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Diffraction of Line-source Radiation by a Metallic Sheet

by

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Summary

The two-dimensional electromagnetic diffraction by a metallic half-plane sheet is investigated. Impedance boundary conditions are used at the sheet. By a method due to LAUWERIER the problem is reduced to a set of Hilbert problems which are solved. Special attention is given to a Hertzian dipole source.

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1. Introduction

In the theory of diffraction exact solutions have been found only for a small number of boundary condition problems, in which the shape of the boundary was especially simple. Amongst these the two-dimensional half-plane diffraction problem, perhaps, is the best-known. The case of incident plane waves and either Dirichlet or Neumann conditions was solved in a famous paper by A. SOMMERFELD [8].
Subsequently the case of a point source, also with either Dirichlet or Neumann boundary conditions, was solved by H.M. MAC DONALD [5].
Since then these solutions have been rederived with the help of other means by many authors.
For boundary conditions of mixed type, i.e. those prescribing a linear combination of the field and its normal derivative at the boundary, T.B.A. SENIOR gave a solution for the case of incident plane waves. In the present paper a solution will be given for a point source. Such a solution, indeed, is implicitly contained in a series of papers on the solution of the equation of Helmholtz in an angle by H.A. LAUWERIER [2], namely in the fourth. LAUWERIER considers boundary conditions which prescribe a linear combination of the field, its normal derivative and its tangential derivatice at the sides of the angle. For an angle 2π, and equating the coefficients of the tangential derivative to zero, the solution of the present problem could be obtained. However these substitutions turn out to be by no means trivial because of the appearance of confluences of poles in the general formula. We therefore prefer to give an independent solution, using the methods which LAUWERIER applied to the more general case.

2. Formulation of the problem

In the Application of electro-magnetic theory to a configuration consisting of several media it will in general be necessary to set up field equations for every medium and then to couple these fields by appropriate conditions at the interfaces of the media. Here we consider a configuration of two media, namely a non-conducting medium and a medium with a large conductivity. In such a case the field in the metal can be taken account of in good approximation by imposing an impedance boundary condition on the field in the non-conducting medium at the interface, and the set of field equations in the conducting medium can be dispensed with. This boundary condition reads (cf. e.g. [7])

$$\vec{n} \times (\vec{n} \times \vec{E} - Z_m \vec{H}) = 0, \qquad (2.1)$$

where $\vec{n}$ is a unit vector, normal to the interface, which is directed outwards as regards the non-conducting medium. As usual $\vec{E}$ is the electric field and $\vec{H}$ the magnetic field. The constant Z_m equals

$$Z_m = \sqrt{\mu_m / (\varepsilon_m + i\,\sigma_m/\omega)}, \qquad (2.2)$$

where a time variation $\exp(-i\omega t)$ of all fields is understood. Here μ_m is the permeability of the conducting medium, ε_m its permittivity and σ_m its conductivity. For a derivation of the above formulae we refer to [7]. In the present context we prefer a slightly different form, which is obtained by applying a Laplace transformation with respect to time. Then (2.1) again holds if $\vec{E}$ and $\vec{H}$ are interpreted as the Laplace transforms of the electric and magnetic fields respectively, but instead of (2.2) we have

$$Z_m = \sqrt{\mu_m / (\varepsilon_m + \sigma_m/p)} \quad , \qquad (2.3)$$

where p is the Laplace variable. Formally (2.2) can be obtained

from (2.3) by replacing p by $-i\omega$.
The Laplace transforms $\vec{E}$ and $\vec{H}$ satisfy the transformed Maxwell equations

$$\nabla\times\vec{H} = (p\varepsilon/c)\vec{E}, \qquad \nabla.\vec{H} = 0,$$
$$\nabla\times\vec{E} = -(p\mu/c)\vec{H}, \qquad \nabla.\vec{E} = 0,$$

in a region without free electric charges. Here ε is the permittivity of the non-conducting medium, μ its permeability and c the velocity of light in vacuum. Introduction of dimensionless variables $(x',y',z')=(p\sqrt{\varepsilon\mu}/c)\,(x,y,z)$, $\nabla=(p\sqrt{\varepsilon\mu}/c)\,\nabla'$ gives the alternative form

(2.4) $\nabla'\times\vec{H} = (1/Z)\vec{E}$, (2.5) $\nabla'.\vec{H} = 0$,

(2.6) $\nabla'\times\vec{E} = -Z\,\vec{H}$, (2.7) $\nabla'.\vec{E} = 0$,

where $Z=\sqrt{\mu/\varepsilon}$. In the sequel we will drop the primes in the above equations.
Substitution of (2.6) in (2.1) gives the alternative form of the boundary condition

(2.8) $$\vec{n}\times\left[\vec{n}\times\vec{E} + (Z_m/Z)\nabla\times\vec{E}\right] = 0.$$

Furthermore it follows from (2.4), (2.6) and (2.7) that

(2.9) $$\nabla^2\vec{E} - \vec{E} = 0.$$

By taking the vector product with $\vec{n}$ we derive from (2.1)

(2.10) $$\vec{n}\times\left[\vec{n}\times\vec{H} + (1/Z_m)\vec{E}\right] = 0,$$

or, substituting (2.4)

(2.11) $$\vec{n}\times\left[\vec{n}\times\vec{H} + (Z/Z_m)\nabla\times\vec{H}\right] = 0.$$

From (2.4), (2.5) and (2.6) it follows that

(2.12) $$\nabla^2\vec{H} - \vec{H} = 0.$$

It is seen that both $\vec{E}$ and $\vec{H}$ satisfy a modified Helmholtz

equation and have the same type of boundary condition.
The above formulae remain valid if, instead of a non-conducting medium, we have a slightly conducting medium with conductivity σ provided we replace ε by $\varepsilon + \sigma/p$.
Still other forms of the boundary conditions are found by taking the divergence of (2.8) and (2.10). This is allowed because these equations essentially contain only vectors in a plane tangential to the interface. We find

$$\left[\vec{n}.\nabla + (Z_m/Z)\right]\vec{n}.\vec{E} = 0, \tag{2.13}$$

$$\left[\vec{n}.\nabla + (Z/Z_m)\right]\vec{n}.\vec{H} = 0. \tag{2.14}$$

These general formulae will be applied to the case that the conducting medium is a metallic sheet, which covers the half-plane $y=0$, $x < 0$. Then $\vec{n}$ has the direction of the negative Y-axis for $y=+0$ and that of the positive Y-axis for $y=-0$. We restrict the discussion to essentially two-dimensional fields, i.e. we only consider fields which are independent of z. The boundary conditions (2.8), (2.10), (2.13) and (2.14) then reduce to

$$E_x = \pm \frac{Z_m}{Z}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right), \qquad H_x = \pm \frac{Z}{Z_m}\left(\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x}\right),$$

$$E_y = \pm \frac{Z}{Z_m}\frac{\partial E_y}{\partial y}, \qquad H_y = \pm \frac{Z_m}{Z}\frac{\partial H_y}{\partial y},$$

$$E_z = \pm \frac{Z_m}{Z}\frac{\partial E_z}{\partial y}, \qquad H_z = \pm \frac{Z}{Z_m}\frac{\partial H_z}{\partial y},$$

for $y = \pm 0$, $x < 0$.

It is seen, that E_y, E_z, H_y, H_z satisfy boundary conditions of the type

$$\left(\frac{\partial}{\partial y} \mp \operatorname{ch}\alpha\right)\varphi = 0.$$

For real values of p, α is either real or purely imaginary. Moreover, in the applications considered here $0 < \operatorname{Im}\alpha < \frac{1}{2}\pi$.

They also satisfy the modified Helmholtz equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - \varphi = 0.$$

The more intricate character of the boundary conditions for E_x and H_x is of no importance since E_x can be calculated from H_y and H_z and H_x from E_y and E_z.
It might be expected that the approximate boundary condition (2.1) will fail at the sharp edge in the origin.
Actually, of course, the metallic sheet will have a small but finite thickness. If such a sheet has a rounded edge, it can be shown that no difficulties will arise. For a detailed discussion of this question we refer to [6].

The diffraction field, or to use a modern term, the scattered field, of course depends on the incident field. The diffraction field can be evaluated for an arbitrary incident field if the appropriate Green's function is known. It hence suffices to determine the Green's function which satisfies the inhomogeneous modified Helmholtz equation

(2.15) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1\right)G = -2\pi\,\delta(x-x_0)\delta(y-y_0),$$

and the boundary conditions

(2.16) $$\left(\frac{\partial}{\partial y} \mp \operatorname{ch}\alpha\right)G = 0 \quad \text{for } x < 0, \quad y = \pm 0 .$$

However, the special case that the incident field is generated by a Hertzian dipole merits a special consideration.
In such a case the magnetic field is the curl of a vector potential $\vec{\pi}$. In ordinary variables we have

$$\vec{H} = \frac{\varepsilon}{c}\,\frac{\partial}{\partial t}\,\nabla\times\vec{\pi}\ , \quad \vec{E} = \nabla\times(\nabla\times\vec{\pi}), \quad \nabla^2\vec{\pi} - \frac{1}{c^2}\frac{\partial^2\vec{\pi}}{\partial t^2} = 0,$$

or taking the Laplace transforms and using non-dimensional variables,

$$\vec{H} = (\eta^2/Z)\nabla\times\vec{\pi}\ , \quad \vec{E} = \eta^2\nabla\times(\nabla\times\vec{\pi}), \quad \nabla^2\vec{\pi} - \vec{\pi} = 0,$$

where $\eta^2 = \varepsilon\mu p^2/c^2$.

For a two-dimensional Hertzian dipole in (x_o,y_o) we have in absence of the metallic sheet

$$\vec{\pi} = Z\ K_o(R)\ \vec{a}\quad ,$$

where $R^2=(x-x_o)^2+(y-y_o)^2$.

This potential satisfies the inhomogeneous equation

$$(2.17)\qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1\right)\vec{\pi} = -2\pi\, z\ \delta(x-x_o)\,\delta(y-y_o)\vec{a}\ .$$

In the presence of the metallic sheet the dipole of course shows the same singularity at (x_o,y_o) and hence the potential satisfies (2.17) in that case too. We consider separately the cases that $\vec{a}$ has the direction of the positive X-, Y- and Z-axis:

1)
$$\vec{\pi}_1 = Z\ \varphi_1(x,y)\ \vec{i}\quad ,$$

then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1\right)\varphi_1 = -2\pi\ \delta(x-x_o)\delta(y-y_o).$$

Moreover

$$\vec{H}_1 = -\eta\ \frac{\partial\varphi_1}{\partial y}\ \vec{k}.$$

Hence we have the boundary condition

$$\left(\frac{\partial^2}{\partial y^2} \mp \frac{Z_m}{Z}\frac{\partial}{\partial y}\right)\varphi_1 = 0 \qquad \text{for } y=\pm 0,\ x<0.$$

2) $$\vec{\pi}_2 = Z\ \varphi_2(x,y)\ \vec{j}\ ,$$

then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1\right) \varphi_2 = -2\pi\,\delta(x-x_o)\,\delta(y-y_o).$$

Moreover

$$\vec{H}_2 = \eta\ \frac{\partial \varphi_2}{\partial x}\ \vec{k}\ .$$

Hence we have the boundary condition

$$\left(\frac{\partial^2}{\partial x \partial y} \mp \frac{Z_m}{Z}\ \frac{\partial}{\partial x}\right) \varphi_2 = 0 \qquad \text{for} \quad y = \pm 0, \quad x < 0,$$

which after integration with respect to x yields

$$\left(\frac{\partial}{\partial y} \mp \frac{Z_m}{Z}\right) \varphi_2 = 0 \qquad \text{for } y = \pm 0, \quad x < 0.$$

3) $$\vec{\pi}_3 = Z\ \varphi_3(x,y)\ \vec{k}\ ,$$

then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1\right) \varphi_3 = -2\pi\,\delta(x-x_o)\,\delta(y-y_o)\ .$$

Moreover

$$\vec{E}_3 = -Z\ \ \varphi_3\ \vec{k}.$$

Hence we have the boundary condition

$$\left(\frac{\partial}{\partial y} \mp \frac{Z}{Z_m}\right) \varphi_3 = 0 \qquad \text{for} \quad y = \pm 0, \quad x < 0.$$

The same type of problem also arises in the theory of sound *). Consider a two-dimensional impulsive source. The transmitted sound is reflected by a semi-soft screen y=0, x < 0. The velocity potential φ then satisfies

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = -2\pi\,\delta(x-x_o)\,\delta(y-y_o)\,\delta(t),$$

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*) Cf. a forthcoming publication by H.A. Lauwerier.

and the boundary conditions are

$$\left(\frac{\partial}{\partial y} \mp \rho_0 \lambda \frac{\partial}{\partial t}\right)\varphi = 0 \qquad \text{for} \quad y = \pm 0, \quad x < 0.$$

The boundary conditions follow from the more general condition for a reflecting medium which reads

$$\left(\frac{\partial}{\partial n} + \rho_0 \lambda \frac{\partial}{\partial t}\right)\varphi = 0,$$

where $\vec{n}$ is normal to the boundary and points into the reflecting medium (cf [1] p.9). Here c is the velocity of sound, ρ_0 the density of the undisturbed medium and λ the rate between the normal velocity of an element of the reflector and the excess pressure on that element. This condition includes the rigid screen ($\lambda = 0$) and the completely soft screen ($\lambda \longrightarrow \infty$).

Application of a Laplace transformation and introduction of dimensionless variables $(x', y', z') = (p/c)\,(x, y, z)$ again yields the equations (2.15) and (2.16) if $\operatorname{ch} \alpha = c\, \rho_0 \lambda$.

3. Reduction to the Hilbert problem

For easy reference we repeat the basic equations

(3.1) $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1\right)G = -2\pi\, \delta(x - x_0)\, \delta(y - y_0),$$

(3.2) $$\left(\frac{\partial}{\partial y} \mp \operatorname{ch} \alpha\right) G = 0 \quad \text{for } y = \pm 0, \quad x < 0.$$

The function G will be composed of the Green's function for the region without screen, which equals $K_0(R)$, $R^2 = (x - x_0)^2 + (y - y_0)^2$, and a regular function φ . These might be called the incident field and the scattered field respectively. We hence put

(3.3) $$G = K_0(R) + \varphi .$$

For φ the following representation will be used

$$(3.4)\qquad \varphi = \begin{cases} \frac{1}{2}\int_{-\infty}^{\infty} \exp(-ix\,\mathrm{sh}w - y\,\mathrm{ch}w) g_1(w)\,dw & \text{for } y > 0, \\ \frac{1}{2}\int_{-\infty}^{\infty} \exp(-ix\,\mathrm{sh}w + y\,\mathrm{ch}w) g_2(w)\,dw & \text{for } y < 0, \end{cases}$$

which representation ensures that G satisfies (3.1).
An analogous representation exists for the modified Bessel function $K_o(R)$. It reads

$$(3.5)\qquad K_o(R) = \frac{1}{2}\int_{-\infty}^{\infty} \exp\left[-i(x-x_o)\mathrm{sh}w - |y-y_o|\mathrm{ch}w\right] dw.$$

Substitution of (3.3), (3.4) and (3.5) in the boundary conditions (3.2) gives

$$\int_{-\infty}^{\infty} \left[(\mathrm{ch}w+\mathrm{ch}\,\alpha)g_1 - (\mathrm{ch}w-\mathrm{ch}\,\alpha)\exp(ix_o\mathrm{sh}w - y_o\mathrm{ch}\,w)\right] e^{-ix\mathrm{sh}\,w} = 0, \qquad x < 0,\ y = +0,$$

$$\int_{-\infty}^{\infty} \left[(\mathrm{ch}w+\mathrm{ch}\,\alpha)g_2 + (\mathrm{ch}\,w+\mathrm{ch}\alpha)\exp(ix_o\mathrm{sh}\,w - y_o\mathrm{ch}\,w)\right] e^{-ix\mathrm{sh}\,w} = 0, \qquad x < 0,\ y = -0,$$

from which we obtain by subtraction and addition

$$(3.6)\qquad \int_{-\infty}^{\infty} \phi^+(w)\, e^{-ix\mathrm{sh}\,w}\,\mathrm{ch}\,w\,dw = 0 \qquad \text{for } x < 0,$$

$$(3.7)\qquad \int_{-\infty}^{\infty} \psi^+(w)\, e^{-ix\mathrm{sh}\,w}\,\mathrm{ch}\,w\,dw = 0 \qquad \text{for } x < 0,$$

where

$$(3.8)\qquad \mathrm{ch}\,w\,\phi^+(w) = (\mathrm{ch}\,w+\mathrm{ch}\alpha)\,(y_1-y_2) - 2\mathrm{ch}\,w\,\exp(ix_o\mathrm{sh}\,w - y_o\mathrm{ch}\,w),$$

$$(3.9)\qquad \mathrm{ch}\,w\,\psi^+(w) = (\mathrm{ch}\,w+\mathrm{ch}\alpha)\,(y_1+y_2) + 2\mathrm{ch}\,\alpha\,\exp(ix_o\mathrm{sh}\,w - y_o\mathrm{ch}\,w).$$

In the following we mostly use polar coordinates r, ϑ, given by $x=r\cos\vartheta$, $y=r\sin\vartheta$, instead of Cartesian coordinates (x,y). It follows from (3.6) and (3.7) that, because of the restriction

$x < 0$, both ϕ^+ and ψ^+ are holomorphic in the strip $0 < \mathrm{Im}\, w < \pi$. Furthermore they are symmetric with respect to $\frac{1}{2}\pi i$.
So for we have taken account of the jump in φ when passing the screen $y=0$, $x < 0$. This jump was implicitly taken care of by the introduction of two functions viz. $g_1(w)$ valid in the upper half of the x-y plane, and $g_2(w)$ valid in the lower half. However φ and $\partial\varphi/\partial y$ should be continuous over the ray $y=0$, $x > 0$. From the continuity of φ we derive

(3.10) $$\int_{-\infty}^{\infty} \phi^-(w) \mathrm{ch}\, w\, e^{-ix \mathrm{sh}\, w}\, dw = 0 \qquad \text{for } x > 0,$$

where

(3.11) $$\phi^-(w) = (g_1 - g_2)/\mathrm{ch}\, w,$$

and from the continuity of $\partial\varphi/\partial y$

(3.12) $$\int_{-\infty}^{\infty} \psi^-(w)\, \mathrm{ch}\, w\, e^{-ix \mathrm{sh}\, w}\, dw = 0 \qquad \text{for } x > 0,$$

where

(3.13) $$\psi^-(w) = g_1 + g_2 \quad ,$$

It follows from (3.10) and (3.12) that, because of the condition $x > 0$, both ϕ^- and ψ^- are holomorphic in the strip $-\pi < \mathrm{Im}\, w < 0$. They also are symmetric with respect to $-\frac{1}{2}\pi i$. Elimination of g_1 and g_2 from (3.8), (3.9), (3.11) and (3.13) and replacement of the Cartesian coordinates by polar ones, $x_0 = r_0 \cdot \cos \vartheta_0$, $y_0 = r_0 \sin \vartheta_0$, yields

(3.14) $$\phi^+(w) = (\mathrm{ch}\, w + \mathrm{ch}\,\alpha)\, \phi^-(w) - 2 \exp\left[-r_0 \mathrm{ch}(w + i\vartheta_0 - \tfrac{1}{2} i\pi)\right]$$

(3.15) $$\psi^+(w) = (\mathrm{ch}\, w + \mathrm{ch}\,\alpha)\, \psi^-(w) + 2\, \mathrm{ch}\,\alpha \exp\left[-r_0 \mathrm{ch}(w + i\vartheta_0 - \tfrac{1}{2} i\pi)\right].$$

The original problem of finding a solution of the inhomogeneous modified Helmholtz equation (3.1) with boundary conditions (3.2) has now been reduced to solving the two Hilbert problems (3.14) and (3.15). To this end we have to factorize $\mathrm{ch}\, w$ and $\mathrm{ch}\, w + \mathrm{ch}\,\alpha$. The former factorization can be carried out by inspection. Indeed

(3.16) $$\operatorname{ch} w = \left[\sqrt{2}\operatorname{ch}\tfrac{1}{2}(w-\tfrac{1}{2}i\pi)\right]\times\left[\sqrt{2}\operatorname{ch}\tfrac{1}{2}(w+i\pi)\right] \quad ,$$

where the first factor in the right-hand member clearly is holomorphic in the strip $0<\operatorname{Im} w<\pi$ and the second factor in the strip $-\pi<\operatorname{Im} w<0$. Formally the latter factorization is given by

(3.17) $$\operatorname{ch} w + \operatorname{ch}\alpha = K^{+}(w)\, K^{-}(w) \quad ,$$

where $K^{+}(w)$ is holomorphic in the strip $0<\operatorname{Im} w<\pi$ and symmetric with respect to $\frac{1}{2}\pi i$ and $K^{-}(w)$ is holomorphic in the strip $-\pi<\operatorname{Im} w<0$ and symmetric with respect to $-\frac{1}{2}\pi i$. The derivation of explicit expressions for K^{+} and K^{-} will be deferred till the next section.
In the sequel we will also use the functions

(3.18) $$L^{\pm}(w) = K^{\pm}(w)/\left[\sqrt{2}\operatorname{ch}\tfrac{1}{2}(w\mp\tfrac{1}{2}i\pi)\right] \quad ,$$

which correspond to a factorization of $1+\operatorname{ch}\alpha/\operatorname{ch} w$.
By aid of (3.16), (3.17) and (3.18) we can write (3.14) and (3.15) in the form

(3.19) $$\phi^{+}/K^{+} - \phi^{-}K^{-} = -2\exp\left[-r_{0}\operatorname{ch}(w+i\vartheta_{0}-\tfrac{1}{2}i\pi)\right]/K^{+} \quad ,$$

(3.20) $$\psi^{+}/L^{+} - \psi^{-}L^{-} = 2\operatorname{ch}\alpha\exp\left[-r_{0}\operatorname{ch}(w+i\vartheta_{0}-\tfrac{1}{2}i\pi)\right]/(\operatorname{ch} w\, L^{+}).$$

The solution of these Hilbert problems are given by

(3.21) $$\phi^{+}(w) = -\frac{K^{+}(w)}{i\pi}\int_{-\infty}^{\infty}\frac{\exp\left[-r_{0}\operatorname{ch}(w_{0}+i\vartheta_{0}-\tfrac{1}{2}i\pi)\right]}{K^{+}(w_{0})}\,\frac{\operatorname{ch} w_{0}\, dw_{0}}{\operatorname{sh} w_{0}-\operatorname{sh} w}, \qquad 0<\operatorname{Im} w<\pi \quad ,$$

(3.22) $$\phi^{-}(w) = -\frac{1}{i\pi K^{-}(w)}\int_{-\infty}^{\infty}\frac{\exp\left[-r_{0}\operatorname{ch}(w_{0}+i\vartheta_{0}-\tfrac{1}{2}i\pi)\right]}{K^{+}(w_{0})}\,\frac{\operatorname{ch} w_{0}\, dw_{0}}{\operatorname{sh} w_{0}-\operatorname{sh} w}, \qquad -\pi<\operatorname{Im} w<0,$$

$$(3.23)\quad \psi^{+}(w)=\frac{\operatorname{ch}\alpha\, L^{+}(w)}{i\pi}\int_{-\infty}^{\infty}\frac{\exp\left[-r_0\operatorname{ch}(w_0+i\vartheta_0-\tfrac{1}{2}i\pi)\right]}{L^{+}(w_0)}\,\frac{dw_0}{\operatorname{sh} w_0-\operatorname{sh} w},$$

$$0<\operatorname{Im} w<\pi\ ,$$

$$(3.24)\quad \psi^{-}(w)=\frac{\operatorname{ch}\alpha}{i\pi L^{-}(w)}\int_{-\infty}^{\infty}\frac{\exp\left[-r_0\operatorname{ch}(w_0+i\vartheta_0-\tfrac{1}{2}i\pi)\right]}{L^{+}(w_0)}\,\frac{dw_0}{\operatorname{sh} w_0-\operatorname{sh} w},$$

$$-\pi<\operatorname{Im} w<0.$$

Elimination of g_1 from (3.11) and (3.12) and substitution of (3.22) and (3.24) gives

$$(3.25)\quad g_2(w)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{H(w_0,w)}{\operatorname{sh} w_0-\operatorname{sh}w}\exp\left[-r_0\operatorname{ch}(w_0+i\vartheta_0-\tfrac{1}{2}i\pi)\right]dw_0,$$

$$-\pi<\operatorname{Im}w<0,$$

where

$$(3.26)\qquad H(w_0,w)=\frac{2\operatorname{ch}\tfrac{1}{2}(w_0+\tfrac{1}{2}i\pi)\operatorname{ch}\tfrac{1}{2}(w-\tfrac{1}{2}i\pi)+\operatorname{ch}\alpha}{L^{+}(w_0)\ L^{-}(w)}\ ,$$

which formulae are valid in the lower half-plane $y<0$. However, for $g_2(w)$ we need a formula which is also valid on the line $\operatorname{Im} w=0$. This is easily obtained by analytical continuation of (3.25) if the path of integration for w_0 is shifted into the strip $0<\operatorname{Im} w_0<\pi$.
Substitution of (3.25) in (3.4) gives, using polar coordinates,

$$\varphi=\frac{1}{4i\pi}\int_{-\infty}^{\infty}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{H(w_0,w)}{\operatorname{sh} w_0-\operatorname{sh} w}$$

$$\exp\left[-r_0\operatorname{ch}(w_0+i\vartheta_0-\tfrac{1}{2}i\pi)-r\operatorname{ch}(w+i\vartheta+\tfrac{1}{2}i\pi)\right]dw_0\ dw,\ -\pi<\vartheta<0.$$

In this integral shift both paths of integration over a distance $+i\varepsilon$.
No poles will be passed and we find the equivalent expression

$$(3.27)\qquad \varphi = \frac{1}{4i\pi} \int_{-\infty - i\varepsilon}^{\infty - i\varepsilon} \int_{-\infty}^{\infty} \frac{H(w_o,w)}{\operatorname{sh} w_o - \operatorname{sh} w}$$
$$\exp\left[-r_o \operatorname{ch}(w_o + i\vartheta_o - \tfrac{1}{2}i\pi) - r\operatorname{ch}(w + i\vartheta + \tfrac{1}{2}i\pi)\right] dw_o\, dw.$$

Other representations for φ can be obtained by expressing g_2 in terms of $\emptyset^+$ and ψ^+ or by using g_1 either in terms of $\emptyset^-$ and ψ^- or $\emptyset^+$ and ψ^+.
The continuity of φ and $\partial\varphi/\partial y$ on $y=0$, $x>0$ ensures that all these expressions represent the same function φ for $-\pi<\vartheta<\pi$.
The extension of (3.27) to the range $0<\vartheta<\pi$ can also be carried out by a suitable continuation. This will be done in section 5.

4. The factorization

In this section we explicitly carry out the factorization

$$(4.1)\qquad \operatorname{ch} w + \operatorname{ch}\alpha = K^+(w)\, K^-(w),$$

which was formally introduced in the preceding section. The factor K^+ is required to be holomorphic in the strip $0<\operatorname{Im} w<\pi$ and to be symmetric with respect to $\frac{1}{2}i\pi$, and K^- to be holomorphic in the strip $-\pi< \operatorname{Im} w < 0$ and symmetric with respect to $-\frac{1}{2}i\pi$. The same factorization has been carried out, by different means, by SENIOR [6] and by HEINS and FESHBACH [3]. By logarithmic derivation we obtain

$$(4.2)\qquad \frac{d}{dw}\ln K^+(w) + \frac{d}{dw}\ln K^-(w) = \frac{d}{dw}\ln(\operatorname{ch} w + \operatorname{ch}\alpha),$$

which equation, again, is a Hilbert problem on w=0. Its solution is

$$\frac{d}{dw}\ln K^{\pm}(w) = \pm\frac{\operatorname{ch} w}{2i\pi}\int_{-\infty}^{\infty}\frac{d}{dw_o}\ln(\operatorname{ch} w_o + \operatorname{ch}\alpha)\frac{dw_o}{\operatorname{sh} w_o - \operatorname{sh} w},$$

which can be brought in the form

$$\frac{d}{dw}\ln K^{\pm}(w) =$$

$$\mp\frac{1}{4i\pi}\left[\frac{\text{ch } w}{\text{sh } w+\text{sh }\alpha}\int_{-\infty}^{\infty}\text{th }\tfrac{1}{2}(w_o-\alpha)dw_o+\frac{\text{ch } w}{\text{sh } w-\text{sh }\alpha}\int_{-\infty}^{\infty}\text{th}\tfrac{1}{2}(w_o+\alpha)dw_o\right]\pm$$

$$\pm\frac{1}{4i\pi}\left[\frac{\text{sh } w}{\text{ch } w-\text{ch }\alpha}\int_{-\infty}^{\infty}\text{th }\tfrac{1}{2}(w_o+w)dw_o+\frac{\text{sh } w}{\text{ch } w+\text{ch }\alpha}\int_{-\infty}^{\infty}\text{cth }\tfrac{1}{2}(w_o-w)dw_o\right].$$

Divergent integrals are rendered meaningful by defining

$$\int_{-\infty}^{\infty} f(w_o)dw_o = \lim_{p\to\infty}\int_{-p}^{p} f(w_o)dw_o .$$

After evaluation of the integrals we arrive (under the condition $-\pi < \text{Im}\,\alpha < \pi$)

$$K^{\pm}(w)=\sqrt{\text{ch } w+\text{ch}\alpha}\ \exp\left[\pm\frac{1}{2i\pi}\left\{\Lambda(w+\alpha)+\Lambda(w-\alpha)\right\}\right], \tag{4.2}$$

where

$$\Lambda(w) = \int_0^w \frac{tdt}{\text{sh}t} . \tag{4.3}$$

This function is related to Legendre's Chi-function by

$$\Lambda(w) = 2\,\chi_2(\text{th }\tfrac{1}{2}\,w),$$

(cf. [3]), but this notation might be misleading because it suggests the functional relation $\Lambda(w+2i\pi)=\Lambda(w)$, which does not hold.

An alternative form of (4.2) is

$$K^{\pm}(w)=\sqrt{2}\ \text{ch }\tfrac{1}{2}(w\pm\alpha)\sqrt{\frac{\text{sh }\tfrac{1}{2}(w\mp\alpha)}{\text{sh }\tfrac{1}{2}(w\pm\alpha)}}\ \exp\left[\mp\frac{1}{2i\pi}\left\{\Lambda(w+\alpha-i\pi)+\Lambda(w-\alpha+i\pi)\right\}\right],$$

which shows clearly the zero's at $w=\mp\alpha+2ki\pi$.

5. Some alternative representations

In this section a few alternative expressions for the regular part φ of the Green's function will be derived. It follows from (3.27) that

$$(5.1)\quad \varphi=-\frac{1}{8\pi}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\int_{-\infty}^{\infty}\frac{H(w_0,w)\exp\left[-r_0\,\mathrm{ch}(w_0+i\vartheta_0-\frac{1}{2}i\pi)-r\,\mathrm{ch}(w+i\vartheta+\frac{1}{2}i\pi)\right]}{\mathrm{ch}\,\frac{1}{2}(w_0+w)\,\mathrm{ch}\,\frac{1}{2}(w_0-w-i\pi)}\,dw_0\,dw.$$

This integral has been derived for $-\pi<\vartheta<0$. However, by shifting the paths of integration its validity can be extended to the full range $-\pi<\vartheta<\pi$. Putting $w=v-i\vartheta-\frac{1}{2}i\pi$ and $w_0=v_0-i\vartheta_0+\frac{1}{2}i\pi$ and shifting the paths of integration to the real v and v_0 axes, we obtain

$$(5.2)\quad \varphi=-\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H(v_0-i\vartheta_0+\frac{1}{2}i\pi,v-i\vartheta-\frac{1}{2}i\pi)\exp(-r_0\mathrm{ch}\,v_0-r\mathrm{ch}\,v)}{\mathrm{ch}\,\frac{1}{2}(v_0+v-i\vartheta_0-i\vartheta)\,\mathrm{ch}\,\frac{1}{2}(v_0-v-i\vartheta_0+i\vartheta)}\,dv_0\,dv,$$

provided no poles are passed. This will not occur if

$$-\pi<\vartheta+\vartheta_0<\pi \qquad \text{and} \qquad -\pi<\vartheta-\vartheta_0<\pi\ .$$

Since the source was taken in the upper half of the X-Y plane, or $0<\vartheta_0<\pi$, the above inequalities may be replaced by the single inequalities

$$\vartheta_0-\pi<\vartheta<-\vartheta_0+\pi\ .$$

The Green's function, hence, equals

$$(5.3)\qquad G(r,\vartheta,r_0,\vartheta_0,\alpha)=$$

$$=K_0(R)-\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H(v_0-i\vartheta_0+\frac{1}{2}i\pi,v-i\vartheta-\frac{1}{2}i\pi)\exp(-r_0\mathrm{ch}v_0-r\mathrm{ch}v)}{\mathrm{ch}(v_0-i\vartheta_0)+\mathrm{ch}(v-i\vartheta)}\,dv_0dv,$$

$$\vartheta_0-\pi<\vartheta<-\vartheta_0+\pi\ .$$

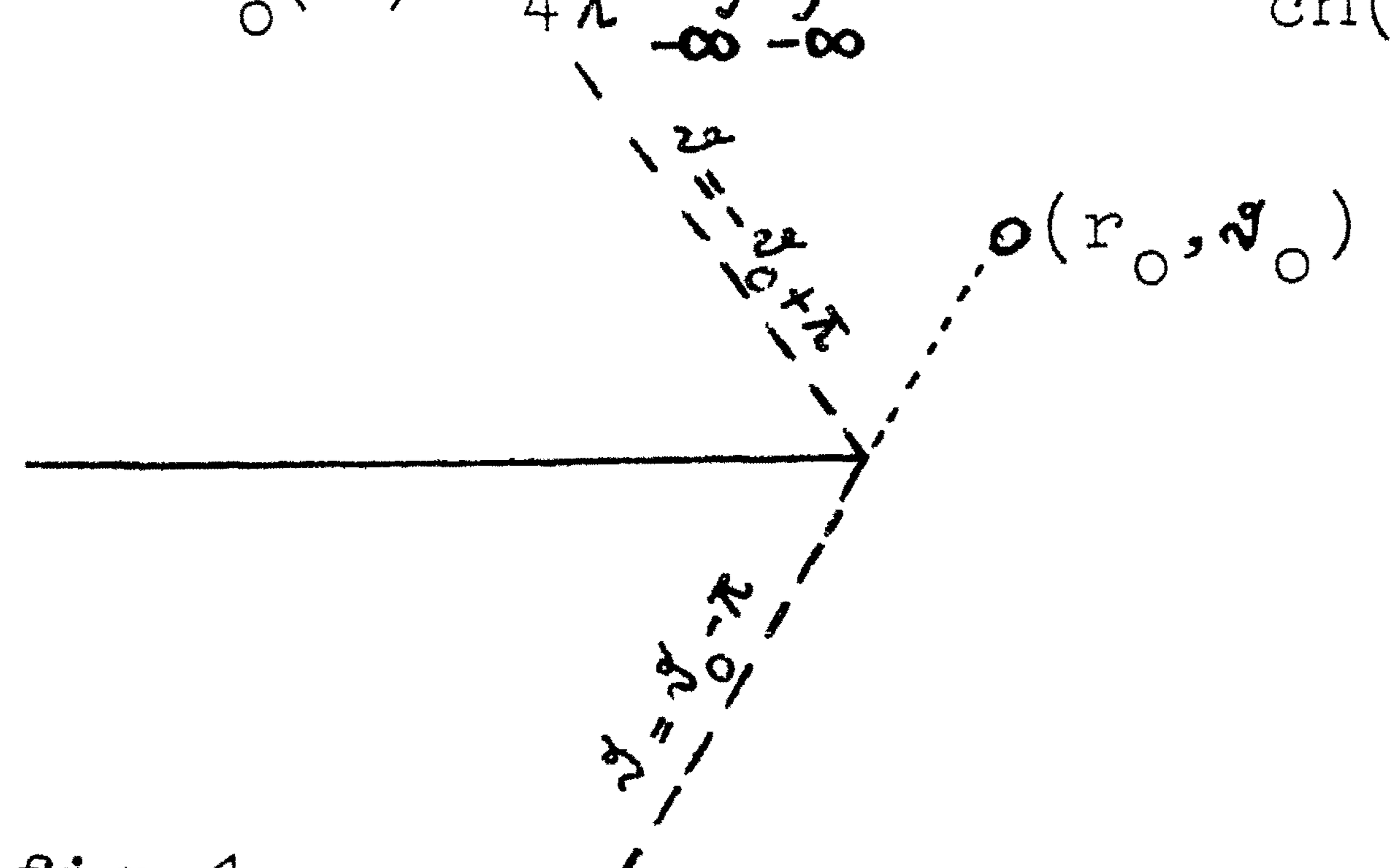

fig.1

Borrowing from the terminology of geometrical optics, one can say that (5.2) is valid in the directly illuminated region (comp. fig. 1).

Expressions valid in the other regions can be found in the following way. Again consider (5.1) and shift the path of integration of w from the strip $-\pi < \mathrm{Im}\, w < 0$ into the strip $0 < \mathrm{Im}\, w < \pi$. Then a pole at $w=w_0$ will be passed and the residue at this pole adds an extra term. At the pole we have

$$H(w_0,w_0) = \mathrm{ch}\, w_0,$$

and the residue hence equals

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \exp\left[-r_0\mathrm{ch}(w_0+i\vartheta_0-\tfrac{1}{2}i\pi)-r\mathrm{ch}(w_0+i\vartheta_0+\tfrac{1}{2}i\pi)\right]dw_0=K_0(R),$$

which term has to be substracted. The main term of φ, again, has the form (5.2) with $-\pi<\varepsilon<0$. Putting $w=v-i\vartheta-\frac{1}{2}i\pi$, $w_0=v_0-i\,\vartheta_0+\frac{1}{2}i\pi$ as before and shifting the paths of integration, we obtain

(5.4) $\quad G(r,\vartheta,r_0,\vartheta_0) =$

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H(v_0-i\vartheta_0+\frac{1}{2}i\pi,v-i\vartheta-\frac{1}{2}i\pi)\exp(-r_0\mathrm{ch}\,v_0-r\mathrm{ch}\,v)}{\mathrm{ch}(v_0-i\vartheta_0)+\mathrm{ch}(v-i\vartheta)}\,dv_0\,dv,$$

$$-\pi<\vartheta<\vartheta_0-\pi,$$

where the region of validity $-\pi<\vartheta<\vartheta_0-\pi$ — the shadow region (comp.fig.2) — is determined by requiring that during the shifting of the paths of integration for v_0 and v no poles will be passed.

Finally shift the path of integration of w from the strip $-\pi < \mathrm{Im}\, w < 0$ into the strip $-2\pi < \mathrm{Im}\, w < \pi$. Then a pole at $w=-w_0-i\pi$ will be passed.
At this pole we have

$$H(w_0,-w_0-i\pi) = -\frac{\mathrm{ch}\,w_0-\mathrm{ch}\,\alpha}{\mathrm{ch}\,w_0+\mathrm{ch}\,\alpha}\,\mathrm{ch}\,w_0,$$

and the residue, hence, equals

$$-\tfrac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{ch}\,w_0-\mathrm{ch}\,\alpha}{\mathrm{ch}\,w_0+\mathrm{ch}\,\alpha}\exp\left[-r_0\mathrm{ch}(w_0+i\vartheta_0-\tfrac{1}{2}i\pi)-r\mathrm{ch}(w+i\vartheta+\tfrac{1}{2}i\pi)\right]dw_0,$$

which term has to be added.
Using the identity

$$\frac{\operatorname{ch} w_0-\operatorname{ch}\alpha}{\operatorname{ch} w_0+\operatorname{ch}\alpha} = 1 + \operatorname{cth}\alpha\left[\operatorname{th}\tfrac{1}{2}(w_0-\alpha)-\operatorname{th}\tfrac{1}{2}(w_0+\alpha)\right],$$

the residue can be written in the form

$$K_0(R') + \operatorname{Res}(\alpha) + \operatorname{Res}(-\alpha),$$

where $R'^2=r^2+r_0^2-2rr_0\cos(\vartheta+\vartheta_0)=(x-x_0)^2+(y+y_0)^2$ and

(5.5) $\qquad \operatorname{Res}(\alpha) =$

$$\tfrac{1}{2}\operatorname{cth}\alpha\int_{-\infty}^{\infty}\operatorname{th}\tfrac{1}{2}(w_0-\alpha)\exp\left[-r_0\operatorname{ch}(w_0+i\vartheta_0-\tfrac{1}{2}i\pi)-r\operatorname{ch}(w_0-i\vartheta+\tfrac{1}{2}i\pi)\right]dw_0.$$

The main term of φ again has the form (5.2). Putting $w_0=v_0-i\vartheta_0+\frac{1}{2}i\pi$ and shifting the paths of integration, we obtain

(5.6) $\qquad G(r,\vartheta,r_0,\vartheta_0) =$

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H(v_0-i\vartheta_0+\frac{1}{2}i\pi,v-i\vartheta-\frac{1}{2}i\pi)\exp(-r_0\operatorname{ch}v_0-r\operatorname{ch}v)}{\operatorname{ch}(v_0-i\vartheta_0)+\operatorname{ch}(v-i\vartheta)}dv_0dv+$$

$$+K(R)+K(R')+\operatorname{Res}(\alpha)+\operatorname{Res}(-\alpha), \qquad -\vartheta_0+\pi<\vartheta<\pi.$$

The region of validity of (5.5) - the region where direct reflection occurs (comp. fig. 1) - again is determined by the requirement that during the shifting of the paths of integration of v and v_0 no poles will be passed.

6. Special cases

1) The case $\alpha=\frac{1}{2}i\pi$.
For $\alpha=\frac{1}{2}i\pi$ the boundary conditions (1.2) simply become $\partial G/\partial y=0$ for $x<0$, $y=0$. This corresponds to the well-known diffraction problem for a half-ray with vanishing of the normal

derivative of the field at the screen $x<0$, $y=0$. In this case $L^{+}=L^{-}=1$ and from (2.24)

$$H(w_0,w)=-2\mathrm{ch}\,\tfrac{1}{2}(w_0+\tfrac{1}{2}i\pi)\mathrm{ch}\,\tfrac{1}{2}(w-\tfrac{1}{2}i\pi)=-\mathrm{ch}\,\tfrac{1}{2}(w_0+w)+\mathrm{ch}\,\tfrac{1}{2}(w_0-w-i\pi).$$

Hence

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H(v_0-i\vartheta_0+\frac{1}{2}i\pi,v-i\vartheta-\frac{1}{2}i\pi)\exp(-r_0\mathrm{ch}\,v_0-r\,\mathrm{ch}\,v)}{\mathrm{ch}\,\frac{1}{2}(v_0+v-i\vartheta_0-i\vartheta)\mathrm{ch}\frac{1}{2}(v_0-v-i\vartheta_0+i\vartheta)}\,dv_0\,dv=$$

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\mathrm{sech}\,\tfrac{1}{2}(v_0+v-i\vartheta_0-i\vartheta)-\mathrm{sech}\,\tfrac{1}{2}(v_0-v-i\vartheta_0+i\vartheta)\right]\exp(-r_0\mathrm{ch}v_0-r\mathrm{ch}v)dv_0\,dv.$$

Furthermore

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp(-r_0\mathrm{ch}\,v_0-r\mathrm{ch}\,v)}{\mathrm{ch}\,\frac{1}{2}(v_0+v-i\beta)}\,dv_0dv=\begin{cases}-\,\Theta(r,r_0,\beta-2\pi), & \pi<\beta<3\pi,\\ \Theta(r,r_0,\beta), & -\pi<\beta<\pi.\end{cases}$$

where $\Theta(r,r_0,\beta)=\int_{s_0}^{\infty}e^{-c\,\mathrm{ch}\,w}\,dw$,

and $c^2=r^2+r_0^2-2rr_0\cos\beta$, $\quad \mathrm{ch}\,w_0=\frac{r+r_0}{c}$ (cf. the Appendix of [2], part III).

It follows that

$$G(r,r_0,\vartheta,\vartheta_0,\tfrac{1}{2}i\pi)=\begin{cases}-\frac{1}{2}\Theta(r,r_0,\vartheta_0+\vartheta-2\pi)-\frac{1}{2}\Theta(r,r_0,\vartheta_0-\vartheta)+K_0(R)+K_0(R'), & -\vartheta_0+\pi<\vartheta<\pi,\\ \frac{1}{2}\Theta(r,r_0,\vartheta_0+\vartheta)-\frac{1}{2}\Theta(r,r_0,\vartheta_0-\vartheta)+K_0(R) & \vartheta_0-\pi<\vartheta<-\vartheta_0+\pi,\\ \frac{1}{2}\Theta(r,r_0,\vartheta_0+\vartheta)+\frac{1}{2}\Theta(r,r_0,\vartheta_0-\vartheta-2\pi) & -\pi<\vartheta<\vartheta_0-\pi,\end{cases}$$

which agrees with the known solution.

2) The case $\alpha\longrightarrow\infty$.

For $\alpha\longrightarrow\infty$ the boundary conditions (1.2) become $G=0$ for $x<0$, $y=0$. This corresponds to the well-known diffraction problem for a half-ray with vanishing field at the screen $x<0$, $y=0$.
In this case

$$H(w_0,w)\longrightarrow-2\mathrm{ch}\,\tfrac{1}{2}(w_0-\tfrac{1}{2}i\pi)\mathrm{ch}\,\tfrac{1}{2}(w+\tfrac{1}{2}i\pi)=-\mathrm{ch}\,\tfrac{1}{2}(w_0+w)-\mathrm{ch}\,\tfrac{1}{2}(w_0-w-i\pi).$$

Hence

$$\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{H(v_0-i\vartheta_0+\frac{1}{2}i\pi,\,v-i\vartheta-\frac{1}{2}i\pi)\exp(-r_0\mathrm{ch}\,v_0-r\mathrm{ch}\,v)}{\mathrm{ch}\,\frac{1}{2}(v_0+v-i\vartheta_0-i\vartheta)\,\mathrm{ch}\,\frac{1}{2}(v_0-v-i\vartheta_0+i\vartheta)}\,dv_0\,dv\longrightarrow$$

$$-\frac{1}{8\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\mathrm{sech}\,\tfrac{1}{2}(v_0+v-i\vartheta_0-i\vartheta)+\mathrm{sech}\,\tfrac{1}{2}(v_0-v-i\vartheta_0+i\vartheta)\right]\exp(-r_0\mathrm{ch}\,v_0-r\,\mathrm{ch}\,v)\,dv_0\,dv.$$

Furthermore

$$\mathrm{ch}\,\alpha\int_{-\infty}^{\infty}\frac{\exp(-R'\mathrm{ch}t)dt}{\mathrm{ch}(t+i\Theta'-\frac{1}{2}i\pi)-\mathrm{ch}\,\alpha}\longrightarrow -2\,K_0(R').$$

It follows that

$$G(r,r_0,\vartheta,\vartheta_0,\infty)=\begin{cases}\frac{1}{2}\Theta(r,r_0,\vartheta_0+\vartheta-2\pi)-\frac{1}{2}\Theta(r,r_0,\vartheta_0-\vartheta)+K_0(R)-K_0(R'), & -\vartheta_0+\pi<\vartheta<\pi,\\ -\frac{1}{2}\Theta(r,r_0,\vartheta_0+\vartheta)-\frac{1}{2}\Theta(r,r_0,\vartheta_0-\vartheta)+K_0(R), & \vartheta_0-\pi<\vartheta<-\vartheta_0+\pi,\\ -\frac{1}{2}\Theta(r,r_0,\vartheta_0+\vartheta)-\frac{1}{2}\Theta(r,r_0,\vartheta_0-\vartheta-2\pi) & ,-\pi<\vartheta<\vartheta_0-\pi,\end{cases}$$

in accordance with the known solution.

It might be surmised that in the case $\alpha=0$ too considerable simplifications would occur. This, however, proves to be false.

3) The case $r_0\longrightarrow\infty$.

The functions, occurring in (5.3), (5.4) and (5.5) are all of the form

$$\tfrac{1}{2}\int_{-\infty}^{\infty} f(t)\,e^{-r_0\,\mathrm{ch}t}\,dt,$$

which asymptotically equals

$$f(0)\,\sqrt{\pi/(2\,r_0)}\;\;e^{-r_0}\,.$$

We apply this formula to (5.3), (5.4) and (5.5).
Furthermore we put $G=\sqrt{\pi/(2\,r_0)}\exp(-r_0)G^{*}$. We then obtain

$$(6.1)\quad G^*(r,\vartheta,\vartheta_0,\alpha)\approx -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(-i\vartheta_0+\tfrac{1}{2}i\pi,\, v-i\vartheta-\tfrac{1}{2}i\pi)\exp(-r\ \mathrm{ch} v)}{\cos\vartheta_0+\mathrm{ch}(v-i\vartheta)}\,dv,$$

$$-\pi<\vartheta<\vartheta_0-\pi\,,$$

$$(6.2)\quad G^*(r,\vartheta,\vartheta_0,\alpha)\approx -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(-i\vartheta_0+\tfrac{1}{2}i\pi,\, v-i\vartheta-\tfrac{1}{2}i\pi)\exp(-r\ \mathrm{ch}\, v)}{\cos\vartheta_0+\mathrm{ch}(v-i\vartheta)}\,dv+$$

$$+\exp\left[r\cos(\vartheta-\vartheta_0)\right],$$

$$\vartheta_0-\pi<\vartheta<-\vartheta_0+\pi.$$

For the asymptotic form of (5.6) we consider Res(α) separately. Putting in (5.5) $w_0=v_0-i\vartheta_0+\frac{1}{2}i\pi$ we find

$$\mathrm{Res}(\alpha)=\tfrac{1}{2}\mathrm{cth}\alpha\int_{-\infty+i\vartheta_0-\frac{1}{2}i\pi}^{\infty+i\vartheta_0-\frac{1}{2}i\pi}\mathrm{th}\,\tfrac{1}{2}(v_0-i\vartheta_0+\tfrac{1}{2}i\pi-\alpha)\exp\left[-r_0\mathrm{ch}\, v_0+r\,\mathrm{ch}(v_0-i\vartheta-i\vartheta_0)\right]dv_0.$$

Since $0<\vartheta_0<\pi$ the path of integration may be shifted to the real v_0-axis provided $-\frac{1}{2}\pi\leqslant \mathrm{Im}\,\alpha\leqslant\frac{1}{2}\pi$. In the case of electromagnetic radiation, and also in the case of diffraction of sound, this inequality indeed holds. If $\mathrm{ch}\alpha>1$ then $\mathrm{Im}\,\alpha=0$, and if $\mathrm{ch}\alpha<1$ then $0<\mathrm{Im}\,\alpha\leqslant\frac{1}{2}\pi$.
We hence have, with $\mathrm{Res}(\alpha)=\mathrm{Res}^*(\alpha)\sqrt{\pi/(2\,r_0)}\exp(-r_0)$:

$$\mathrm{Res}^*(\alpha)\approx\ \mathrm{cth}\alpha\ \mathrm{th}\,\tfrac{1}{2}(-i\vartheta_0+\tfrac{1}{2}i\pi-\alpha)\exp\left[r\ \mathrm{ch}\ s(\vartheta+\vartheta_0)\right],$$

and

$$\mathrm{Res}^*(\alpha)+\mathrm{Res}^*(-\alpha)=-\frac{2\ \mathrm{ch}\alpha}{\mathrm{ch}\alpha+\sin\vartheta_0}\exp\left[r\cos(\vartheta+\vartheta_0)\right].$$

Applying these results we find for the asymptotic form of (5.6)

$$(6.3)\quad G^*(r,\vartheta,\vartheta_0,\alpha)\approx -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{H(-i\vartheta_0+\tfrac{1}{2}i\pi,\, v-i\vartheta-\tfrac{1}{2}i\pi)\exp(-r\,\mathrm{ch}\, v)}{\cos\vartheta_0+\mathrm{ch}(v-i\vartheta)}\,dv\ +$$

$$+\exp\left[-r\cos(\vartheta-\vartheta_0)\right]-\frac{\mathrm{ch}\alpha-\sin\vartheta_0}{\mathrm{ch}\alpha+\sin\vartheta_0}\exp\left[-r\cos(\vartheta+\vartheta_0)\right],$$

$$-\tfrac{1}{2}\pi\leqslant\mathrm{Im}\alpha<\tfrac{1}{2}\pi.\quad -\vartheta_{\_}+\pi<\vartheta<\pi\ .$$

4) The case $r \longrightarrow \infty$

This case is analogous to the former. We confine ourselves to giving the pertinent formulae.

(6.4) $$G(r,\vartheta,r_o,\vartheta_o,\alpha) \approx -\frac{\exp(-r)}{\sqrt{8\pi r}} \int_{-\infty}^{\infty} \frac{H(v_o - i\vartheta_o + \frac{1}{2}i\pi, -i\vartheta - \frac{1}{2}i\pi)}{\operatorname{ch}(v_o - i\vartheta_o) + \cos\vartheta}\, dv_o,$$

$$-\pi < \vartheta < \vartheta_o - \pi \,.$$

(6.5) $$G(r,\vartheta,r_o,\vartheta_o,\alpha) \approx -\frac{\exp(-r)}{\sqrt{8\pi r}} \int_{-\infty}^{\infty} \frac{H(v_o - i\vartheta_o + \frac{1}{2}i\pi, -i\vartheta - \frac{1}{2}i\pi)}{\operatorname{ch}(v_o - i\vartheta_o) + \cos\vartheta}\, dv_o +$$

$$+ \sqrt{\frac{\pi}{2r}} \exp\left[-r + r_o \cos(\vartheta - \vartheta_o) \right],$$

$$\vartheta_o - \pi < \vartheta < -\vartheta_o + \pi .$$

(6.6) $$G(r,\vartheta,r_o,\vartheta_o,\alpha) \approx -\frac{\exp(-r)}{\sqrt{8\pi r}} \int_{-\infty}^{\infty} \frac{H(v_o - i\vartheta_o + \frac{1}{2}i\pi, -i\vartheta - \frac{1}{2}i\pi)}{\operatorname{ch}(v_o - i\vartheta_o) + \cos\vartheta}\, dv_o +$$

$$+ \sqrt{\frac{\pi}{2r}} \exp(-r)\left[\exp\left\{ r_o \cos(\vartheta - \vartheta_o) - \frac{\operatorname{ch}\alpha - \sin\vartheta}{\operatorname{ch}\alpha + \sin\vartheta} \exp\{r_o \cos(\vartheta + \vartheta_o)\right\}\right],$$

$$-\tfrac{1}{2}\pi < \operatorname{Im}\alpha < \tfrac{1}{2}\pi, \quad -\vartheta_o + \pi < \vartheta < \pi .$$

7. A related boundary value problem

In the discussion on the Hertzian dipole in section 2 we encountered the boundary condition.

$$\left(\frac{\partial^2}{\partial y^2} \mp \operatorname{ch}\alpha \frac{\partial}{\partial y}\right) G^* = 0, \quad \operatorname{ch}\alpha = Z_m/Z, y = \pm 0, \; x < 0,$$

where G^* again satisfies the modified inhomogeneous Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1\right) G^* = -2\pi\, \delta(x - x_o)\delta(y - y_o).$$

This problem can be solved by the method given in the preceding sections. We will give the salient points of the analysis only.
We put

$$G^* = K_0(R) + \varphi^*,$$

$$\varphi^* = \begin{cases} \frac{1}{2} \int_{-\infty}^{\infty} \exp(-ix\,\mathrm{sh}\, w - y\,\mathrm{ch}\, w) g_1^*(w)\,dw, \\ \frac{1}{2} \int_{-\infty}^{\infty} \exp(-ix\,\mathrm{sh}\, w + y\,\mathrm{ch}\, w) g_2^*(w)\,dw. \end{cases}$$

This leads to

$$\int_{-\infty}^{\infty} \mathrm{ch}\, w\left[(\mathrm{ch}\, w+\mathrm{ch}\,\alpha)g_1^*+(\mathrm{ch}\, w-\mathrm{ch}\,\alpha)\exp(ix_0\mathrm{sh}\, w-y_0\mathrm{ch}\, w)\right]e^{-ix\mathrm{sh}\, w}=0$$

$$\int_{-\infty}^{\infty} \mathrm{ch}\, w(\mathrm{ch}\, w+\mathrm{ch}\,\alpha)\left[g_2^* + \exp(ix_0\mathrm{sh}\, w-y_0\mathrm{ch}\, w)\right] e^{-ix\mathrm{sh}\, w} = 0.$$

Then

$$\emptyset^{+*}(w)=(\mathrm{ch}\, w+\mathrm{ch}\alpha)(g_1^*-g_2^*)-2\,\mathrm{ch}\alpha\, \exp(ix_0\mathrm{sh}\, w-y_0\mathrm{ch}\, w),$$

and

$$\psi^{+*}(w) =(\mathrm{ch}\, w+\mathrm{ch}\alpha)(g_1^*+g_2^*)+2\,\mathrm{ch}\, w\, \exp(ix_0\mathrm{sh}\, w-y_0\mathrm{ch}\, w),$$

are holomorphic in the strip $0 < \mathrm{Im}\, w < \pi$.
From the continuity of φ and $\partial\varphi/\partial y$ for $y=0$, $x > 0$ we derive that

$$\emptyset^{-*}(w) = (g_1^*-g_2^*)\,/\mathrm{ch}\, w,$$

and

$$\psi^{-*}(w) = g_1^*+g_2^*,$$

are holomorphic in the strip $-\pi < \mathrm{Im}\, w < 0$.
On the line $w=0$ we hence have the Hilbert problems

$$\emptyset^{+*}=(1+\mathrm{ch}\,\alpha\,/\mathrm{ch}\, w)\emptyset^{-*}-2\mathrm{ch}\,\alpha\, \exp(ix_0\mathrm{sh}\, w-y_0\mathrm{ch}\, w),$$

$$\psi^{+*}=(\mathrm{ch}\, w+\mathrm{ch}\,\alpha)\,\psi^{-*}+2\mathrm{ch}\, w\, \exp(ix_0\mathrm{sh}\, w-y_0\mathrm{ch}\, w),$$

or, after factorization of $\operatorname{ch} w+\operatorname{ch}\alpha$ and $1+\operatorname{ch}\alpha/\operatorname{ch} w$,

$$\frac{\emptyset^{+*}}{L^{+}} - \emptyset^{-*} L^{-} = -2\operatorname{ch}\alpha \exp(ix_o \operatorname{sh} w - y_o \operatorname{ch} w)/L^{+} ,$$

$$\frac{\psi^{+*}}{K^{+}} - \psi^{-*} K^{-} = 2 \operatorname{ch} w \exp(ix_o \operatorname{sh} w - y_o \operatorname{ch} w)/K^{+}.$$

Solving $\emptyset^{-*}$ and ψ^{-*} from these problems leads to the following expression for φ^{*}

$$(7.1)\quad \varphi^{*} = -\frac{1}{8\pi} \int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \int_{-\infty}^{\infty} \frac{H^{*}(w_o,w)\exp\left[-r_o \operatorname{ch}(w_o+i\vartheta_o-\tfrac{1}{2}i\pi)-r\operatorname{ch}(w+i\vartheta+\tfrac{1}{2}i\pi)\right]}{\operatorname{ch}\tfrac{1}{2}(w_o+w)\operatorname{ch}\tfrac{1}{2}(w_o-w-i\pi)} dw_o\, dw,$$

where

$$(7.2)\qquad H^{*}(w_o,w) = \frac{\operatorname{ch}^2 w_o + 2\operatorname{ch}\alpha \operatorname{ch}\tfrac{1}{2}(w_o+\tfrac{1}{2}i\pi)\operatorname{ch}\tfrac{1}{2}(w-\tfrac{1}{2}i\pi)}{K^{+}(w_o)\, K^{-}(w)} .$$

In the same way as before expressions for G^{*} can be obtained from (7.1) and (7.2).

8. Conclusion

In the preceding section we have derived Green's functions for the modified Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 1\right)G = -2\pi\,\delta(x-x_o)\,\delta(y-y_o),$$

with either the boundary conditions

$$\left(\frac{\partial}{\partial y} \mp \operatorname{ch}\alpha\right) G = 0,$$

or

$$\left(\frac{\partial^2}{\partial y^2} \mp \operatorname{ch}\alpha \frac{\partial}{\partial y}\right) G = 0.$$

These equations were obtained from Maxwell's equations by introduction of non-dimensional variables and application of a Laplace transformation. Returning to the original variables it follows from the preceding sections that the Laplace transform of a Hertzian dipole is given by:

1) if it has the direction of the positive X-axis,

$$(8.1) \qquad \vec{\pi}_1 = Z\ G^*(pr\sqrt{\varepsilon\mu}/c,\ \vartheta, pr_0\sqrt{\varepsilon\mu}/c, \vartheta_0, \alpha)\ \vec{i}\ ,$$

with $Z = \sqrt{\mu/(\varepsilon + \sigma/p)}$; $Z_m = \sqrt{\mu_m/(\varepsilon_m + \sigma_m/p)}$, $\operatorname{ch}\alpha = Z_m/Z$;

2) if it has the direction of the positive Y-axis

$$(8.2) \qquad \vec{\pi}_2 = Z\ G(pr\sqrt{\varepsilon\mu}/c, \vartheta, pr_0\sqrt{\varepsilon\mu}/c, \vartheta_0, \alpha)\ \vec{j}\ ,$$

with $\operatorname{ch}\alpha = Z_m/Z$;

3) if it has the direction of the positive Z-axis

$$(8.3) \qquad \vec{\pi}_3 = Z\ G(pr\sqrt{\varepsilon\mu}/c, \vartheta, pr_0\sqrt{\varepsilon\mu}/c, \vartheta_0, \alpha)\ \vec{k}\ ,$$

with $\operatorname{ch}\alpha = Z/Z_m$.

For large values of σ_m and small values of σ we find in the former two cases $\operatorname{ch}\alpha > 1$, hence $\operatorname{Im}\alpha = 0$. In the latter case, however, $\operatorname{ch}\alpha < 1$, hence $0 < \operatorname{Im}\alpha < \frac{1}{2}i\pi$.

The formulae in this paper have been derived for real values of p. However all functions are analytic in p and hence can be continued to complex values. This property is important if all fields are harmonic in time. Taking a timefactor $\exp(-i\omega t)$, the amplitude functions can be obtained by replacing p by $-i\omega$. If this is done in particular in the special case $r_0 \longrightarrow \infty$ (section 6) the formulae describe the diffraction of a plane monochromatic wave. This is the problem investigated by SENIOR [6]. His solution agrees with ours if in (6.1), (6.2) and (6.3) G^* is replaced by ZG^*, r and r_0 are multiplied by $-i\omega\sqrt{\varepsilon\mu}/c$ and if we take $Z_m = \sqrt{\mu_m/(\varepsilon_m + i\sigma_m/\omega)}$ and $Z = \sqrt{\mu/\varepsilon}$.

It would be of interest to perform the inverse Laplace transformation. In this way transient phenomena could be investigated. The inverse transformation, however, proves to

be practicable only approximately. This is due to the dependence of Z_m and hence of ch α, on p.
In the case of sound waves (cf. section 2) ch α is independent of p, and it turns out that the inverse transformation can be performed in closed form. This will be shown in the forthcoming paper by H.A. LAUWERIER already referred to.

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