y_o\to+0} X_1\right] \text{ and } \delta_-(m^2+P)=F^{-1}\left[\lim_{y_o\to-0} X_2\right] \quad \text{or}$$

$$F\left[\delta_+(m^2+P)\right]=\lim_{y_o\to+0} X_1 \text{ and } F\left[\delta_-(m^2+P)\right]=\lim_{y_o\to-0} X_2 \quad .$$

Hence to obtain the Fourier transforms of $\delta_+(m^2+P)$ and $\delta_-(m^2+P)$ we have to make a Hilbert splitting of the distribution $X(x_1,x_2,x_3,x_o)$ into two holomorphic

functions with the above mentioned properties.
In chapter 5 we have shown that the distribution $X(x_1,x_2,x_3,x_o)$ can be written as:

$$X(r,x_o)=F[\delta(m^2+P)] = 4\pi m \frac{K_1(m\, Q^{\frac{1}{2}})}{Q^{\frac{1}{2}}} \quad \text{for } r>|x_o| \tag{5.19}$$

and

$$X(r,x_o)=F[\delta(m^2+P)]=i\pi^2 m \frac{H_1^{(2)}\{m(-Q)^{\frac{1}{2}}\}-H_1^{(1)}\{m(-Q)^{\frac{1}{2}}\}}{(-Q)^{\frac{1}{2}}} \quad \text{for } |x_o|>r \tag{5.20}$$

where $Q= r^2-x_o^2= x_1^2+x_2^2+x_3^2-x_o^2$ (5.12)

We consider now the complex z_o-plane with $z_o=x_o+iy_o$.
Since $Q^{\frac{1}{2}}$ has a cut along the negative Q-axis of the complexe Q-plane, we have to introduce two cuts along the real axis of the z_o plane, viz: $x_o > r$ and $x_o < -r$.
We denote the function Q with z_o instead of x_o by

$$Q(r,z_o) = r^2-x_o^2+y_o^2-2\, i\, x_o y_o \tag{6.1}$$

Since the asymptotic behaviour of the modified Bessel function $K_1(z)$ is given by

$$K_1(z) \sim (\tfrac{1}{2}\pi/z)^{+\frac{1}{2}}\, e^{-z} \tag{6.2}$$

it is clear that

$X(r,z_o)=4\pi m \dfrac{K_1[m\{Q(r,z_o)\}^{\frac{1}{2}}]}{\{Q(r,z_o)\}^{\frac{1}{2}}}$ is holomorphic in the whole strip $|x_o|< r$ of the z_o plane.
Hence for $|x_o|< r$ we have:

$$X_1(r,z_o)=X_2(r,z_o)=2\pi m \frac{K_1[m\{Q(r,z_o)\}^{\frac{1}{2}}]}{\{Q(r,z_o)\}^{\frac{1}{2}}} \tag{6.3}$$

We continue now the function $X_1(r,z_o)$ analytically to the regions $x_o > r, y_o > 0$ and $x_o < -r, y_o > 0$, while the function $X_2(r,z_o)$ is continued analytically to the regions $x_o > r, y_o < 0$ and $x_o < -r, y_o < 0$.
Using the relations:

$$K_1(z\, e^{i\frac{\pi}{2}}) = -\tfrac{1}{2}\pi H_1^{(2)}(z)$$
$$K_1(z\, e^{-i\frac{\pi}{2}}) = -\tfrac{1}{2}\pi H_1^{(1)}(z) \qquad (6.4)$$

and remembering the asymptotic formulae:

$$H_1^{(1)}(z) \sim (\tfrac{1}{2}\pi z)^{-\frac{1}{2}}\, e^{-i\frac{3}{4}\pi}\, e^{iz} \quad \text{for } -\pi < \arg z < +2\pi$$
$$H_1^{(2)}(z) \sim (\tfrac{1}{2}\pi z)^{-\frac{1}{2}}\, e^{+i\frac{3}{4}\pi}\, e^{-iz} \quad \text{for } -2\pi < \arg z < +\pi \qquad (6.5)$$

we find that the analytical continuations of $X_1(r,z_o)$ are:

$$X_1(r,z_o) = -\pi^2 m \frac{H_1^{(1)}\left[m\left\{e^{+\pi i}Q(r,z_o)\right\}^{\frac{1}{2}}\right]}{Q(r,z_o)^{\frac{1}{2}}} \quad \text{in the region } x_o > r, y_o > 0$$

and (6.6)

$$X_1(r,z_o) = -\pi^2 m \frac{H_1^{(2)}\left[m\left\{e^{-\pi i}Q(r,z_o)\right\}^{\frac{1}{2}}\right]}{Q(r,z_o)^{\frac{1}{2}}} \quad \text{in the region } x_o < -r, y_o > 0$$

while those of $X_2(r,z_o)$ are

$$X_2(r,z_o) = -\pi^2 m \frac{H_1^{(2)}\left[m\left\{e^{-\pi i}Q(r,z_o)\right\}^{\frac{1}{2}}\right]}{Q(r,z_o)^{\frac{1}{2}}} \quad \text{in the region } x_o > r, y_o < 0$$

and (6.7)

$$X_2(r,z_o) = -\pi^2 m \frac{H_1^{(1)}\left[m\left\{e^{+\pi i}Q(r,z_o)\right\}^{\frac{1}{2}}\right]}{Q(r,z_o)^{\frac{1}{2}}} \quad \text{in the region } x_o < -r, y_o < 0.$$

It can be shown that the functions $X_1(r,z_o)$ and $X_2(r,z_o)$ satisfye the conditions of the lemma; in particular the existence of the distributional limits for $y_o \longrightarrow +0$ will become clear, when in the following these limits are actually calculated.

Taking the limit for $y_o \longrightarrow +0$ respectively $y_o \longrightarrow -0$ we obtain with $Q = r^2 - x_o^2 = x_1^2 + x_2^2 + x_3^2 - x_o^2$ the results:

$$\lim_{y_o \to +0} X_1(r,z_o) = 2\pi m \frac{K_1(m\, Q^{\frac{1}{2}})}{Q^{\frac{1}{2}}} \quad \text{for } |x_o| < r \text{, i.e. outside the light cone}$$

$$\lim_{y_o \to +0} X_1(r,z_o) = -i\pi^2 m \frac{H_1^{(1)}\left\{m(-Q)^{\frac{1}{2}}\right\}}{(-Q)^{\frac{1}{2}}} \quad \text{for } x_o > r \text{, i.e. inside the forward l.c.}$$

$$\lim_{y_o \to +0} X_1(r,z_o) = +i\pi^2 m \frac{H_1^{(2)}\left\{m(-Q)^{\frac{1}{2}}\right\}}{(-Q)^{\frac{1}{2}}} \quad \text{for } x_o < -r \text{, i.e. inside the backward l.c.} \qquad (6.8)$$

$$\lim_{y_o\to -0} X_2(r,z_o)=2\pi m \frac{K_1(m\,Q^{\frac{1}{2}})}{Q^{\frac{1}{2}}} \quad \text{for } |x_o|<r, \text{ i.e. outside the l.c.}$$

$$\lim_{y_o\to -0} X_2(r,z_o)=+i\pi^2 m \frac{H_1^{(2)}\{m(-Q)^{\frac{1}{2}}\}}{(-Q)^{\frac{1}{2}}} \quad \text{for } x_o > r, \text{ i.e. inside the forward l.c.}$$

$$\lim_{y_o\to -0} X_2(r,z_o)=-i\pi^2 m \frac{H_1^{(1)}\{m(-Q)^{\frac{1}{2}}\}}{(-Q)^{\frac{1}{2}}} \quad \text{for } x_o < -r, \text{ i.e. inside the backward l.c.} \tag{6.9}$$

The whole situation is illustrated in figure 2.

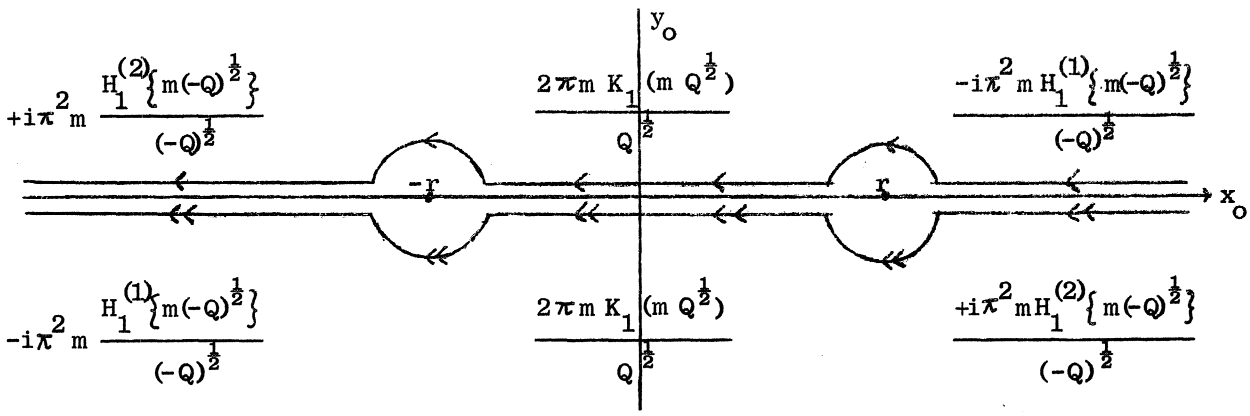

fig.2

It can be verified without difficulty that $\lim_{y_o\to +0} X_1$ and $\lim_{y_o\to -0} X_2$ satisfye the homogeneous Klein-Gordon equation and hence we can conclude:

$$F[\delta_+(m^2+P)] = \lim_{y_o\to +0} X_1(r,z_o)$$

and (6.10)

$$F[\delta_-(m^2+P)] = \lim_{y_o\to -0} X_2(\vec{r},z_o)$$

The behaviour of the distribution $F[\delta_{\pm}(m^2+P)]$ in the neighbourhood of the lightcone can be obtained by expanding the right hand side of (6.3) for small values of $|Q(r,z_o)|$.

It follows from formula (6.3) that we have for small values of $|Q(r,z_o)|$

$$X_1(r,z_o)=X_2(r,z_o)=2\pi \frac{1}{Q(r,z_o)} + O\{\log|Q(r,z_o)|\} = \frac{2\pi}{r^2-(x_o+iy_o)^2}+O(\log|Q|).$$

Hence:

$$F[\delta_+(m^2+P)] = \lim_{y_o \to +0} \frac{2\pi}{r^2-(x_o+iy_o)^2} + O(\log|Q|) \qquad (6,11)$$

and

$$F[\delta_-(m^2+P)] = \lim_{y_o \to -0} \frac{2\pi}{r^2-(x_o+iy_o)^2} + O(\log|Q|) \qquad (6.12)$$

The reduction of the right hand sides of (6.11) and (6.12) can be carried out as follows:

$$\lim_{y_o \to +0} \frac{2\pi}{r^2-(x_o+iy_o)^2} = 2\pi \lim_{y_o \to +0} \frac{\partial}{\partial x_o}\left[\frac{1}{2r}\log\frac{r+(x_o+iy_o)}{r-(x_o+iy_o)}\right] =$$

(6.13)

$$2\pi \frac{\partial}{\partial x_o} \lim_{y_o \to +0} \frac{1}{2r}\left[\log\frac{r+(x_o+iy_o)}{r-(x_o+iy_o)}\right] = 2\pi \frac{\partial}{\partial x_o}\left[\frac{1}{2r}\log\left|\frac{r+x_o}{r-x_o}\right| + \frac{i\pi}{2r}\left\{1-\Theta(r^2-x_o^2)\right\}\right]$$

By aid of partial integration it can be shown, that

$$\frac{\partial}{\partial x_o}\frac{1}{2r}\log\left|\frac{r+x_o}{r-x_o}\right| = \frac{1}{r^2-x_o^2} \qquad (6.14)$$

where $\left\langle \frac{1}{r^2-x_o^2}, \varphi(x)\right\rangle$ has to be conceived as a principal value of Cauchy.

By aid of the chain-rule one has the formula:

$$\frac{\partial}{\partial x_o}\Theta(r^2-x_o^2) = -2x_o\,\delta(r^2-x_o^2) \qquad (6.15)$$

Substitution of (6.14) and (6.15) into (6.13) yields:

$$\lim_{y_o \to +0} \frac{2\pi}{r^2-(x_o+iy_o)^2} = \frac{2\pi}{r^2-x_o^2} + 2i\pi^2\,\delta_+(r^2-x_o^2) - 2i\pi^2\,\delta_-(r^2-x_o^2) \qquad (6.16)$$

and we obtain in the same way

$$\lim_{y_o \to -0} \frac{2\pi}{r^2-(x_o+iy_o)^2} = \frac{2\pi}{r^2-x_o^2} - 2i\pi^2\,\delta_+(r^2-x_o^2) + 2i\pi^2\,\delta_-(r^2-x_o^2) \qquad (6.17)$$

Hence the behaviour of the distributions $F[\delta_\pm(m^2+P)]$ can finally be expressed as:

$$F\left[\delta_{\pm}(m^2+P)\right] = \frac{2\pi}{r^2-x_o^2} \pm 2i\pi^2\, \delta_+(r^2-x_o^2) \mp 2i\pi^2\, \delta_-(r^2-x_o^2) + O(\log|r^2-x_o^2|) \tag{6.18}$$

Collecting the results (6.8), (6.9) and (6.18) we may write:

$$F\left[\delta_+(m^2+P)\right] = 2i\pi^2\, \delta_+(Q)-2i\pi^2\, \delta_-(Q) + \Theta(Q)2\pi m\, \frac{K_1(m\sqrt{Q})}{\sqrt{Q}} -$$

$$-\Theta(-Q)i\pi^2 m\left\{\Theta(x_o)\, \frac{H_1^{(1)}(m\sqrt{-Q})}{\sqrt{-Q}} - \Theta(-x_o)\, \frac{H_1^{(2)}(m\sqrt{-Q})}{\sqrt{-Q}}\right\} \tag{6.19}$$

and

$$F\left[\delta_-(m^2+P)\right] =-2i\pi^2\, \delta_+(Q)+2i\pi^2\, \delta_-(Q) + \Theta(Q)2\pi m\, \frac{K_1(m\sqrt{Q})}{\sqrt{Q}} +$$

$$+\Theta(-Q)i\pi^2 m\left\{\Theta(x_o)\, \frac{H_1^{(2)}(m\sqrt{-Q})}{\sqrt{-Q}} - \Theta(-x_o)\, \frac{H_1^{(1)}(m\sqrt{-Q})}{\sqrt{-Q}}\right\} \tag{6.20}$$

7. The solutions of the Klein-Gordon equation

7.1. The solutions of the homogeneous equation

It has been shown in section 3 that the general Lorentz invariant solution of the homogeneous Klein-Gordon equation can be written as:

$$f(x_1,x_2,x_3,x_o)=c_+\, F^{*-1}\left[\delta_+(k^2-m^2)\right] +c_-\, F^{*-1}\left[\delta_-(k^2-m^2)\right] \tag{7.1}$$

Putting

$$\Delta^+(x)= -2\pi i\, F^{*-1}\left[\delta_+(k^2-m^2)\right] \tag{7.2}$$

and

$$\Delta^-(x)= +2\pi i\, F^{*-1}\left[\delta_-(k^2-m^2)\right] \tag{7.3}$$

we can also write

$$f(x_1,x_2,x_3,x_o) = c_1\, \Delta^+(x) + c_2\, \Delta^-(x) \tag{7.4}$$

According to equation (3.9) the distribution $\Delta^{\pm}(x)$ satisfies the relation

$$\Delta^{\pm}(x_1,x_2,x_3,-x_o) = \mp\, \frac{i}{(2\pi)^3}\, F\left[\delta_{\pm}(k^2-m^2)\right] \tag{7.5}$$

It follows from section (6.2) that the introduced holomorphic functions $x_1(r,z_o)$ and $x_2(r,z_o)$ are correlated to each other by

$$X_1(r,+z_o) = X_2(r,-z_o)$$

and hence

$$X_1(r,x_o+i0) = X_2(r,-x_o-i0)$$

From this and equation (7.5) it is deduced that:

$$\Delta^+(x_1,x_2,x_3,+x_o) = -\frac{i}{(2\pi)^3} X_1(r,-x_o+i0) = -\frac{i}{(2\pi)^3} X_2(r,+x_o-i0) =$$

$$= -\frac{i}{(2\pi)^3} F\left[\delta_-(k^2-m^2)\right]$$

and in the same way:

$$\Delta^-(x_1,x_2,x_3,+x_o) = +\frac{i}{(2\pi)^3} F\left[\delta_+(k^2-m^2)\right]$$

and so we have:

$$\Delta^\pm(x) = \mp\frac{i}{(2\pi)^3} F\left[\delta_\mp(k^2-m^2)\right] .$$

Applying the results (6.19) and (6.20) we obtain the formulae:

$$\Delta^+(x) = -\frac{1}{4\pi}\delta_+(R^2) + \frac{1}{4\pi}\delta_-(R^2) + + \Theta(R^2)\frac{m}{8\pi}\left\{\Theta(x_o)\frac{H_1^{(2)}(m\,R)}{R} - \Theta(-x_o)\frac{H_1^{(1)}(m\,R)}{R}\right\} - - \Theta(-R^2)\frac{im}{4\pi^2}\frac{K_1(m\sqrt{-R^2})}{\sqrt{-R^2}} \quad (7.6)$$

and

$$\Delta^-(x) = -\frac{1}{4\pi}\delta_+(R^2) + \frac{1}{4\pi}\delta_-(R^2) + + \Theta(R^2)\frac{m}{8\pi}\left\{\Theta(x_o)\frac{H_1^{(1)}(m\,R)}{R} - \Theta(-x_o)\frac{H_1^{(2)}(m\,R)}{R}\right\} + + \Theta(-R^2)\frac{im}{4\pi^2}\frac{K_1(m\sqrt{-R^2})}{\sqrt{-R^2}} \quad (7.7)$$

with

$$R^2=x_o^2-x_1^2-x_2^2-x_3^2=x_o^2-r^2 \quad \text{and } R= +\sqrt{x_o^2-r^2} \tag{7.8}$$

Other important solutions of the homogeneous differential equation are the solutions $\Delta(x)$ and $\Delta^{(1)}(x)$, which are respectively uneven and even. They are defined as:

$$\Delta(x)= \Delta^+(x)+ \Delta^-(x) \quad \text{and} \quad \Delta^{(1)}(x)=i\left\{\Delta^+(x)- \Delta^-(x)\right\} \tag{7.9}$$

By aid of the well known relations

$$H_1^{(1)}(z)+H_1^{(2)}(z) = 2\ J_1(z)$$

and (7.10)

$$H_1^{(1)}(z)-H_1^{(2)}(z) = 2\ i\ Y_1(z)$$

it follows immediately from the formulae (7.6) and (7.7), that

$$\Delta(x)=-\frac{1}{2\pi}\,\delta_+(R^2)+\frac{1}{2\pi}\,\delta_-(R^2)+\Theta(R^2)\,\frac{m}{4\pi}\left\{\Theta(x_o)\,\frac{J_1(m\ R)}{R}-\Theta(-x_o)\,\frac{J_1(m\ R)}{R}\right\} \tag{7.11}$$

and

$$\Delta^{(1)}(x)=\Theta(R^2)\,\frac{m}{4\pi}\,\frac{Y_1(m\ R)}{R}+\Theta(-R^2)\,\frac{m}{2\pi^2}\,\frac{K_1(m\sqrt{-R^2})}{\sqrt{-R^2}} \tag{7.12}$$

7.2 <u>The propagator function $\Delta(\vec{x},x_o)$</u>

The function $\Delta(x)$, which is called the Pauli-Jordan function, has the important property that it according to (7.11) vanishes outside the lightcone.

We consider now $\Delta(x)$ as a distribution in the space variables x_1,x_2,x_3, while the time x_o is taken as a parameter. We take only positive values for x_o.

This distribution is denoted by $\Delta(\vec{x},x_o)$.

$$\Delta(\vec{x},x_o)= -\frac{1}{2\pi}\,\delta(x_o^2-r^2)+\Theta(x_o^2-r^2)\,\frac{m}{4\pi}\,\frac{J_1(m\sqrt{x_o^2-r^2})}{\sqrt{x_o^2-r^2}} \tag{7.13}$$

Since by virtue of (2.19) and (2.20) the differentiation of the right hand side of (7.13) to x_o is formally the same whether x_o is variable or a parameter, the three dimensional distribution $\Delta(\vec{x},x_o)$ with parameter x_o satisfies also the Klein-Gordon equation which we write in the form:

$$\frac{\partial^2}{\partial x_o^2} \Delta(\vec{x},x_o)=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}-m^2\right)\Delta(\vec{x},x_o) \qquad (7.14)$$

By aid of (2.8) it is clear, that $\lim\limits_{x_o \to +0} \Delta(\vec{x},x_o)=0$ and so we write

$$\Delta(\vec{x},0)=0 \qquad (7.15)$$

Since the order of the processes of differentiation and taking a limit is not essential for distributions we have also

$$\lim_{x_o \to +0} \frac{\partial^2}{\partial x_o^2} \Delta(\vec{x},x_o)=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}-m^2\right)\lim_{x_o \to +0} \Delta(\vec{x},x_o)=0$$

and hence

$$\ddot{\Delta}(\vec{x},0)=0 \qquad (7.16)$$

To investigate $\lim\limits_{x_o \to +0} \frac{\partial}{\partial x_o} \Delta(\vec{x},x_o)$ we write according to the chain-rule (2.20):

$$\lim_{x_o \to +0} \frac{\partial}{\partial x_o} \Delta(\vec{x},x_o)=\frac{-1}{\pi} \lim_{x_o \to +0}\left[x_o\, \delta^{(1)}(x_o^2-r^2)\right]+$$

$$\lim_{x_o \to +0}\left[2x_o\, \delta(x_o^2-r^2)\, \frac{m}{4\pi}\, \frac{J_1(m\sqrt{x_o^2-r^2})}{\sqrt{x_o^2-r^2}}\right]+\lim_{x_o \to +0}\left[\theta(x_o^2-r^2)\frac{m}{4\pi}\, \frac{\partial}{\partial x_o}\, \frac{J_1(m\sqrt{x_o^2-r^2})}{\sqrt{x_o^2-r^2}}\right] \qquad (7.17)$$

Expanding $\frac{J_1(m\sqrt{x_o^2-r^2})}{\sqrt{x_o^2-r^2}}$ into a power series of $(x_o^2-r^2)$ and using the relation $P\,\delta(P)=0$ and the formula (2.8), it is evident that the second and third term of the right hand side of (7.17) vanish; hence

$$\lim_{x_o \to +0} \frac{\partial}{\partial x_o} \Delta(\vec{x},x_o)= -\frac{1}{\pi} \lim_{x_o \to +0}\left[x_o\, \delta^{(1)}(x_o^2-r^2)\right] \qquad (7.18)$$

By aid of (2.18) we may write:

$$\left\langle x_o\, \delta^{(1)}(x_o - r^2), \varphi \right\rangle = \frac{x_o}{2} \int_\Omega \left[\frac{1}{2r} \frac{\partial}{\partial r} \left\{ \varphi(r\omega_1, r\omega_2, r\omega_3) r \right\} \right]_{r=x_o} d\Omega =$$

$$\frac{x_o}{4} \int_\Omega \left(\frac{\partial \varphi}{\partial x_1} \omega_1 + \frac{\partial \varphi}{\partial x_2} \omega_2 + \frac{\partial \varphi}{\partial x_3} \omega_3 \right)_{r=x_o} d\Omega + \frac{1}{4} \int_\Omega \varphi(x_o\omega_1, x_o\omega_2, x_o\omega_3)\, d\Omega$$

and therefore $\lim\limits_{x_o \to +0} \left\langle x_o\, \delta^{(1)}(x_o^2 - r^2), \varphi \right\rangle = \pi\, \varphi(0,0,0)$ and consequently

$$-\frac{1}{\pi} \lim_{x_o \to +0} \left[x_o\, \delta^{(1)}(x_0^2 - r^2) \right] = -\, \delta(x_1, x_2, x_3) \qquad (7.19)$$

Hence $\lim\limits_{x_o \to +0} \frac{\partial}{\partial x_o} \Delta(\vec{x}, x_o) = -\, \delta(x_1, x_2, x_3)$

or

$$\dot{\Delta}(\vec{x}, 0) = -\, \delta(\vec{x}) \qquad (7.20)$$

By virtue of the properties (7.15), (7.16) and (7.20) the solution $\varphi_o(\vec{x}, x_o)$ of the homogeneous Klein-Gordon equation with the initial conditions

$$\begin{aligned} \varphi_o(\vec{x}, 0) &= f(\vec{x}) \\ \dot{\varphi}_o(\vec{x}, 0) &= g(\vec{x}) \end{aligned} \qquad (7.21)$$

can be written as:

$$\varphi_o(\vec{x}, x_o) = -\left[\Delta(\vec{x}, x_o) * g(\vec{x}) + \dot{\Delta}(\vec{x}, x_o) * f(\vec{x}) \right] \qquad (7.22)$$

where the convolution-products have to be taken with respect to three variables x_1, x_2 and x_3. Due to the representation of the function $\varphi_o(\vec{x}, x_o)$ in the form (7.22), the distribution $\Delta(\vec{x}, x_o)$ is called a propagator-function.

7.3 The solutions of the inhomogeneous equation

We have shown in section 3 that the general Lorentz-invariant solution of the inhomogeneous Klein-Gordon equation (1.3) can be written as:

$$g(x_1, x_2, x_3, x_o) = F^{*-1}\left[\frac{1}{m^2 - k^2} \right] + c_+ \, F^{*-1}\left[\delta_+(k^2 - m^2) \right] + c_- \, F^{*-1}\left[\delta_-(k^2 - m^2) \right] \qquad (7.23)$$

Special solutions are obtained by taking the following values of c_+ and c_-.

1^e $c_+ = c_- = 0$

The solution g(x) is now denoted by $\bar{\Delta}(x)$ and we obtain according to formula (3.9)

$$\bar{\Delta}(x_1,x_2,x_3,-x_o) = \frac{1}{(2\pi)^4} F\left[\frac{1}{m^2-k^2}\right]$$

Applying the result (5.26) we obtain:

$$\bar{\Delta}(x_1,x_2,x_3,-x_o) = -\frac{m}{8\pi} \Theta(R^2) \frac{J_1(m R)}{R} + \frac{\delta(R^2)}{4\pi}$$

and since this distribution is even in x_o, we have the result

$$\bar{\Delta}(x_1,x_2,x_3,x_o) = -\frac{m}{8\pi} \Theta(R^2) \frac{J_1(m R)}{R} + \frac{\delta(R^2)}{4\pi} \qquad (7.24)$$

This distribution has its support within and on the forward and backward lightcone and vanishes outside the cone.

2^e $c_+ = c_- = \pi i$

We denote g(x) by $\Delta_C(x)$ and we obtain:

$$\Delta_C(x) = F^{*-1}\left[\frac{1}{m^2-k^2} + \pi i\, \delta(m^2-k^2)\right] = F^{*-1}\left[\frac{1}{m^2-k^2-i0}\right] \qquad (7.25)$$

By aid of section 7.1, formulae (7.2), (7.3) and (7.9), we may write:

$$\Delta_C(x) = \bar{\Delta}(x) - \tfrac{1}{2}(\Delta^+(x) - \Delta^-(x)) = \bar{\Delta}(x) + \tfrac{1}{2}\, i\, \Delta^{(1)}(x)$$

Using the formulae (7.12) and (7.24) the result becomes:

$$\Delta_C(x) = \frac{\delta(R^2)}{4\pi} - \frac{m}{8\pi} \Theta(R^2) \frac{H_1^{(2)}(m R)}{R} + \frac{m}{4\pi^2}\, i\, \Theta(-R^2) \frac{K_1(m\sqrt{-R^2})}{\sqrt{-R^2}} \qquad (7.26)$$

3^e $c_+ = c_- = -i\pi$

g(x) is denoted by $\Delta_{AC}(x)$ and we obtain:

$$\Delta_{AC}(x) = F^{*-1}\left[\frac{1}{m^2-k^2} - \pi i\, \delta(m^2-k^2)\right] = F^{*-1}\left[\frac{1}{m^2-k^2+i0}\right] \qquad (7.27)$$

In the same way as in the foregoing case we write

$$\Delta_{AC}(x) = \bar{\Delta}(x) + \tfrac{1}{2}(\Delta^{+}(x) - \Delta^{-}(x)) = \bar{\Delta}(x) - \tfrac{1}{2}\, i\, \Delta^{(1)}(x)$$

Using again (7.12) and (7.24), we get the result:

$$\Delta_{AC}(x) = \frac{\delta(R^2)}{4\pi} - \frac{m}{8\pi}\,\theta(R^2)\,\frac{H_1^{(1)}(m\,R)}{R} - \frac{m}{4\pi^2}\, i\,\theta(-R^2)\,\frac{K_1(m\sqrt{-R^2})}{\sqrt{-R^2}} \tag{7.28}$$

4e $c_+ = -c_- = +i\pi$

g(x) is denoted by $\Delta_R(x)$ and we obtain:

$$\Delta_R(x) = F^{*-1}\left[\frac{1}{m^2-k^2} + i\pi\,\delta_+(m^2-k^2) - i\pi\,\delta_-(m^2-k^2)\right] \tag{7.29}$$

or by aid of (7.2), (7.3) and (7.9)

$$\Delta_R(x) = \bar{\Delta}(x) - \tfrac{1}{2}(\Delta^{+}(x) + \Delta^{-}(x)) = \bar{\Delta}(x) - \tfrac{1}{2}\,\Delta(x)$$

Using now (7.11) and (7.24), the result becomes:

$$\begin{aligned} \Delta_R(x) &= \frac{1}{2\pi}\,\delta_+(R^2) - \frac{m}{4\pi}\,\theta(R^2)\,\frac{J_1(m\,R)}{R} && \text{for } x_o > 0 \\ \Delta_R(x) &= 0 && \text{for } x_o < 0 \end{aligned} \tag{7.30}$$

5e $c_+ = -c_- = -i\pi$

g(x) is denoted by $\Delta_A(x)$ and we obtain:

$$\Delta_A(x) = F^{*-1}\left[\frac{1}{m^2-k^2} - i\pi\,\delta_+(m^2-k^2) + i\pi\,\delta_-(m^2-k^2)\right] \tag{7.31}$$

or

$$\Delta_A(x) = \bar{\Delta}(x) + \tfrac{1}{2}\,\Delta(x).$$

Using again (7.11) and (7.24), we get the result:

$$\begin{aligned} \Delta_A(x) &= 0 && \text{for } x_o > 0 \\ \Delta_A(x) &= \frac{1}{2\pi}\,\delta_-(R^2) - \frac{m}{4\pi}\,\theta(R^2)\,\frac{J_1(m\,R)}{R} && \text{for } x_o < 0 \end{aligned} \tag{7.32}$$

From these results it follows immediately that $\bar{\Delta}(x)$ has its support within and on the lightcone, $\Delta_R(x)$ within and on the forward lightcone and $\Delta_A(x)$ within and on the backward lightcone, whereas the supports of $\Delta_C(x)$ and $\Delta_{AC}(x)$ have no boundaries at all.

The following relations are verified easily:

$$\Delta_C(x) = \begin{cases} -\Delta^+(x) & \text{for } x_o > 0 \\ \Delta^-(x) & \text{for } x_o < 0 \end{cases} \qquad \Delta_{AC}(x) = \begin{cases} -\Delta^-(x) & \text{for } x_o > 0 \\ \Delta^+(x) & \text{for } x_o < 0 \end{cases} \tag{7.33}$$

$$\Delta_R(x) = \begin{cases} -\Delta(x) & \text{for } x_o > 0 \\ 0 & \text{for } x_o < 0 \end{cases} \qquad \Delta_A(x) = \begin{cases} 0 & \text{for } x_o > 0 \\ \Delta(x) & \text{for } x_o < 0 \end{cases} \tag{7.34}$$

$$2\bar{\Delta}(x) = \Delta_R(x) + \Delta_A(x) = \Delta_C(x) + \Delta_{AC}(x) \tag{7.35}$$

$$i\,\Delta^{(1)}(x) = \Delta^-(x) - \Delta^+(x) = \Delta_C(x) - \Delta_{AC}(x) \tag{7.36}$$

From the formulae (7.25), (7.27), (7.29) and (7.31) follow useful and interesting relations for the positive and negative frequency parts of the distributions $\Delta_C(x)$, $\Delta_{AC}(x)$, $\Delta_R(x)$ and $\Delta_A(x)$. Denoting them by $\Delta_C^+(x)$, $\Delta_C^-(x)$ etc, we have:

$$\Delta_C^+(x) = \Delta_R^+(x) \quad , \quad \Delta_C^-(x) = \Delta_A^-(x)$$
$$\Delta_{AC}^+(x) = \Delta_A^+(x) \quad , \quad \Delta_{AC}^-(x) = \Delta_R^-(x) \tag{7.37}$$

The distributions $\Delta_R(x)$ and $\Delta_A(x)$ are called the retarded and advanced Green's functions; the distributions $\Delta_C(x)$ and $\Delta_{AC}(x)$ the causal and the anti-causal Green's functions.

In conclusion we consider once more the inhomogeneous Klein-Gordon equation

$$(\Box - m^2)\,\varphi(x) = -j(x) \tag{7.39}$$

The Lorentz-invariant solution of this equation can be represented in many ways, e.g.:

$$\varphi(x) = \varphi_\alpha(x) + \Delta_\alpha(x) * j(x) \tag{7.40}$$

where we may write instead of α : R,A,C,AC; the convolution-product is taken with respect to all variables x_1, x_2, x_3 and x_o.

Taking $\alpha = R$ the function $\varphi_R(x)$ is the potential of the incoming field, whereas for $\alpha = A$ the function $\varphi_A(x)$ is the potential of the outgoing field.

Taking $\alpha = C$ it can be shown by aid of (7.37) that $\varphi_C(x)$ is the sum of the positive frequency part of the potential of the incoming field and the negative frequency part of the potential of the outgoing field.

For $\alpha = AC$ the potential $\varphi_{AC}(x)$ is the sum of the negative frequency part of the potential of the incoming field and the positive frequency part of the potential of the outgoing field.

For these and many other interesting physical considerations the reader is referred to lit.3 and 15.

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