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On the Numerical Calculation of Elliptic Integrals of the first and second kind and the Elliptic Functions of Jacobi

by

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Introduction

In this paper formulae will be given, which can be used for accurate and fast computation of Elliptic Integrals and Jacobian Elliptic Functions.
Extensive theory can be found in a monograph by A.V. King [1].
All formulae are based upon Gauss theory of the arithmetico-geometrical means [2], and Legendre's method of computation.
Programs to obtain 12 significant figures were set up in ALGOL-60, and run on the X1 computer of the Mathematical Centre.
Recently a series expansion method has been proposed by DiDonati and Hershey [3], which they claim to be superior to Legendre's method. Their ALGOL program [4] was also run on the X1.
A judicious application of Gauss' theory, however resulted in a more compact and faster program than the series expansion program.

We have the usual definitions for the Incomplete Elliptic integrals

(1.1) $$F(k,\varphi)=\int_0^{\varphi}(1-k^2\sin^2\varphi)^{-\frac{1}{2}}\,d\varphi$$ we assume $0\leqq k\leqq 1$,

(1.2) $$E(k,\varphi)=\int_0^{\varphi}(1-k^2\sin^2\varphi)^{+\frac{1}{2}}\,d\varphi\,.$$

Moreover we consider the function *)

(1.3) $$B(k,\varphi)=\int_0^{\varphi}\cos^2\varphi(1-k^2\sin^2\varphi)^{-\frac{1}{2}}\,d\varphi\,,$$ which is

related to F and E by

(1.4) $$B(k,\varphi)=\frac{1}{k^2}\left[E(k,\varphi)-k'^2F(k,\varphi)\right],\quad k'=(1-k^2)^{\frac{1}{2}}\,.$$

For small values of k both numerator and denominator of (1.4) become small, which results in a loss of significant figures if B is calculated from tables of E and F.

*) In the Technical note TN 28, Mathematical Centre, November 1962, a table in 11 decimals is given of $B(k,\varphi)$ with $\varphi = 1^\circ(1^\circ)90^\circ$, $k = \sin[1^\circ(1^\circ)45^\circ]$.

The Complete Elliptic Integrals are defined by

(1.5) $$K(k) = F(k, \tfrac{\pi}{2}) ,$$

(1.6) $$\mathcal{E}(k) = E(k, \tfrac{\pi}{2}) ,$$

(1.7) $$B(k) = B(k, \tfrac{\pi}{2}) .$$

If $F(k,\varphi)=u$, the inverse function is called the amplitude function $\varphi=\mathrm{am}(u,k)$.
The Jacobian Elliptic functions are defined by

(1.8) $$\mathrm{sn}(u,k) = \sin \mathrm{am}(u,k) ,$$

(1.9) $$\mathrm{cn}(u,k) = \cos \mathrm{am}(u,k) ,$$

(1.10) $$\mathrm{dn}(u,k) = (1-k^2 \, \mathrm{sn}^2(u,k))^{\frac{1}{2}}.$$

<u>Arithmetico-geometrical means</u>

Let a_o and b_o be two positive numbers. We then define the following sequences: $\{a_n\}$, $\{b_n\}$ as follows

$$a_{n+1} = \frac{a_n+b_n}{2} \; ; \; b_{n+1} = (a_n \, b_n)^{\frac{1}{2}}; \; n=0,1,2,\ldots \; .$$

It can easily be shown that these sequences converge quadratically to the same limit, which is denoted by ag $M(a_o,b_o)$.

In the following the letters a_n and b_n will be used exclusively if they occur in the just mentioned sequences.

<u>The Incomplete Elliptic Integrals</u>

Let us define the following functions

(2.1) $$F^*(a,b;\varphi)= \frac{1}{a} F((1- \frac{b^2}{a^2})^{\frac{1}{2}},\varphi)= \int_o^{\varphi} (a^2\cos^2\varphi +b^2\sin^2 \varphi)^{-\frac{1}{2}}d\varphi,$$

(2.2) $$E^*(a,b;\varphi)=a\ E\left(\left(1-\frac{b^2}{a^2}\right)^{\frac{1}{2}},\varphi\right)=\int_0^{\varphi}(a^2\cos^2\varphi+b^2\sin^2\varphi)^{\frac{1}{2}}\,d\varphi,$$

(2.3) $$F'(a,b;\varphi)=\frac{1}{a}\ F\left(\frac{b}{a},\ \varphi\right)=\int_0^{\varphi}(a^2-b^2\sin^2\varphi)^{-\frac{1}{2}}\,d\varphi,$$

(2.4) $$E'(a,b;\varphi)=a\ E\left(\frac{b}{a},\ \varphi\right)=\int_0^{\varphi}(a^2-b^2\sin^2\varphi)^{\frac{1}{2}}\,d\varphi.$$

The Landen transformation [5] is given by

(2.5) $$\tan\ \varphi_1=\frac{(1+k')\tan\ \varphi}{1-k'\ \tan^2\varphi},$$

or if we write $k'=b_o/a_o$, $0<a\leq 1$, $0\leq b\leq 1$, by

(2.6) $$\tan\ \varphi_1=\frac{(a_o+b_o)\tan\ \varphi}{a_o-b_o\ \tan^2\varphi}.$$

Substitution in (2.1) and (2.2) gives

(2.7) $$F^*(a_o,b_o;\varphi)=\tfrac{1}{2}\ F^*(a_1,b_1;\varphi_1)\ \text{where}\ a_1=\frac{a_o+b_o}{2}\ ;\ b_1=(a_o\ b_o)^{\frac{1}{2}},$$

(2.8) $$E^*(a_o,b_o;\varphi)=E^*(a_1,b_1;\varphi_1)-\frac{b_oa_o}{2}\ F^*(a_1,b_1;\varphi_1)+\frac{a_o-b_o}{2}\sin\varphi_1.$$

If we transform in the opposite direction writing

$$\tan\ \varphi=\frac{(1+k_1')\tan\ \varphi_1}{1-k_1'\ \tan^2\varphi_1}\quad\text{where}\ k_1=\frac{2\sqrt{k}}{1+k},$$

we get with $k=b_o/a_o$

(2.9) $$a_1\ \cot\ \varphi_1=\tfrac{1}{2}\left\{a_o\ \cot\varphi+(a_o^2\ \cot^2\varphi+a_o^2-b_o^2)^{\frac{1}{2}}\right\}.$$

Substitution in (2.3) and (2.4) gives

(2.10) $$F'(a_o,b_o;\varphi)=F'(a_1,b_1;\varphi_1)\ \text{where}\ a_1=\frac{a_o+b_o}{2}\ ;\ b_1=(a_o\ b_o)^{\frac{1}{2}},$$

(2.11) $$E'(a_o,b_o;\varphi)=2\ E'(a_1,b_1;\varphi_1)+(a_o-b_o)a_1F'(a_1,b_1;\varphi_1)-b_o\sin\varphi.$$

Gauss' transformation [2] is given by

$$\sin \varphi_1 = \frac{(1+k)\sin \varphi}{1+k \sin^2 \varphi}$$ which becomes if we substitute $k=b_o/a_o$

(2.12) $$\sin \varphi_1 = \frac{(a_o+b_o)\sin \varphi}{a_o+b_o \sin^2 \varphi} .$$

Substitution in (2.3) and (2.4) gives

(2.13) $F'(a_o,b_o;\varphi) = \frac{1}{2} F'(a_1,b_1;\varphi_1)$ with $a_1 = \frac{a_o+b_o}{2}$; $b_1 = (a_o\ b_o)^{\frac{1}{2}}$,

(2.14) $E'(a_o,b_o;\varphi) = E'(a_1,b_1;\varphi_1) + \frac{a_o^2-b_o^2}{4} F'(a_1,b_1;\varphi_1) - b_o \sin\varphi \cos\varphi_1 .$

An iterative program based on (2.14) may lead to the loss of significant figures, as $\cos\varphi_1$ must be calculated from $\sin\varphi_1$. Reversion of this transformation results in

$$\sin \varphi = \frac{(1+k_1)\sin \varphi_1}{1+k_1 \sin^2 \varphi_1}$$ with $k_1 = \frac{1-k'}{1+k'}$, or if we write $k'=b_o/a_o$

$$\sin \varphi = \frac{2a_o \sin \varphi_1}{(a_o+b_o)+(a_o-b_o)\sin^2 \varphi_1}$$

Substitution in (2.1) and (2.2) gives

(2.15) $$F^*(a_o,b_o;\varphi) = F^*(a_1,b_1;\varphi_1),$$

(2.16) $$E^*(a_o,b_o;\varphi) = 2\ E^*(a_1,b_1;\varphi_1) - b_1 a_1\ F^*(a_1,b_1;\varphi_1) + (a_o-b_o)\sin \varphi_1 \cos \varphi .$$

It leads however to the same difficulty.
We therefore based our programs on iterative use of formulae (2.7), (2.8), (2.10) and (2.11).

One can easily verify that the calculation of ag M (1,b), where $1 \geq b > 0,3$ needs at most three cycles, and for $b > 0.9539 = (1-0.3^2)^{\frac{1}{2}}$ only two cycles, to obtain 12 significant figures.

Thus when $k < 0.9539$ we choose $a_0=1$, $b_0=k'$ and base our program on formulae (2.7) and (2.8).

With $\tan \varphi_1 = \dfrac{(a_0+b_0)\tan\varphi}{a_0-b_0\tan^2\varphi}$ or $a_1 \cot \varphi_1 = \frac{1}{2}(a_0 \cot\varphi - \dfrac{b_0\, a_0}{a_0 \cot\varphi})$

and $a_0=1$, $b_0=k'$, we get, after rearranging terms, so that loss of significant figures can not occur,

(2.17) $$F(k,\varphi) = F^*(1,k';\varphi) \approx \frac{\varphi_4}{8(a_3+b_3)} \quad ,$$

(2.18) $$E(k,\varphi) = E^*(1,k';\varphi) \approx$$

$$\frac{1}{4}\left\{\frac{2\varphi_4}{a_3+b_3}\left[\frac{1}{4} - \sum_{i=0}^{3} 2^{i-3}(a_i^2-b_i^2)\right] + \sum_{i=0}^{3}(a_i^2-b_i^2)\frac{\sin\varphi_{i+1}}{a_{i+1}}\right\},$$

(2.19) $$B(k,\varphi) \approx \left\{\frac{1}{2} - \sum_{i=1}^{3} 2^{i-1}p_i\right\}\frac{\varphi_4}{8(a_3+b_3)} + \sum_{i=0}^{3}\frac{p_i}{4}\frac{\sin\varphi_{i+1}}{a_{i+1}} \, ,$$

where $$p_i = \frac{a_{i-1}-b_{i-1}}{4(a_{i-1}+b_{i-1})} \cdot p_{i-1} \, , \quad p_0=1.$$

φ_4 is calculated with the formula

$$\varphi_4 = \arctan\left(\frac{1}{\cot\varphi_4}\right) + \left\{\sum_{j=0}^{4} 2^{4-j}\,\frac{1-\mathrm{sgn}(\cot\varphi_j)}{2} + \frac{1-\mathrm{sgn}(\cot\varphi_4)}{2}\right\}\cdot\frac{\pi}{2}.$$

When $k \geq 0.9539$ we choose $a_0=1$, $b_0=k$ and base our program on formulae (2.10) and (2.11).

Iterating two times, we get

$$(2.20)\quad F(k,\varphi)=F'(1,k;\varphi)\approx \frac{2}{a_2+b_2}\ \ln \frac{1+(1+\cot^2\varphi_3)^{\frac{1}{2}}}{\cot\varphi_3}\ ,$$

$$(2.21)\quad E(k,\varphi)=E'(1,k;\varphi)\approx \frac{1}{a_2+b_2}\left\{k'^2+\sum_{i=1}^{2} 2^i(a_i^2-b_i^2)\right\}\ .$$

$$.\ \ln \frac{1+(1+\cot^2\varphi_3)^{\frac{1}{2}}}{\cot\varphi_3}+\sum_{i=0}^{2} 2^i\ b_i(\sin\varphi_3-\sin\varphi_i)+\sin\varphi_3\ ,$$

$$(2.22)\quad B(k,\varphi)\approx \frac{1}{k^2}\left[\frac{1}{a_2+b_2}\left\{-k'^2+\sum_{i=1}^{2} 2^i(a_i^2-b_i^2)\right\}\ \ln \frac{1+(1+\cot^2\varphi_3)^{\frac{1}{2}}}{\cot\varphi_3}+\right.$$

$$\left.+\sum_{i=0}^{2} 2^i\ b_i(\sin\varphi_3-\sin\varphi_i)+\sin\varphi_3\right].$$

A fast and compact ALGOL-60 program was easily obtained, and run on the X1. Comparing tested values with a 12 decimal table of Legendre - Emde [6], we found a relative error of the order 10^{-12}.
The ALGOL program of the series expansion method by DiDonati and Hershey ([3], [4]) was also run on the X1.
It turned out that our program is about four times faster.
Further on we observed that for $\varphi=1^\circ$ and $k=\sin 89^\circ$, the series expansion program did not give an answer within an acceptable time, in fact the calculation needed more then 10 minutes.

The Complete Elliptic Integrals

We only write down the formulae, which can be easily derived from the foregoing formulae.
We mention here that Morgan Ward [7] derived similar formulae for the Complete Elliptic Integral of the third kind.

When $k < 0.9539$ we get

$$K(k) \approx \frac{\pi}{a_3+b_3} \quad \text{where } a_o=1;\ b_o=k' ,$$

$$\mathcal{E}(k) \approx \left\{\frac{a_o^2+b_o^2}{2} - \sum_{i=1}^{3} 2^{i-1}(a_i^2-b_i^2)\right\} \frac{\pi}{a_3+b_3} ,$$

$$B(k) \approx \left\{\tfrac{1}{2} - \sum_{i=1}^{3} 2^{i-1} p_i\right\} \frac{\pi}{a_3+b_3} \quad \text{where } p_i = \frac{a_{i-1}-b_{i-1}}{4(a_{i-1}+b_{i-1})}\cdot p_{i-1}, p_o=1$$

When $k \gtrsim 0.9539$ we get if we use

$$K(k) = \sum_{n=0}^{\infty} \left\{\frac{(\frac{1}{2})_n}{n!}\right\}^2 \left\{\psi(n+1) - \psi(n+\tfrac{1}{2}) - \ln k'\right\} k'^{2n} ,$$

$$K(k) \approx \frac{1}{4\ a_3} \ln \frac{a_3\ a_2\ a_1^2}{(\frac{k'}{4})^4} ,$$

$$\mathcal{E}(k) \approx a_3 + \sum_{i=0}^{2} 2^i (a_i^2-b_i^2) \frac{1}{8\ a_3} \ln \frac{a_3\ a_2\ a_1^2}{(\frac{k'}{4})^4},$$

$$B(k) \approx \frac{1}{k^2}\left\{a_3 + \left[-(a_o^2-b_o^2)+2(a_1^2-b_1^2)+4(a_2^2-b_2^2)\right] \frac{1}{8\ a_3} \ln \frac{a_3\ a_2\ a_1^2}{(\frac{k'}{4})^4}\right\} .$$

Results were checked with a 15 decimal table [8].
The relative errors were all of the order 10^{-12}.

<u>The Elliptic Functions of Jacobi</u>

Inverting formula (2.15) by which we can obtain $F(k,\varphi)$, we get $\sin \varphi_o$ with known $F(k,\varphi)$.

As $\frac{\varphi_4}{a_4} \approx u=F(k,\varphi)$ we find $\sin \varphi_o$ by $\sin \varphi_i = \dfrac{2a_i \sin \varphi_{i+1}}{(a_i+b_i)+(a_i-b_i)\sin^2 \varphi_{i+1}}$,

with $a_o=1$, $b_o=k'$ for $k < 0.9539$.
(the same is done by Salzer [9])

When $\varphi_4 \leq \frac{\pi}{4}$ then $\varphi_0 \leq \frac{\pi}{4}$, we use

$cn(u,k) \approx (1-\sin^2 \varphi_0)^{\frac{1}{2}}$ (which is not suitable when $\varphi_0 > \frac{\pi}{4}$!).

When $\varphi_4 > \frac{\pi}{4}$ we use the well-known relation

$$(3.1) \quad cn(u,k)=k' \frac{sn(K-u,k)}{dn(K-u,k)} \approx \frac{k' \sin \varphi_0^*}{(1-k'^2 \sin^2 \varphi_0^*)^{\frac{1}{2}}} \text{ where } K=K(u),$$

and $\sin \varphi_0^*$ is obtained from $\varphi_4^* = a_4(K-u) = \frac{\pi}{2} - a_4 u$.

For $dn(u,k)$ we use similar formulae

$$dn(u,k) \approx (1-k^2 \sin^2 \varphi_0)^{\frac{1}{2}} \quad \text{for } \varphi_4 \leq \frac{\pi}{4} ,$$

$$(3.2) \quad dn(u,k) = \frac{k'}{dn(K-u,k)} \approx \frac{k'}{(1-k^2 \sin^2 \varphi_0^*)^{\frac{1}{2}}} \quad \text{for } \varphi_4 > \frac{\pi}{4} .$$

When $k > 0.9539$ we invert formula (2.10) and obtain $\tan \varphi_3 = \sinh(a_3 u)$, then $\tan \varphi_0$ can be calculated with

$$\tan \varphi_i = \frac{2 a_i \tan \varphi_{i+1}}{(a_i+b_i)-(a_i-b_i)\tan^2 \varphi_{i+1}} .$$

Then $sn(u,k) = \sin \varphi_0 = \frac{\tan \varphi_0}{(1+\tan^2 \varphi_0)^{\frac{1}{2}}}$, $cn(u,k) = \frac{1}{(1+\tan^2 \varphi_0)^{\frac{1}{2}}}$.

When $\frac{u}{K} > \frac{1}{2}$ a numerically better formula can be obtained if we use (3,1); then

$$\tan \varphi_3^* = \sinh \{a_3(K-u)\}$$

and we get

$$cn(u,k) \approx \frac{k' \tan \varphi_0^*}{(1+k'^2 \tan^2 \varphi_0^*)^{\frac{1}{2}}} .$$

In the same way we get

$$\mathrm{dn}(u,k) \approx \left(\frac{1+k'^2\ \tan^2 \varphi_0}{1+\tan^2\ \varphi_0}\right)^{\frac{1}{2}} \text{ when } \frac{u}{K} \lessapprox \tfrac{1}{2},$$

$$\mathrm{dn}(u,k) \approx k' \left(\frac{1+\tan^2\ \varphi_0^*}{1+k'^2\ \tan^2\ \varphi_0^*}\right)^{\frac{1}{2}} \text{ when } \frac{u}{K} > \tfrac{1}{2}.$$

Test values could be checked with a 12 decimal table [8]. Again the relative error was of the order 10^{-12}.

The ALGOL-60 procedures will now be described.

```
comment procedure for the Complete Elliptic Integral of the first kind
(formula (1,5)), A = arcsin (k), 0 ≤ A ≤π/2;
real procedure K (A); value A; real A;
begin   real a1, a2, b, k1; integer n; b:= abs (sin (A)); if b ≥ .9539
        then    begin   a1:= (1+b)/2; b:= sqrt(b); a2:= (a1+b)/2;
                        b:=sqrt (a1×b); K:= ln (128×(a2+b)×a2×a1↑2/
                        k1↑4)/(a2+b)/2
                end
        else    begin   b:= abs (cos (A)); a1:=1; for n:=1, 2, 3 do
                        begin a2:= (a1+b)/2; b:= sqrt (a1×b); a1:= a2
                        end; K:= 3.14159265359/(a1+b)
                end
end K;

comment procedure for the Complete Elliptic Integral of the second kind
(formula (1,6)), A = arcsin (k), 0 ≤ A ≤π/2;
real procedure E (A); value A; real A;
begin   real a1, a2, b, s, k1; integer n; b:= abs (sin (A));
        k1:= abs (cos (A)); if b ≥ .9539
        then    begin   a1:= (1+b)/2; s:= k1×k1/2+a1×a1-b; b:= sqrt (b);
                        a2:= (a1+b)/2; A:= a1×b; s:= s/2+a2×a2-A;
                        b:= sqrt (A); E:= (a2+b)/2+(s/(a2+b))× ln (128×
                        (a2+b)×a2×a1×a1/(k1↑4))
                end
        else    begin   b:= k1; a1:= 1; s:= 1 + b × b;
                        for n:= 1, 2, 3 do
                        begin a2:= (a1+b)/2; A:= a1×b; a1:= a2;
                              s:= s/2-a1×a1+A; b:= sqrt (A)
                        end; E:= 12.5663706144 × s /(a1+b)
                end
end E;

comment procedure for the Complete Elliptic Integral (formula (1,7)),
A = arcsin (k), 0 < A ≤π/2;
real procedure B (A); value A; real A;
begin   real a1, a2, b, b1, s, k1; integer n; b:= abs (sin (A));
        k1:= abs (cos (A)); b1:= b; if b ≥ .9539
        then    begin   a1:= (1+b)/2; s:= a1×a1 - k1×k1 /2 -b;
                        b:= sqrt (b); a2:= (a1+b)/2; A:= a1×b;
                        b:= sqrt (A); s:= s/2+a2×a2-A;
                        B:= ((a2+b)/2+(s/(a2+b))× ln (128×(a2+b)×a2×a1×
                        a1/(k1↑4)))/(b1×b1)
                end
        else    begin   b:= k1; a1:= 1; s:= 0; k1:= 1;
                        for n:= 1, 2, 3 do
                        begin k1:= .25 × k1 × (a1 - b)/(a1 + b);
                              a2:= (a1+b)/2; A:= a1×b; a1:= a2;
                              s:= s/2 - k1; b:= sqrt (A)
                        end; B:= 3.14159265359 × (.5 + s × 4)/(a1 + b)
                end
end B;
```

```
comment procedure for the Incomplete Elliptic Integral of the first kind
(formula (1,1)), A = arcsin (k), P = φ, 0 ≤ A, P ≤ π/2;
real procedure  F (A, P); value A, P; real  A, P;
begin   real  a, a1, b, b1, si; integer  n, m; b:= abs (cos (A));
        b1:= abs (sin (A)); if  b1 < .9539
        then     begin   a:= 1; si:= cos (P)/sin (P); n:=0;  A:= b;
                         for m:= 1, 2, 3 do
                         begin  si:= ( si-A/si)/2; n:= 2×n + (1 - sign (si))
                                /2; a:= (a+b)/2; b:=  sqrt (A); A:= a×b
                         end; si:= (si - A/si)/2; n:= 2×n + 1 - sign (si);
                         a:= (a+b)/2; F:= ((2 × arctan ( a/si) + n ×
                         3.14159265359)/a)/32
                 end
        else     begin   a:= 1; b:= b1; si:= cos (P)/sin (P); a1:= b×b;
                         for m:= 1, 2 do
                         begin A:= a×b; si:= a×si + sqrt (a×a×(si×si + 1)
                               - a1); a:= (a+b)/2; si:= si/(a×2);
                               b:= sqrt (A); a1:= A
                         end; si:= a×si + sqrt (a×a×(si×si + 1) - A);
                         a:= (a+b)/2; si:= si/(a×2); F:= ln ((1 + sqrt
                         (1 + si×si))/si)/a
                 end
end F;

comment procedure for the Incomplete Elliptic Integral of the second
kind (formula (1,2)), A = arcsin (k), P = φ, 0 ≤ A, P ≤ π/2;
real procedure  E (A, P); value A, P; real  A, P;
begin   real U, V;
        procedure  EA;
        begin   real  a, a1, b, S1, S2, si, co; integer n, m;
                b:= U; a:= 1; co:= cos(P)/sin (P); S1:= S2:= 0; a1:= b×b;
                n:= 0; for m:= 1, 2, 3, 4 do
                begin   A:= a×b; co:= (co - A/co)/2; n:= 2×n + (1 - sign
                        (co))/2; P:= a×a; a:= (a+b)/2; P:= P - a1;
                        S2:= S2 + P × sign (1.5 - n + (n:4)×4)/sqrt
                        a×a +co×co); S1:= S1/2 + P; a1:= A; b:= sqrt (A)
                end; E:= (((2 × arctan (a/co) + (n +(1 - sign (co))/2) ×
                3.14159265359) × (.250 - S1)/a)/2 + S2)/4
        end;
        procedure  EB;
        begin   real a, S1, co, si; real array b [0:3], SIN [0:3];
                integer m; a:= 1; S1:= U × U; b [0]:= V; si:= sin (P);
                co:= cos (P)/si; SIN [0]:= si; b [3]:= 1 + V;
                co:= (co + sqrt (co × co + 1 - V × V))/b [3]; si:= co×co;
                for m:= 0, 1, 2 do
                begin   SIN [m+1]:= 1/sqrt (1 + si); A:= a × b[m];
                        a:= b [3]/2; b [m+1]:= sqrt (A); P:= a × a;
                        S1:= S1/2 + P - A; b [3]:= a + b [m+1];
                        co:= (a × co + sqrt (P × (si + 1) - A))/b [3];
                        si:= co × co
                end; si:= sqrt (1 + si); P:= 1/si; U:= 0; b [3]:= a;
                for m:= 3, 2, 1, 0 do U:= 2 × U + b [m] × (P - SIN [m]);
                E:= 4 × S1 × ln ((1 + si)/co)/a + P + U
        end;
        U:= abs (cos (A)); V:= abs (sin (A)); if V < .9539 then EA else EB
end E;
```

```
comment procedure for the Incomplete Elliptic Integral (formula (1,3)),
A = arcsin (k), P = φ, 0 ≤ A, P ≤ π/2;
real procedure B (A, P); value A, P; real A, P;
begin   real U, V;
        procedure BA;
        begin   real a,b, S1, S2, co; integer n, m; array p [0:4];
                b:= U; a:= 1; co:= cos(P)/sin (P); S2:= S1:= 0; n:= 0;
                p [0]:= 1; for m:= 1, 2, 3, 4 do
                begin   A:= a×b; co:= (co - A/co)/2; n:= 2×n + (1 - sign
                        (co))/2; p [m]:= .25 × p [m - 1] × (a - b)/(a + b);
                        a:= (a+b)/2;S1:= S1/2 + p [m]; S2:= S2 + p [m - 1]
                        × sign (1.5 - n + (n:4)×4)/sqrt (a×a +co×co);
                        b:= sqrt (A)
                end; B:= (2 × arctan (a/co) + (n +(1 - sign (co))/2) ×
                3.14159265359) × (.015625 - S1/4)/a + S2/4
        end;
        procedure BB;
        begin   real a, S1, co, si; real array b [0:3], SIN [0:3];
                integer m; a:= 1; S1:= - U × U; b [0]:= V; si:= sin (P);
                co:= cos (P)/si; SIN [0]:= si; b [3]:= 1 + V; V:= V × V;
                co:= (co + sqrt (co × co + 1 - V))/b [3]; si:= co × co;
                for     m:= 0, 1, 2 do
                begin   SIN [m+1]:= 1/sqrt (1 + si); A:= a × b[m];
                        a:= b [3]/2; b [m+1]:= sqrt (A); P:= a × a;
                        S1:= S1/2 + P - A; b [3]:= a + b [m+1]; co:= (a × co
                        + sqrt (P × (si + 1) - A))/b [3]; si:= co × co
                end; si:= sqrt (1 + si); P:= 1/si; U:= 0; b [3]:= a;
                for m:= 3, 2, 1, 0 do U:= 2 × U + b [m] × (P - SIN [m]);
                B:= (4 × S1 × ln ((1 + si)/co)/a + P + U)/V
        end;
        U:= abs (cos (A)); V:= abs (sin (A)); if V < .9539 then BA else BB
end B;

comment procedure for the Jacobian Elliptic Function (formula (1,8)),
A = arcsin (k), 0 ≤ A ≤ π/2, 0 ≤ u ≤ K (k);
real procedure sn (u, A); value A, u; real A, u;
begin   real array a [0:3], b [0:3]; real a1, t; integer i;
        a [0]:= 1; b [0]:= abs (sin (A)); if b [0] ≥ .9539
        then    begin   for i:= 1, 2 do
                        begin a1:= (a [i-1] + b [i-1])/2;
                              b [i]:= sqrt (a [i-1] × b [i-1]); a [i]:= a1
                        end; a1:= exp ((a [2] + b [2]) × u); t:= (a1 - 1)/
                        (2 × sqrt (a1)); for i:= 2, 1, 0 do t:= 2 × a [i] ×
                        t/(a [i] + b [i] - (a [i] - b [i]) × t × t);
                        sn:= t/sqrt (1 + t × t)
                end
        else    begin   b [0]:= abs (cos (A)); for i:= 1, 2, 3 do
                        begin a1:= (a [i-1] + b [i-1])/2; b [i]:= sqrt (
                              a [i-1] × b [i-1]); a [i]:= a1
                        end; a1:= (a [3] + b [3]) × u/2; t:= sin (a1);
                        for i:= 3, 2, 1, 0 do t:= 2 × a [i] × t/(a [i] +
                        b [i] + (a [i] - b [i]) × t × t); sn:= t
                end
end sn;
```

```
comment procedure for the Jacobian Elliptic Function (formula (1,9)),
A = arcsin (k), 0 ≤ A ≤ π/2, 0 ≤ u ≤ K (k);
real procedure  cn (u, A); value A, u; real A, u;
begin   real array a [0:3], b [0:3]; real a1, b1, t,k1; integer i;
        a [0]:= 1; b1:= b [0]:= abs (sin (A)); k1:= abs (cos (A));
        if b [0] ≥ .9539 then
        begin   for i:= 1, 2 do
                begin   a1:= (a [i-1] + b [i-1])/2; b [i]:= sqrt
                        (a [i-1] × b [i-1]); a [i]:= a1
                end; a1:= (ln ((a [2] + b [2]) × a [2] × a [1] × a [1])/2
                + 2.42601513194 - 2 × ln (k1))/(a [2] + b [2]);
                if u/a1 > .5 then
                begin   a1:= exp ((a [2] + b [2]) × (a1 - u));
                        t:= (a1 - 1)/(2 × sqrt (a1)); for i:=  2, 1, 0 do
                        t:= 2 × a [i] × t /(a [i] + b [i] - (a [i] - b [i])
                        × t × t); cn:= k1 × t /sqrt (1 + (k1 × t)↑2)
                end else
                begin   a1:= exp ((a [2] + b [2]) × u); t:= (a1 - 1)/(2 ×
                        sqrt (a1)); for i:= 2, 1, 0 do
                        t:= 2 × a [i] × t /(a [i] + b [i] - (a [i] - b [i])
                        × t × t); cn:= 1/sqrt (1 + t × t)
                end
        end else
        begin   b [0]:= k1; for i:= 1, 2, 3 do
                begin a1:= (a [i-1] + b [i-1])/2; b [i]:= sqrt
                      (a [i-1] × b [i-1]); a [i]:= a1
                end; a1:= (a [3] + b [3]) × u/2; if a1 < .7853981634O then
                begin   t:= sin (a1); for i:= 3, 2, 1, 0 do
                        t:= 2 × a [i] × t/(a [i] + b [i] + (a [i] - b [i])
                        × t × t); cn:= sqrt (1 - t × t)
                end else
                begin   t:= sin (1.57079632679 - a1);
                        for i:= 3, 2, 1, 0 do t:= 2 × a [i] × t/(a [i] +
                        b [i] + (a [i] - b [i]) × t × t);
                        cn:= b [0] × t /sqrt (1 - (b1 × t)↑2)
                end
        end
end cn;
```

```
comment procedure for the Jacobian Elliptic Function (formula (1,10)),
A = arcsin (k), 0 ≤ A ≤ π/2, 0 ≤ u ≤ K (k);
real procedure  dn (u, A); value A, u; real A, u;
begin   real array a [0:3], b [0:3]; real a1, b1, t, k1; integer i;
        a [0]:= 1; b [0]:= b1:= abs (sin (A)); k1:= abs (cos (A));
        if b [0] ≥ .9539 then
        begin   for i:= 1, 2 do
                begin   a1:= (a [i-1] + b [i-1])/2; b [i]:= sqrt
                        (a [i-1] × b [i-1]); a [i]:= a1
                end; a1:= (ln ((a [2] + b [2]) × a [2] × a [1] × a [1])/2
                + 2.42601513194 - 2 × ln (k1))/(a [2] + b [2]);
                if u/a1 > .5 then
                begin   a1:= exp ((a [2] + b [2]) × (a1 - u));
                        t:= (a1 - 1)/(2 × sqrt (a1)); for i:= 2, 1, 0 do
                        t:= 2 × a [i] × t /(a [i] + b [i] - (a [i] - b [i])
                        × t × t); dn:= k1 × sqrt ((1 + t × t)/
                        (1 + (k1 × t)↑2))
                end else
                begin   a1:= exp ((a [2] + b [2]) × u); t:= (a1 - 1)/
                        (2 × sqrt (a1)); for i:= 2, 1, 0 do
                        t:= 2 × a [i] × t /(a [i] + b [i] - (a [i] - b [i])
                        × t × t); dn:= sqrt ((1 + (k1 × t)↑2)/(1 + t × t))
                end
        end else
        begin   b [0]:= k1; for i:= 1, 2, 3 do
                begin   a1:= (a [i-1] + b [i-1])/2; b [i]:= sqrt
                        (a [i-1] × b [i-1]); a [i]:= a1
                end; a1:= (a [3] + b [3]) × u/2;
                if a1 < .7853981634 then
                begin   t:= sin (a1); for i:= 3, 2, 1, 0 do
                        t:= 2 × a [i] × t/(a [i] + b [i] + (a [i] - b [i])
                        × t × t); dn:= sqrt (1 - (b1 × t)↑2)
                end else
                begin   t:= sin (1.57079632679 - a1); for i:= 3, 2, 1, 0 do
                        t:= 2 × a [i] × t/(a [i] + b [i] + (a [i] - b [i])
                        × t × t); dn:= b [0]/sqrt (1 - (b1 × t)↑2)
                end
        end
end dn;
```

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