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Uniformly valid approximations and the singular perturbation method

by

J. Grasman

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Contents

1. Introduction

In this report we investigate the differential equation

$$(1.1) \qquad \varepsilon\Delta\phi - \frac{\partial\phi}{\partial x} = 0 \ , \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \ , \quad 0 < \varepsilon << 1;$$

$\phi(x,y;\varepsilon)$ holds for $x^2 + y^2 \leq 1$. The boundary value is given for $x^2 + y^2 = 1$. It is known (see [7]), that for this problem the singular perturbation method can be applied. It yields a boundary-layer at the part of the boundary for $x > 0$.
In the usual singular perturbation methods the points $(x,y) = (0,1)$ and $(x,y) = (0,-1)$ are excluded in this case. Before we examine these points we introduce the notion of a reduced differential equation.

The reduced equation of a differential equation with a small parameter ε is constructed by posing $\varepsilon = 0$ in the differential equation. For example the reduced equation of (1.1) is $\frac{\partial\phi}{\partial x} = 0$. If we introduce other coordinates by some transformation, then it remains that $\varepsilon = 0$ is substituted. Before this can be done, the terms of the equation need to be $O(\varepsilon^{\alpha})$ with $\alpha \geq 0$ and for at least one term there has to hold $\alpha = 0$. This condition can be satisfied by multiplying the equation with a suitable power in ε.

In the points (0,1) and (0,-1) the characteristics of the reduced equation of (1.1) are tangent to the boundary. The behaviour of the solution $\phi(x,y;\varepsilon)$ is such that an asymptotic expansion of the solution, which holds on a great part of the domain, will not hold in a neighbourhood of these points.
Therefore a special investigation is made by introducing local coordinates in the neighbourhood of these points. As appears from solving this local problem, the behaviour of the solution in the points, where the boundary-layer begins, can be described very well by an intermediate- and an interior boundary-layer expansion, see (2.27) and (2.44).
Finally we compose a uniform valid expansion by taking together the first terms of the outer expansion, the boundar-layer expansion and the intermediate boundary-layer expansion.

We prove that the accuracy is $O(\varepsilon^{1/3})$ in the first and $O(\varepsilon^{4/7})$ in the second approximation.

In this report we take the results of Eckhaus [1] as a starting-point. It is professor W. Eckhaus, who suggested the investigations of this report. In [4] and [5], respectively, Frankena and Mauss have obtained uniform approximations in a different way.
Applying the method of Frankena we obtain for the accuracy $O(\varepsilon^{1/4})$ in the first approximation of $\phi(x,y;\varepsilon)$, as it is defined by (2.1) and (2.2).

2. Local asymptotic approximations

2.1. Introductory remarks

We consider the function $\phi(x,y;\varepsilon)$, satisfying the differential equation

$$(2.1)\qquad L_\varepsilon(\phi) \equiv \varepsilon \left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\} - \frac{\partial\phi}{\partial x} = 0$$

for $x^2 + y^2 \leq 1$ with the boundary condition

$$(2.2)\qquad \phi(x,y;\varepsilon) = \psi(\theta), \text{ where } x = \sin\theta,\ y = \cos\theta.$$

ε is a small positive parameter. We assume that $\phi(x,y;\varepsilon)$ has the expansion

$$(2.3)\qquad \phi(x,y;\varepsilon) = \sum_{n=0}^{\infty} U_n(x,y)\varepsilon^n.$$

By substitution of (2.3) in (2.1) we obtain, after equalization of the coefficients of equal powers of ε, equations for $U_n(x,y)$, $n = 0, 1, 2, \ldots$. It turns out that $U_0(x,y)$ cannot satisfy both the equation $\frac{\partial U_0}{\partial x} = 0$ and the boundary condition $U_0(x,y) = \psi(\theta)$. Moreover, $U_n(x,y)$ for $n = 1, 2, \ldots$ has singularities in (0,1) and (0,-1). In order to investigate the behaviour of the solution in a neighbourhood of the boundary and especially in a neighbourhood of the points (0,1) and (0,-1) we introduce the local coordinates ξ, η:

$$(2.4a)\qquad x = (1-\rho)\sin\theta,\ y = (1-\rho)\cos\theta;$$

$$(2.4b)\qquad \rho = \varepsilon^\nu\xi,\ \theta = \varepsilon^\mu\eta \text{ or } \pi - \theta = \varepsilon^\mu\eta \text{ for } 0 \leq \theta \leq \pi.$$

Substitution of (2.4) in (2.1) and (2.2) leads to

$$(2.5)\qquad L_\varepsilon[\phi] \equiv \varepsilon^{1-2\nu}\frac{\partial^2\phi}{\partial\xi^2} + \frac{\varepsilon^{1-2\mu}}{(1-\varepsilon^\nu\xi)^2}\frac{\partial^2\phi}{\partial\eta^2} + \left[\varepsilon^{\mu-\nu}\left\{\frac{\sin\varepsilon^\mu\eta}{\varepsilon^\mu}\right\} - \frac{\varepsilon^{1-\nu}}{(1-\varepsilon^\nu\xi)}\right]\frac{\partial\phi}{\partial\xi} - \varepsilon^{-\mu}\frac{\cos\varepsilon^\mu\eta}{(1-\varepsilon^\nu\xi)}\frac{\partial\phi}{\partial\eta} = 0,$$

$$(2.6)\qquad \phi[0,\eta;\varepsilon] = \psi(\varepsilon^\mu\eta).$$

For $\nu > 0$, $\mu = 0$ the reduced equation of (2.5) represents the behaviour of the solution in the neighbourhood of the boundary and for $\nu > 0$, $\mu > 0$ in the neighbourhood of (0,1) or (0,-1). We refer to [1] for a description of all possible reduced equations.

We restrict ourselves to four local equations (see fig. 1)

a. $\nu = 0$, $\mu = 0$: the equation (2.1) in x,y-coordinates;

b. $\nu = 1$, $\mu = 0$: the boundary-layer equation;

c. $\nu = 2/3$, $\mu = 1/3$: the intermediate boundary-layer equation;

d. $\nu = 1$, $\mu = 1$: the interior boundary-layer equation;

In the following sections we shall construct with aid of these equations expansions for $\phi(x,y;\varepsilon)$. Each of these expansions will be valid in a part of the domain $x^2 + y^2 \leq 1$; the parts fill up the domain completely.

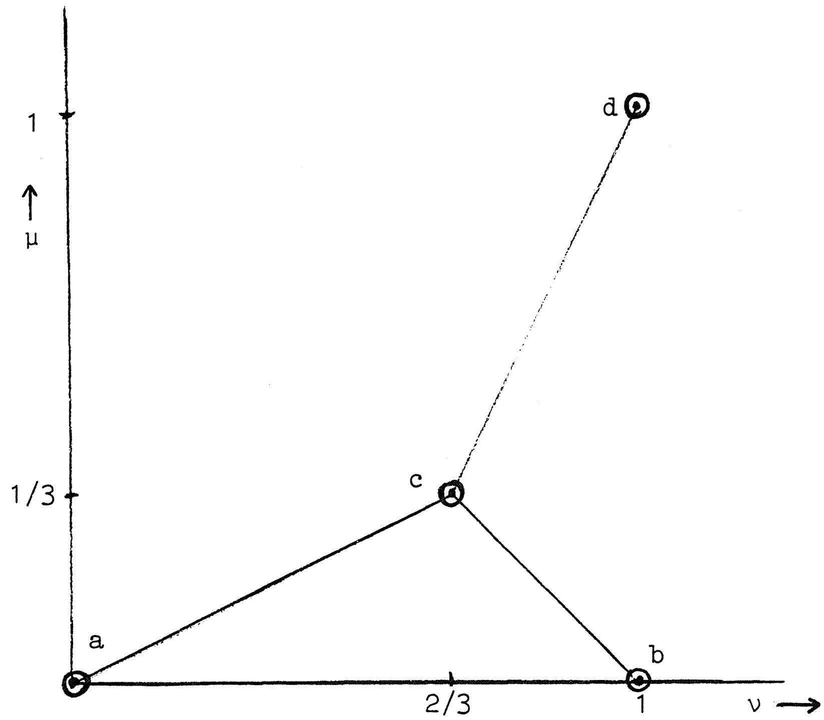

fig. 1

2.2. Expansion of solution outside the boundary layers

We consider the expansion (2.3). $U_n(x,y)$ satisfies

$$\frac{\partial U_0}{\partial x} = 0 \text{ and } \frac{\partial U_n}{\partial x} = \left\{\frac{\partial^2 U_{n-1}}{\partial x^2} + \frac{\partial^2 U_{n-1}}{\partial y^2}\right\}, \quad n = 1, 2, \ldots .$$

In this way we obtain

$$(2.7) \qquad U_0(x,y) = f(y),$$

$$(2.8) \qquad U_n(x,y) = \int_{-\sqrt{1-y^2}}^{x} \left\{\frac{\partial^2 U_{n-1}}{\partial \bar{x}^2} + \frac{\partial^2 U_{n-1}}{\partial y^2}\right\} d\bar{x}, \quad n = 1, 2, \ldots .$$

The function f(y) can satisfy the boundary condition only along a part of the boundary. Theorem IV of [3] shows that this part consists only of the left half of the circle $x^2 + y^2 = 1$, so that we have

$$f(y) = \psi(-\arccos y).$$

$$(2.9) \quad U_1(x,y) = (x + \sqrt{1 - y^2})\,\frac{d^2 f}{dy^2}, \quad \frac{d^2 f}{dy^2} = \frac{1}{1 - y^2}\,\psi''(-\arccos y) + \frac{y}{(1 - y^2)^{3/2}}\,\psi'(-\arccos y).$$

The method of induction delivers after many calculations

$$(2.10) \quad U_n(x,y) = \sum_{k=1}^{2n} \psi^{(k)}(-\arccos y) \cdot \sum_{m=1}^{n} R_{k,m,n}(x,y)\, \frac{(x+\sqrt{1-y^2})^m}{(1-y^2)^{\frac{1}{2}(3n+m-k)}} \; ;$$

$R_{k,m,n}(x,y)$ is a bounded function for $x^2 + y^2 \leq 1$.
There exists a positive number M, such that

$$|\psi^{(k)}(-\arccos y) R_{k,m,n}(x,y)| \leq M$$

for all k, m and n, as $x^2 + y^2 \leq 1$. We make an estimation for $U_n(x,y)$ in the domain $x^2 + y^2 \leq 1$ with exception of the parts A_1 and A_2.

$$A_1: \ 0 \le \rho \le K\varepsilon^{2\mu}, \ 0 \le |\theta| \le K\varepsilon^{\mu};$$

$$A_2: \ 0 \le \rho \le K\varepsilon^{2\mu}, \ 0 \le |\pi-\theta| \le K\varepsilon^{\mu};$$

$$0 < \mu \le 1/3.$$

K is an arbitrary large positive number. It appears that

$$|U_n(x,y)| \le 2nM(K\varepsilon^{\mu})^{1-3n}, \tag{2.11}$$

so that the expansion (2.3) converges.

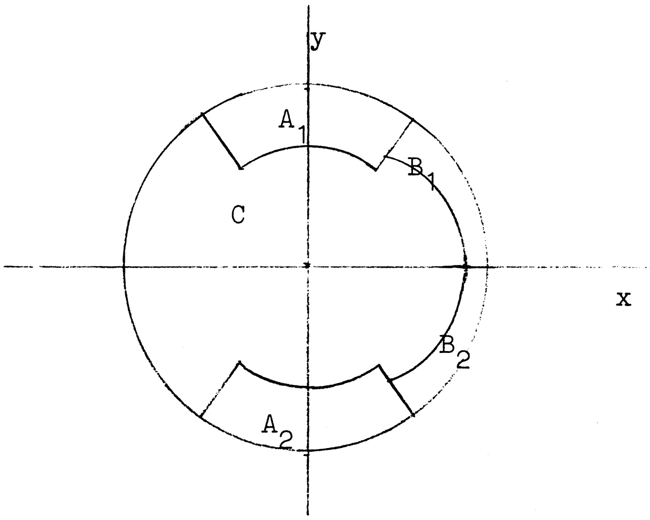

fig. 2a

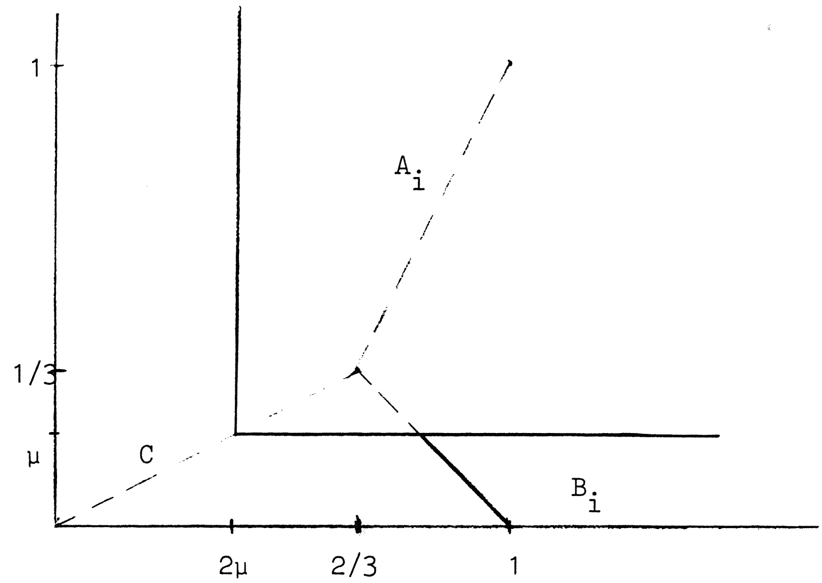

fig. 2b

2.3. Boundary-layer expansion

Equation (2.5) is, for $\nu = 1$, $\mu = 0$ and $\delta \leq \theta \leq \pi-\delta$, where δ is an arbitrary small positive number,

$$(2.12) \qquad \frac{\partial^2\phi}{\partial\xi^2} + \sin\theta\,\frac{\partial\phi}{\partial\xi} = \varepsilon\left[\frac{1}{(1-\varepsilon\xi)}\left\{\frac{\partial\phi}{\partial\xi} + \cos\theta\,\frac{\partial\phi}{\partial\theta}\right\} - \frac{\varepsilon}{(1-\varepsilon\xi)^2}\right].$$

We introduce the boundary-layer expansion

$$(2.13) \qquad \phi(x,y;\varepsilon) = \sum_{n=0}^{\infty} V_n(\xi,\theta)\varepsilon^n.$$

Substitution of (2.13) in (2.12) gives

$$(2.14) \qquad \frac{\partial^2 V_0}{\partial\xi^2} + \sin\theta\,\frac{\partial V_0}{\partial\xi} = 0$$

and

$$(2.15) \qquad \frac{\partial^2 V_1}{\partial\xi^2} + \sin\theta\,\frac{\partial V_1}{\partial\xi} = \frac{\partial V_0}{\partial\xi} + \cos\theta\,\frac{\partial V_0}{\partial\theta}.$$

The general solutions of (2.14) and (2.15) are

$$(2.16) \qquad V_0(\xi,\theta) = A_0(\theta)e^{-\xi\sin\theta} + B_0(\theta)$$

and

$$(2.17) \quad V_1(\xi,\theta) = \left[\frac{A_0(\theta)\cos^2\theta}{2\sin\theta}\,\xi^2 + \left\{\frac{A_0'(\theta)\cos\theta}{\sin\theta} + \frac{A_0(\theta)}{\sin^2\theta}\right\}\xi + A_1(\theta)\right]e^{-\xi\sin\theta} + \frac{B_0'(\theta)\cos\theta}{\sin\theta}\,\xi + B_1(\theta).$$

$A_0(\theta)$, $A_1(\theta)$, $B_0(\theta)$ and $B_1(\theta)$ are determined by the boundary condition and the matching condition.

a. The boundary condition involves

$$(2.18) \qquad V_0(0,\theta) = \psi(\theta), \quad V_1(0,\theta) = 0.$$

b. Satisfying the matching condition means that the expansion (2.13) has to match (2.3) for $\xi >> 1$. Substituting the boundary-layer coordinates in (2.3) and expanding the terms in the neighbourhood of $\rho = 0$, we find

$$(2.19) \qquad \phi[\xi,\theta;\varepsilon] = U_0[0,\theta] + \varepsilon\{\xi\left[\frac{\partial U_0}{\partial\rho}\right]_{\rho=0} + U_1[0,\theta]\} + \varepsilon^2 \ldots .$$

The matching condition is satisfied, if

$$(2.20) \qquad \lim_{\xi\to\infty} V_0(\xi,\theta) = U_0[0,\theta]$$

and

$$(2.21) \qquad \lim_{\xi\to\infty} \{V_1(\xi,\theta) - \xi\left[\frac{\partial U_0}{\partial\rho}\right]_{\rho=0}\} = U_1[0,\theta].$$

From (2.18), (2.20) and (2.21) we deduce

(2.22a) $\quad A_0(\theta) = \psi(\theta) - \psi(-\theta),\quad$ (2.22c) $\quad A_1(\theta) = 0,$

(2.22b) $\quad B_0(\theta) = \psi(-\theta),\quad$ (2.22d) $\quad B_1(\theta) = 0.$

$V_n(\xi,\theta)$ can be written as

$$(2.23) \qquad V_n(\xi,\theta) = P_n(\xi,\theta)e^{-\xi \sin\theta} + Q_n(\xi,\theta),\ n = 1, 2, \ldots .$$

After some laborous calculations we obtain for $n = 2, 3, \ldots$, using (2.12),

$$(2.24a) \qquad P_n(\xi,\theta) = \sum_{k=1}^{2n} S_{pk}(\xi,\theta)\ \xi^k\ \theta^{-3n+k+1},$$

$$(2.24b) \qquad Q_n(\xi,\theta) = \sum_{k=1}^{n} S_{qk}(\xi,\theta)\ \xi^k\ \theta^{-3n+k+1}.$$

$S_{pk}(\xi,\theta)$ and $S_{qk}(\xi,\theta)$ are bounded functions for $0 \le \xi$ and $0 \le \theta \le \pi$. There exists a number M such that

$$\max\{|S_{pk}(\rho/\varepsilon,\theta)|,\ |S_{qk}(\rho/\varepsilon,\theta)|\} \le M \text{ for } k = 1, 2, \ldots .$$

We observe that it is possible to make the following estimation

$$|V_n(\rho/\varepsilon,\theta)| \leq R(M,K)\; n(K\varepsilon^{\mu})^{(1-3n)} \tag{2.25}$$

in the domains B_1 and B_2.

$$B_1:\; 0 \leq \rho \leq K^2\varepsilon/\theta,\; K\varepsilon^{\mu} < \theta \leq \pi/2;$$

$$B_2:\; 0 \leq \rho \leq K^2\varepsilon/(\pi-\theta),\; K\varepsilon^{\mu} < \pi-\theta \leq \pi/2;$$

$$0 < \mu \leq 1/3, \text{ see fig. 2.}$$

In formula (2.25) $R(M,K)$ is a positive number depending on M and K. The expansion (2.13) converges in B_1 and B_2, on account of the validity of estimate (2.25).

2.4. Intermediate boundary-layer expansion

In the present and the following section we only consider the neighbourhood of (0,1). For (0,-1) the computations are completely analogous.

In (2.5) we take $\nu = 2/3$, $\mu = 1/3$; $\xi \geq \delta > 0$ for $0 \leq |\eta| \leq \delta$ and $\xi \geq 0$ for $\delta \leq |\eta|$, where δ is an arbitrary small positive number. In this case the solution of the reduced equation of (2.5) represents the behaviour of $\phi(x,y;\varepsilon)$ in the neighbourhood of (0,1). In the same way as in the preceding section we assert an expansion for $\phi(x,y;\varepsilon)$: the intermediate boundary-layer expansion.

The name "intermediate boundary-layer" has to do with the fact that another boundary-layer exists: the so-called "interior boundary-layer" ($\mu = \nu = 1$). The intermediate boundary-layer is singular in $\xi = \eta = 0$, therefore we construct this interior boundary-layer expansion in the following section.

For $\nu = 2/3$, $\mu = 1/3$ equation (2.5) is

$$(2.26)\quad \frac{\partial^2\phi}{\partial\xi^2} + \eta\frac{\partial\phi}{\partial\xi} - \frac{\partial\phi}{\partial\eta} = \varepsilon^{2/3}\left[\frac{-1}{(1-\varepsilon^{2/3}\xi)^2}\frac{\partial^2\phi}{\partial\eta^2} - \left\{-\eta^3 + \ldots + \frac{1}{(1-\varepsilon^{2/3}\xi)}\right\}\frac{\partial\phi}{\partial\xi} - \frac{\frac{1}{2}\eta^2 + \ldots}{(1-\varepsilon^{2/3}\xi)}\frac{\partial\phi}{\partial\eta}\right].$$

We pose that $\phi(x,y;\varepsilon)$ can be expanded as

$$(2.27)\qquad \phi(x,y;\varepsilon) = \sum_{n=0}^{\infty} Y_n(\xi,\eta)\varepsilon^{n/3}.$$

The terms $Y_n(\xi,\eta)$ have to satisfy matching conditions. They have to match with:

a. the outer expansion (2.3);

b. the boundary-layer expansion (2.13);

c. the interior boundary-layer expansion (2.44).

The usual way of solving the problem for $Y_n(\xi,\eta)$ is: firstly, to deduce the general solution from (2.26) and (2.27); and secondly, to choose the undetermined terms of the general solution such that the matching conditions are satisfied. This method generates a lot of difficult additional problems. We therefore have searched for another method to solve this very problem.

Inspired by P. Roberts [2] we assert that the intermediate boundary-layer has to satisfy the boundary condition

$$Y_n(0,\eta) = \psi^{(n)}(0)\eta^n/n!,\ n = 0,\ 1,\ 2,\ \dots \quad (2.28)$$

This method makes it necessary to verify the matching conditions afterwards. It is obvious that

$$Y_0(\xi,\eta) = \psi(0) \quad (2.29)$$

and

$$\frac{\partial^2 Y_1}{\partial \xi^2} + \eta \frac{\partial Y_1}{\partial \xi} - \frac{\partial Y_1}{\partial \eta} = 0. \quad (2.30)$$

Introduction of a new dependent variable $\phi_1(\xi,\eta)$,

$$Y_1(\xi,\eta) = \exp(-\ 1/2\ \xi\eta - 1/12\ \eta^3)\phi_1(\xi,\eta),$$

leads to

$$\frac{\partial^2 \phi_1}{\partial \xi^2} + \frac{1}{2}\xi\phi_1 = \frac{\partial \phi_1}{\partial \eta}, \quad (2.30)$$

$$\phi_1(0,\eta) = \exp(\tfrac{1}{12}\eta^3)\psi'(0)\eta. \quad (2.31)$$

We note that

$$P_1(\xi,\eta;p) = e^{m^2\eta p} \cdot \mathrm{Ai}(p - m\xi), \quad (2.32a)$$

$$P_2(\xi,\eta;p) = e^{m^2\eta p} \cdot \mathrm{Ai}(\omega p - m\omega\xi), \quad (2.32b)$$

$$P_3(\xi,\eta;p) = e^{m^2\eta p} \cdot \mathrm{Ai}(\omega^2 p - m\omega^2\xi) \quad (2.32c)$$

are solutions of (2.30), where Ai(z) is the Airy-function and p is an arbitrary constant; $m = 2^{-1/3}$, $\omega = \exp(2/3\ \pi i)$, $\omega^2 = \exp(-\ 2/3\ \pi i)$.

Two of the three solutions (2.32) can be chosen independently. By considering these three solutions together we may make use of some special properties of the Airy-functions.
In [2] Roberts introduces the solution

$$(2.33)\quad Y_1(\xi,\eta) = -m^{-2}\psi'(0)\exp(-\tfrac{1}{2}\xi\eta - \tfrac{1}{12}\eta^3)\Big[\omega\int_0^\infty \frac{Ai'(x)}{Ai(x)} Ai(x-m\omega\xi)\,.$$

$$e^{m^2\eta x\omega^2}\,dx + \omega^2\int_0^\infty \frac{Ai'(x)}{Ai(x)} Ai(x-m\omega^2\xi)e^{m^2\eta x\omega}\,dx$$

$$-\,\omega^2\int_0^\infty \frac{Ai'(\omega x)}{Ai(\omega x)} Ai(\omega x - m\omega\xi)e^{m^2\eta x}\,dx\; +$$

$$-\,\omega\int_0^\infty \frac{Ai'(\omega^2 x)}{Ai(\omega^2 x)} Ai(\omega^2 x - m\omega^2\xi)e^{m^2\eta x}\,dx\Big].$$

We investigate the boundary value

$$(2.34)\quad Y_1(0,\eta) = -m^{-2}\psi'(0)\exp(-\tfrac{1}{12}\eta^3)\int_0^\infty \{Ai'(x)(\omega e^{m^2\eta x\omega^2} + \omega^2 e^{m^2\eta x\omega}) +$$

$$-\,(Ai'(\omega x)\omega^2 + Ai'(\omega^2 x)\omega)e^{m^2\eta x}\}dx.$$

From the theory of Airy-functions it is known that for all x there holds

$$(2.35)\qquad Ai(x) + \omega Ai(\omega x) + \omega^2 Ai(\omega^2 x) = 0,$$

so that from (2.34) there follows

$$(2.36)\quad Y_1(0,\eta) = -m^{-2}\psi'(0)\exp(-\tfrac{1}{12}\eta^3)\int_0^\infty Ai'(x)(e^{m^2\eta x} + \omega e^{m^2\eta x\omega^2} +$$

$$+\,\omega^2 e^{m^2\eta x\omega})dx,$$

$$Y_1(0,\eta) = \psi'(0)\eta.$$

For the computation of (2.36) we refer to [2].
This result agrees with (2.28).

We verify the matching conditions. In a certain sense it has been done in [2] for matching the outer expansion and the boundary-layer expansion. Since we approach the problem from another point of view, we give a proof of it in a rather different way. Matching the interior boundary-layer expansion will be investigated in the following section.

a. Matching the outer expansion

In accordance with the principle applied in (2.19) we substitute the coordinates corresponding to $\nu = 2/3$, $\mu = 1/3$ into (2.3):

$$(2.37)\quad \phi[\rho,\theta;\varepsilon] = U_0[0,0] + \varepsilon^{1/3}\sum_{n=0}^{\infty} U_{n,1/3} + \varepsilon^{2/3}\sum_{n=0}^{\infty} U_{n,2/3} + \varepsilon \ldots .$$

$U_{n,1/3}\varepsilon^{1/3}$ represents the part of $U_n\varepsilon^n$ that is $O(\varepsilon^{1/3})$ in the coordinates corresponding to $\nu = 2/3$, $\mu = 1/3$. For example

$$U_0(x,y) = \psi(-\arccos y) = \psi(0) - \arccos\{(1-\varepsilon^{2/3}\xi)\cos \varepsilon^{1/3}\eta\}\psi'(0) + \ldots$$

$$U_0(x,y) = \psi(0) - \varepsilon^{1/3}\sqrt{2\xi + \eta^2}\,\psi'(0) + \varepsilon^{2/3}\ldots,$$

so

$$(2.38)\qquad U_{0,1/3}[\xi,\eta] = -\sqrt{2\xi + \eta^2}\cdot\psi'(0).$$

As W. Eckhaus in [1] suggests, we match along the line $\nu = 2\mu$, $0 \le \mu \le 1/3$, see fig. 1. This means that the expansions (2.27) and (2.38) for $\xi = C\eta^2$ $(C > 0)$ and $|\eta| >> 1$ are equal:

$$(2.39)\qquad \lim_{\substack{\xi=C\eta^2\\|\eta|\to\infty}} \{Y_1(\xi,\eta) - \sum_{n=0}^{\infty} U_{n,1/3}[\xi,\eta]\} = 0.$$

It is easily seen that

$$(2.40)\qquad \lim_{\substack{\xi=C\eta^2\\|\eta|\to\infty}} U_{n,1/3}[\xi,\eta] = 0,\ n = 1, 2, \ldots .$$

We derive from (2.33) for $\xi = C\eta^2$, $|\eta| >> 1$:

(2.41) $$Y_1(\xi,\eta) \approx -\sqrt{2\xi + \eta^2}\,\psi'(0),$$

which is proved in appendix a. So now we can see that (2.38), (2.40) and (2.41) do indeed yield (2.39).

b. Matching the boundary-layer expansion

In the coordinates corresponding to $\nu = 2/3$, $\mu = 1/3$ the expansion (2.13) transforms into

$$\phi[\xi,\eta;\varepsilon] = \psi(0) + \varepsilon^{1/3} \sum_{n=0}^{\infty} V_{n,1/3}[\xi,\eta] + \varepsilon^{2/3} \sum_{n=0}^{\infty} V_{n,2/3} + \varepsilon \ldots,$$

$$V_{0,1/3}[\xi,\eta] = (2\eta e^{-\xi\eta} - \eta)\psi'(0).$$

In this case we match along the line $\nu = 1 - \mu$, $0 \leq \mu \leq 1/3$, see fig. 1. The matching condition takes the form

(2.42) $$\lim_{\substack{\xi = C/\eta \\ \eta\to\infty}} \{Y_1(\xi,\eta) - \sum_{n=0}^{\infty} V_{n,1/3}[\xi,\eta]\} = 0, \; C > 0.$$

Because in [2] it has been proved for $\xi = C/\eta$ and $\eta >> 1$ that

$$Y_1(\xi,\eta) \approx (2\eta e^{-\xi\eta} - \eta)\psi'(0)$$

(2.42) is indeed valid.

Moreover, it is easily seen that

$$\lim_{\substack{\xi = C/\eta \\ \eta\to\infty}} V_{n,1/3}[\xi,\eta] = 0, \; n = 1, 2, \ldots .$$

2.5. Interior boundary-layer expansion

We consider the case $\nu = \mu = 1$, $\xi \geq 0$, $|\eta| \geq 0$. Equation (2.5) then takes the form

$$(2.43)\quad \frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} - \frac{\partial\phi}{\partial\eta} = \varepsilon\Big[(2\xi+\varepsilon\ldots)\frac{\partial^2\phi}{\partial\eta^2} - \Big\{\frac{\sin\,\xi\eta}{\varepsilon} - \frac{1}{(1-\varepsilon\xi)}\Big\}\frac{\partial\phi}{\partial\xi}$$

$$- \Big\{(\xi + \tfrac{1}{2}\eta^2) + \varepsilon\ \ldots\Big\}\frac{\partial\phi}{\partial\eta}\Big].$$

The interior boundary-layer expansion is

$$(2.44)\qquad \phi[\xi,\eta;\varepsilon] = \sum_{n=0}^{\infty} W_n(\xi,\eta)\varepsilon^n$$

and satisfies the boundary condition

$$(2.45)\qquad \sum_{n=0}^{\infty} W_n(0,\eta)\varepsilon^n = \psi(\varepsilon\eta).$$

As a first approximation we obtain

$$W_0(\xi,\eta) = \psi(0).$$

For the next term in the approximation we have the equation

$$(2.46)\qquad \frac{\partial^2 W_1}{\partial\xi^2} + \frac{\partial^2 W_1}{\partial\eta^2} - \frac{\partial W_1}{\partial\eta} = 0.$$

It is known (see [1]) that $W_1(\xi,\eta)$, as it has the form

$$\varepsilon W_1(\xi,\eta) = \frac{\xi}{2\pi}\int_{-\infty}^{+\infty} f(\varepsilon p)e^{-\frac{1}{2}(p-\eta)}\ \frac{K_1(R/2)}{R}\ dp,$$

is a solution of this equation, if $\varepsilon W_1(0,\eta) = f(\varepsilon p)$.
$K_1(z)$ is a modified Bessel function of the first kind and

$$R = \{(\eta - p)^2 + \xi^2\}^{\frac{1}{2}}.$$

(2.45) has been satisfied by the choice $f(\varepsilon p) = \psi(\varepsilon p) - \psi(0)$. It is impossible to use a Taylor-expansion of $\psi(\varepsilon p)$ and to change the order of integration and summation, because the integrals diverge for $p \to -\infty$.

Matching the intermediate boundary-layer expansion

We transform $\varepsilon^{1/3}Y_1(\xi,\eta)$ into $\varepsilon^{1/3}Y_1(\xi\varepsilon^{1/3},\eta\varepsilon^{2/3})$ and develop this form into a power series in ε. We now match along the line $\nu = \frac{1}{2}(1+\mu)$, $\frac{1}{3} \leq \mu \leq 1$, see fig. 1. This means that $\varepsilon^{1/3}Y_1(\xi\varepsilon^{1/3},\eta\varepsilon^{2/3})$ and $\varepsilon W_1(\xi,\eta)$ have to be asymptotically equivalent for $\eta = C\xi^2$ $(C \neq 0)$, $\xi >> 1$.
In appendix b it is proved, that in this case

(2.47) $$\varepsilon^{1/3}Y_1(\xi\varepsilon^{1/3},\eta\varepsilon^{2/3}) \approx \varepsilon\psi'(0)(\eta + \frac{1}{2}\xi^2).$$

Further, we investigate $W_1(\xi,\eta)$ and make the following estimate

$$\varepsilon W_1(\xi,\eta) = \frac{\xi}{2\pi}\int_{-N}^{\infty} \{\psi(\varepsilon p) - \psi(0)\}e^{-\frac{1}{2}(p-\eta)} \frac{K_1(R/2)}{R}\,dp + O(\varepsilon^M),$$

where N and M are arbitrarily large positive numbers.
The asymptotic behaviour of $W_1(\xi,\eta)$ can be determined by the saddle-point method. We also refer to [6], in which in order to prevent a loss of higher order approximations the computations are rather complicated. In both cases we obtain

(2.48) $$\varepsilon W_1(\xi,\eta) \approx \sqrt{\frac{2}{\pi}}\int_{\delta(N)}^{\infty} \{\psi[\varepsilon(\eta - \frac{\xi^2}{2t^2})] - \psi(0)\}e^{-\frac{1}{2}t^2}\,dt,$$

where $\delta(N) \downarrow 0$ for $N \to \infty$. We introduce the functions

(2.49) $$S_1(t;\xi,\eta) = \int_{\delta(N)}^{t} \{\psi[\varepsilon(\eta - \frac{\xi^2}{2\tau^2})] - \psi(0)\}\,d\tau$$

and

(2.50)

$$S_2(t;\xi,\eta) = \int_{\delta(N)}^{t} \tau S_1(\tau;\xi,\eta)d\tau.$$

Repeated partial integration of (2.48) yields

$$(2.51)\quad \varepsilon W_1(\xi,\eta) = \Big[\{S_1(t;\xi,\eta) + S_2(t;\xi,\eta)\}e^{-\frac{1}{2}t^2}\Big]_{\delta(N)}^{\infty} + $$

$$+ \int_{\delta(N)}^{\infty} tS_2(t;\xi,\eta)e^{-\frac{1}{2}t^2}\, dt.$$

The integrand of (2.49) may be written in a truncated development

$$(2.52)\quad S_1(t;\xi,\eta) = \int_{\delta(N)}^{t} \{\varepsilon(\eta - \frac{\xi^2}{2\tau^2})\psi'(0) + \frac{1}{2}\,\varepsilon^2(\eta - \frac{\xi^2}{2\tau^2})^2\psi''(g)\}dt,$$

$$0 < g < t.$$

So that now (2.49) and (2.50) can be integrated. Moreover,

$$\Big[\{S_1(t;\xi,\eta) + S_2(t;\xi,\eta)\}e^{-\frac{1}{2}t^2}\Big]_{\delta(N)}^{\infty} \to 0, \text{ as } N \to \infty.$$

Thus we obtain

$$(2.53)\qquad \varepsilon W_1(\xi,\eta) \approx \varepsilon\psi'(0)(\eta + \frac{1}{2}\,\xi^2).$$

After examination of (2.47) and (2.53) our conclusion is that the terms $\varepsilon^{1/3}Y_1(\rho/\varepsilon^{2/3},\theta/\varepsilon^{1/3})$ and $\varepsilon W_1(\rho/\varepsilon,\theta/\varepsilon)$ indeed match.

Comment:

It is impossible to expand $\{\psi[\varepsilon(\eta - \frac{\xi^2}{2t^2})] - \psi(0)\}$ in (2.48) directly in terms of ε, because partial integration would lead to singular terms.

3. Uniformly valid approximations

3.1. Introductory remarks

In the preceding chapter we constructed approximations of $U(x,y;\varepsilon)$ which are locally valid in the domain $x^2 + y^2 \leq 1$. Our aim is to construct a uniformly valid approximation; therefore we compose the local expansions to one expansion. This is done in such a way, that the asymptotic equivalence of the composite expansion and each local expansion in its part of validity can be proved.
In 3.2 we reproduce a theorem, that will be applied in the following sections.

The accuracy of the uniform approximation will be estimated by choosing the boundaries between the parts in such a manner that the remainder tersm are optimal.

3.2. Application of the maximum principle

Let

$$M[\phi] \equiv a(\xi,\eta)\frac{\partial^2\phi}{\partial\xi^2} + 2b(\xi,\eta)\frac{\partial\phi}{\partial\xi\partial\eta} + c(\xi,\eta)\cdot\frac{\partial^2\phi}{\partial\eta^2} + d(\xi,\eta)\frac{\partial\phi}{\partial\xi} +$$

$$+ e(\xi,\eta)\frac{\partial\phi}{\partial\eta} + f(\xi,\eta)\,\phi$$

be a differential expression elliptic in a bounded domain G while the coefficients a, b, etc. are continuous within G with $a(\xi,\eta) > 0$.

Theorem:

Let $\phi(\xi,\eta)$ be the solution of the differential equation

$$M[\phi] = h(\xi,\eta)$$

valid in G, while along the boundary Γ of G the relation

$$\phi(\xi,\eta)\big|_\Gamma = k(\xi,\eta)\big|_\Gamma$$

holds. $e(\xi,\eta)$ is either positive or negative and $f(\xi,\eta) \leq 0$ in G. If there exists a constant m with the properties:

$$|h(\xi,\eta)| \leq m \quad \text{in } \overline{G}$$

$$|k(\xi,\eta)| \leq m \quad \text{along } \Gamma,$$

then there exists also a real number M independent of m such that

$$|\phi(\xi,\eta)| \leq mM \quad \text{in } \overline{G}.$$

For the proof of this theorem we refer to [3], where it has been derived from the maximum principle. We investigate the parts A_1 and A_2 for the elliptic differential equation

$$(3.1)\quad L[\phi(\xi,\eta)] \equiv \varepsilon^{1-3\mu}\frac{\partial^2\phi}{\partial\xi^2} + \frac{\varepsilon^{2\mu}}{(1-\varepsilon^{2\mu}\xi)^2}\frac{\partial^2\phi}{\partial\eta^2} + \left[\left\{\frac{\sin\varepsilon^\mu\eta}{\varepsilon^\mu}\right\} - \frac{\varepsilon^{2\mu}}{(1-\varepsilon^{2\mu}\xi)}\right]\frac{\partial\phi}{\partial\xi}$$

$$- \frac{\cos\varepsilon^\mu\eta}{(1-\varepsilon^{2\mu}\xi)}\frac{\partial\phi}{\partial\eta} = 0,$$

$$\rho = \xi\varepsilon^{2\mu},\ \theta = \eta\varepsilon^\mu,\ 0 < \mu \leq 1/3,\ \xi \geq 0,\ |\eta| \geq 0.$$

Obviously $L[\phi(\xi,\eta)]$ satisfies the conditions for $M[\phi(\xi,\eta)]$. It is easily verified that: if $L[R(\xi,\eta)] = O(\varepsilon^{\alpha})$ in G and $R(\xi,\eta) = O(\varepsilon^{\beta})$ on Γ, then $R(\xi,\eta) = O(\varepsilon^{\min(\alpha,\beta)})$ in $\overline{G}$. We shall make use of this property in 3.3 and 3.4 for the parts A_1 and A_2.

3.3. First uniform approximation

We consider the first term of each local expansion.

In domain C we have

$$\phi(x,y;\varepsilon) = U_0(x,y) + Z_C(x,y;\varepsilon).$$

From (2.11) we derive the estimate

$$|Z_C| \leq \sum_{n=1}^{\infty} |U_n|\varepsilon^n = O(\varepsilon^{1-2\mu}).$$

In domain B_1, B_2 we have

$$\phi(x,y;\varepsilon) = V_0(\rho/\varepsilon,\theta) + Z_B(\rho,\theta;\varepsilon).$$

Using (2.25) we obtain

$$|Z_B| \leq \sum_{n=1}^{\infty} |V_n|\varepsilon^n = O(\varepsilon^{1-2\mu}).$$

In the domain A_1 we apply the theorem of the preceding section to

$$\phi(\xi,\eta;\varepsilon) = \psi(0) + Z_{A_1}(\xi,\eta;\varepsilon).$$

Substitution in (3.1) yields

$$L[Z_{A_1}(\xi,\eta;\varepsilon)] = 0. \tag{3.2}$$

On the boundary of A_1 $Z_{A_1}(\xi,\eta;\varepsilon)$ satisfies:

$$Z_{A_1}(0,\eta;\varepsilon) = \psi(\varepsilon^\mu\eta) - \psi(0) = O(\varepsilon^\mu) \qquad \text{for } 0 \leq |\eta| \leq K;$$

$$Z_{A_1}(K,\eta;\varepsilon) = Z_C[K\varepsilon^{2\mu},\eta\varepsilon^\mu;\varepsilon] = O(\varepsilon^{1-2\mu}) \qquad \text{for } 0 \leq |\eta| \leq K;$$

$$Z_{A_1}(\xi,-K;\varepsilon) = Z_C[\xi\varepsilon^{2\mu},-K\varepsilon^\mu;\varepsilon] = O(\varepsilon^{1-2\mu}) \text{ for } 0 \leq \xi \leq K;$$

$$Z_{A_1}(\xi,K;\varepsilon) = Z_B[\xi\varepsilon^{2\mu},K\varepsilon^{\mu};\varepsilon] = O(\varepsilon^{1-2\mu}) \quad \text{for } 0 \leq \xi \leq K\varepsilon^{1-2\mu};$$

$$Z_{A_1}(\xi,K;\varepsilon) = Z_C[\xi\varepsilon^{2\mu},K\varepsilon^{\mu};\varepsilon] = O(\varepsilon^{1-2\mu}) \quad \text{for } K\varepsilon^{1-2\mu} < \xi \leq K.$$

From (3.2) and the boundary conditions it follows with aid of the theorem

$$Z_{A_1}(\xi,\eta;\varepsilon) = O(\varepsilon^{\min(1-2\mu,\mu)}).$$

In domain A_2 we find, as in domain A_1,

$$\phi(\xi,\eta;\varepsilon) = \psi(\pi) + Z_{A_2}(\xi,\eta;\varepsilon),$$

where

$$Z_{A_2}(\xi,\eta;\varepsilon) = O(\varepsilon^{\min(1-2\mu,\mu)}).$$

We make the optimal choice $\mu = 1/3$. Further we introduce the form

$$W_0(x,y;\varepsilon) = U_0(x,y) + V_0(\rho/\varepsilon,\theta) - \psi(-\theta).$$

Now it is easy to deduce that

$$\phi(x,y;\varepsilon) = W_0(x,y;\varepsilon) + O(\varepsilon^{1/3})$$

in the whole domain $x^2 + y^2 \leq 1$.

3.4. Second uniform approximation

As in 3.3 we may write immediately:

In domain C:

$$\phi(x,y;\varepsilon) = U_0(x,y) + \varepsilon U_1(x,y) + Z_C(x,y;\varepsilon),$$

$$|Z_C| \leq \sum_{n=2}^{\infty} |U_n|\varepsilon^n = O(\varepsilon^{2-5\mu}).$$

In domain B_1, B_2:

$$\phi(x,y;\varepsilon) = V_0(\rho/\varepsilon,\theta) + \varepsilon V_1(\rho/\varepsilon,\theta) + Z_B(\rho,\theta;\varepsilon),$$

$$|Z_B| \leq \sum_{n=2}^{\infty} |V_n|\varepsilon^n = O(\varepsilon^{2-5\mu}).$$

In domain A_1:

(3.3) $$\phi(\xi,\eta;\varepsilon) = \psi(0) + \varepsilon^{1/3}Y_1(\rho/\varepsilon^{2/3},\theta/\varepsilon^{1/3}) + Z_{A_1}(\xi,\eta;\varepsilon).$$

We shall prove in appendix c that

(3.4) $$L[Z_{A_1}(\xi,\eta;\varepsilon)] = O(\varepsilon^{9\mu-2}).$$

Moreover, $Z_{A_1}(\xi,\eta;\varepsilon)$ satisfies on the boundary of A_1:

$$Z_{A_1}(0,\eta;\varepsilon) = \psi(\varepsilon^\mu\eta) - \psi(0) - \varepsilon^\mu\eta\psi'(0) = O(\varepsilon^{2\mu}) \text{ for } 0 \leq |\eta| \leq K;$$

$$Z_{A_1}(K,\eta;\varepsilon) = Z_C[K\varepsilon^{2\mu},\eta\varepsilon^\mu;\varepsilon] = O(\varepsilon^{2-5\mu}) \qquad \text{for } 0 \leq |\eta| \leq K;$$

$$Z_{A_1}(\xi,-K;\varepsilon) = Z_C[\xi\varepsilon^{2\mu},-K\varepsilon^\mu;\varepsilon] = O(\varepsilon^{2-5\mu}) \qquad \text{for } 0 \leq \xi \leq K;$$

$$Z_{A_1}(\xi,K;\varepsilon) = Z_B[\xi\varepsilon^{2\mu},K\varepsilon^\mu,\varepsilon] = O(\varepsilon^{2-5\mu}) \qquad \text{for } 0 \leq \xi \leq K\varepsilon^{1-3\mu};$$

$$Z_{A_1}(\xi,K;\varepsilon) = Z_C[\xi\varepsilon^{2\mu},K\varepsilon^\mu;\varepsilon] = O(\varepsilon^{2-5\mu}) \qquad \text{for } K\varepsilon^{1-3\mu} \leq \xi \leq K.$$

In this case we conclude:

$$Z_{A_1}(\xi,\eta;\varepsilon) = O(\varepsilon^{\min(9\mu-2,2\mu,2-5\mu)}).$$

In domain A_2:

$$\phi(\xi,\eta;\varepsilon) = \psi(\pi) + \varepsilon^{1/3}\overline{Y}_1\{\rho/\varepsilon^{2/3},(\pi-\theta)/\varepsilon^{1/3}\} + Z_{A_2}(\xi,\eta;\varepsilon).$$

In the same way as in the preceding case we find

$$Z_{A_2}(\xi,\eta;\varepsilon) = O(\varepsilon^{\min(9\mu-2,2\mu,2-5\mu)}).$$

The remainder term is estimated as accurately as possible for $\mu = 2/7$.

The uniformly valid approximation now takes the form:

$$\phi(x,y;\varepsilon) = W_0(x,y;\varepsilon) + \varepsilon W_1(x,y;\varepsilon) + \varepsilon^{1/3}W_{1/3}(\rho,\theta;\varepsilon) + O(\varepsilon^{4/7}),$$

$$W_1(x,y;\varepsilon) = U_1(x,y) + V_1(\rho/\varepsilon,\theta) + \frac{\cos\theta\psi'(-\theta)}{\sin\theta}\cdot\rho/\varepsilon,$$

$$W_{1/3}(\rho,\theta;\varepsilon) = Y_1\{\rho/\varepsilon^{2/3},\theta/\varepsilon^{1/3}\} - \psi'(0)\ \{2\theta e^{-\theta\rho/\varepsilon} - \sqrt{2\rho+\theta^2}\}\varepsilon^{-1/3} +$$

$$+ \overline{Y}_1\{\rho/\varepsilon^{2/3},(\pi-\theta)/\varepsilon^{1/3}\} - \psi'(\pi)\{2(\pi-\theta)e^{-(\pi-\theta)\rho/\varepsilon}$$

$$- \sqrt{2\rho + (\pi-\theta)^2}\}\varepsilon^{-1/3}.$$

4. Conclusions

Summarizing the results of the preceding sections we make the following conclusions. We have investigated the differential equation

$$\varepsilon\Delta\phi - \frac{\partial\phi}{\partial x} = 0 \ , \quad 0 < \varepsilon << 1,$$

valid inside the circle $x^2 + y^2 = 1$ with $\phi(x,y;\varepsilon)$ given on the boundary, and we have obtained:

a. In every point of the domain $x^2 + y^2 \leq 1$ there exists an asymptotic expansion of the solution $\phi(x,y;\varepsilon)$.
   However, an explicit computation of the higher order approximations for the intermediate - and interior - expansion appears almost impossible. We only computed the first two terms of both expansions.
b. The neighbourhoods of (0,1) and (0,-1) need to be explored into details in order to get a comprehensive insight in the character of the singularities of the outer expansion at these points.
c. We have made a uniformly valid approximation with accuracy $O(\varepsilon^{1/3})$ and also a better one with accuracy $O(\varepsilon^{4/7})$.

Finally, we remark that the way of matching the expansions, as it has been done, needs to be prescribed into more details, than has been done by the matching principle in the literature.
An extensive study of the matching technique in two dimensions would probably reveal more about the behaviour of the function $\phi(x,y;\varepsilon)$ near the singular points.

5. Appendices

appendix a

We prove:

if $\xi = C\eta^2$ $(C > 0)$ and $|\eta| >> 1$, then

(a.1) $$Y_1(\xi,\eta) \approx -\sqrt{2\xi + \eta^2}\,\psi'(0)$$

holds. After a changing of integration variables (see [2] pag. 102) (2.33) takes the form:

(a.2) $$Y_1(\xi,\eta) = -m^{-2}\psi'(0)\exp(-1/12\,\eta^3)\left[\Phi_A+\Phi_B+\Phi_C+\Phi_D\right],$$

(a.2a) $$\Phi_A = \omega\int_0^\infty \frac{\dot{Ai}(m\omega\xi+x)}{\underset{\cdot}{Ai}(m\omega\xi+x)}\cdot Ai(x)e^{m^2\eta\omega^2 x}\,dx,$$

(a.2b) $$\Phi_B = \omega^2\int_0^\infty \frac{\dot{Ai}(m\omega\xi+x)}{\underset{\cdot}{Ai}(m\omega\xi+x)}\,Ai(x)e^{m^2\eta\omega x}\,dx,$$

(a.2c) $$\Phi_C = -\omega^2\int_0^\infty \frac{\dot{Ai}(m\omega\xi+\omega x)}{\underset{\cdot}{Ai}(m\omega\xi+\omega x)}\,Ai(\omega x)e^{m^2\eta x}\,dx,$$

(a.2d) $$\Phi_D = -\omega\int_0^\infty \frac{\dot{Ai}(m\omega^2\xi+\omega^2 x)}{Ai(m\omega^2\xi+\omega^2 x)}\,Ai(\omega^2 x)e^{m^2\eta x}\,dx.$$

For $\eta > 0$ the largest contribution comes from Φ_C and Φ_D. Taking these terms together and making use of the asymptotic property

$$\frac{\dot{Ai}(p)}{\underset{\cdot}{Ai}(p)} \approx -\sqrt{p} + O(p^{-1}) \quad (\text{for } |p| >> 1),$$

we obtain

$$\Phi_C + \Phi_D \approx -\int_0^\infty \sqrt{m\xi+x}\,\{\omega \underset{\cdot}{Ai}(\omega x) + \omega^2 \underset{\cdot}{Ai}(\omega^2 x)\}e^{m^2\eta x}\,dx.$$

With (2.35) this form reduces to

$$\Phi_C + \Phi_D \approx \int_0^\infty \sqrt{m\xi+x}\, Ai(x) e^{m^2 \eta x}\, dx.$$

Application of the saddle-point method produces the asymptotic formula (a.1).

For $\eta < 0$ Φ_A and Φ_B dominate and their sum has the same asymptotic behaviour as Φ_C and Φ_D.

appendix b

We investigate the behaviour of $Y_1(\xi,\eta)$ for $\eta = C\xi^2$ $(C \neq 0)$.

$$Y_1(\xi,\eta) = - m^{-2}\psi'(0)\exp(-\tfrac{1}{2}\xi\eta - \tfrac{1}{12}\eta^3)\{\Phi_A + \Phi_B + \Phi_C + \Phi_D\},$$

$$\Phi_A = \omega \int_0^\infty Ai'(x)\{1 - m\omega\xi \frac{Ai'(x)}{Ai(x)} + \frac{1}{2} m^2\omega^2\xi^2 \frac{Ai''(x)}{Ai(x)} \ldots\}e^{m^2\eta x\omega^2}\, dx,$$

$$\Phi_B = \omega^2 \int_0^\infty Ai'(x)\{1 - m\omega^2\xi \frac{Ai'(x)}{Ai(x)} + \frac{1}{2} m^2\omega\xi^2 \frac{Ai''(x)}{Ai(x)} \ldots\}e^{m^2\eta x\omega}\, dx,$$

$$\Phi_C = - \omega^2 \int_0^\infty Ai'(\omega x)\{1 - m\omega\xi \frac{Ai'(\omega x)}{Ai(\omega x)} + \frac{1}{2} m^2\omega^2\xi^2 \frac{Ai''(\omega x)}{Ai(\omega x)} \ldots\}e^{m^2\eta x}\, dx,$$

$$\Phi_D = - \omega \int_0^\infty Ai'(\omega^2 x)\{1 - m\omega^2\xi \frac{Ai'(\omega^2 x)}{Ai(\omega^2 x)} + \frac{1}{2} m^2\omega\xi^2 \frac{Ai''(\omega^2 x)}{Ai(\omega^2 x)} \ldots\}e^{m^2\eta x}\, dx.$$

We call the first term of each development respectively Φ_{A_1}, Φ_{B_1}, Φ_{C_1} and Φ_{D_1}, and define

$$Y_{1,1}(\xi,\eta) = - m^{-2}\psi'(0)\exp(-\tfrac{1}{2}\xi,\eta - \tfrac{1}{12}\eta^3)\{\Phi_{A_1}+\Phi_{B_1}+\Phi_{C_1}+\Phi_{D_1}\}.$$

In the same way $Y_{1,2}(\xi,\eta)$, $Y_{1,3}(\xi,\eta)$, etc. .... .
Using (2.35) and (2.36) we obtain

$$Y_{1,1}(\xi,\eta) = \psi'(0) + O(\xi^3).$$

Proceeding in the same way for the second terms we find

$$\Phi_{A_2} + \Phi_{B_2} = m\xi \int_0^\infty \frac{\{Ai'(x)\}^2}{Ai(x)} \{- \omega^2 e^{m^2\eta x\omega^2} - \omega e^{m^2\eta x\omega}\}dx,$$

$$\Phi_{C_2} + \Phi_{D_2} = m\xi \int_0^\infty \left[\frac{\{Ai'(\omega x)\}^2}{Ai(\omega x)} + \frac{\{Ai'(\omega^2 x)\}^2}{Ai(\omega^2 x)}\right]e^{m^2\eta x}\, dx.$$

We define $P(x) = - \int_x^\infty \frac{\{Ai'(s)\}^2}{Ai(s)}\, ds$, so that

$$P(z) = -\int_z^\infty \left\{\frac{Ai'(s)}{Ai(s)}\right\}^2 Ai(s)ds \approx -\int_z^\infty s\, Ai(s)ds = -\int_z^\infty Ai''(s)ds = Ai'(z),$$

for $z = Re^{i\phi}$ with $R >> 1$ and $0 \leq |\phi| < \pi$.

$$\Phi_{A_2} + \Phi_{B_2} = m\xi \int_0^\infty \{-\omega^2 e^{m^2\eta x\omega^2} - \omega e^{m^2\eta x\omega}\} dP(x),$$

$$\Phi_{C_2} + \Phi_{D_2} = m\xi \int_0^\infty e^{m^2\eta x} \{\omega^2 dP(\omega x) + \omega dP(\omega^2 x)\}.$$

$$\Phi_{A_2} + \Phi_{B_2} = m\xi \Big[\{-\omega^2 e^{m^2\eta x\omega^2} - \omega e^{m^2\eta x\omega}\} P(x)\Big]_0^\infty$$

$$- m^3\xi\eta \int_0^\infty P(x)\{-\omega e^{m^2\eta x\omega^2} - \omega^2 e^{m^2\eta\omega}\}dx,$$

$$\Phi_{C_2} + \Phi_{D_2} = m\xi \Big[e^{m^2\eta x}\{\omega^2 P(\omega x) + \omega P(\omega^2 x)\}\Big]_0^\infty$$

$$- m^3\xi\eta \int_0^\infty \{P(\omega x)\omega^2 + P(\omega^2 x)\omega\} e^{m^2\eta x}\, dx.$$

It is immediately seen that

$$Y_{1,2}(\xi,\eta) = O(\xi^3).$$

$$\Phi_{A_3} + \Phi_{B_3} = \frac{1}{2} m^2\xi^2 \int_0^\infty x\, Ai'(x)\{e^{m^2\eta x\omega^2} + e^{m^2\eta x\omega}\}dx,$$

$$\Phi_{C_3} + \Phi_{D_3} = \frac{1}{2} m^2\xi^2 \int_0^\infty x\{-\omega^2\, Ai'(\omega x) - \omega Ai'(\omega^2 x)\} e^{m^2\eta x}\, dx.$$

$$Y_{1,3}(\xi,\eta) = -\frac{1}{2}\xi^2\psi'(0)\exp(-\frac{1}{2}\xi\eta - \frac{1}{12}\eta^3) \int_0^\infty xAi'(x)\{e^{m^2\eta x} +$$

$$+ e^{m^2\eta x\omega} + e^{m^2\eta x\omega^2}\}dx.$$

Partial integration makes

$$Y_{1,3}(\xi,\eta) = \frac{1}{2}\,\xi^2\psi'(0)\,\exp(-\frac{1}{12}\,\eta^3)\int_0^\infty Ai(x)\{e^{m^2\eta x} + e^{m^2\eta x\omega} +$$

$$+ e^{m^2\eta x\omega^2}\}dx + O(\xi^3).$$

Application of the same method of computation as in (2.36) yields

$$Y_{1,3}(\xi,\eta) = \frac{1}{2}\,\xi^2\psi'(0) + O(\xi^3).$$

Finally we mention that

$$Y_{1,n}(\xi,\eta) = O(\xi^3) \quad \text{for} \quad n = 4,\ 5,\ 6,\ \ldots\ .$$

In this appendix we have demonstrated that

$$Y_1(\xi,\eta) \approx \psi'(0)(\eta + \frac{1}{2}\,\xi^2)$$

for $\eta = C\xi^2$ $(C \neq 0)$ and $0 < \xi << 1$.
In the coordinates $\theta = \varepsilon\eta$ and $\rho = \varepsilon\xi$ the matching takes the form

$$\varepsilon^{1/3}Y_1(\xi,\eta) \approx \varepsilon\psi'(0)(\eta + \frac{1}{2}\,\xi^2).$$

appendix c

Substitution of (3.3) in (3.1) yields

$$\text{(c.1)}\quad L[Z_{A_1}(\xi,\eta)] = -\,\varepsilon^{1/3+2\mu}\Big[\frac{1}{(1-\varepsilon^{2\mu}\xi)^2}\frac{\partial^2 Y_1}{\partial\eta^2} + \{\eta^3 + \ldots - \frac{1}{(1-\varepsilon^{2\mu}\xi)}\}\frac{\partial Y_1}{\partial\eta} - \frac{\eta^2 + \ldots}{(1-\varepsilon^{2\mu}\xi)}\frac{\partial Y_1}{\partial\eta}\Big].$$

In accordance with (2.33) for $Y_1 = Y_1(\xi\varepsilon^{2\mu-2/3}, \eta\varepsilon^{\mu-1/3})$ we obtain

$$\text{(c.2a)}\quad \frac{\partial Y_1}{\partial\xi} = -\frac{1}{2}\,\varepsilon^{3\mu-1}Y_1 + m^{-1}\varepsilon^{2\mu-2/3}\,R_1(\omega^2,\omega,1,1,\xi,\eta),$$

$$\text{(c.2b)}\quad \frac{\partial Y_1}{\partial\eta} = (-\frac{1}{2}\,\xi - \frac{1}{4}\,\eta^2)\varepsilon^{3\mu-1}Y_1 - \varepsilon^{\mu-1/3}\,R_0(x,x,-\omega^2x,-\omega x,\xi,\eta),$$

$$\text{(c.2c)}\quad \frac{\partial^2 Y_1}{\partial\eta^2} = -\frac{1}{2}\,\eta\varepsilon^{3\mu-1}Y_1 + (-\frac{1}{2}\,\xi - \frac{1}{4}\,\eta^2)\varepsilon^{3\mu-1}\{\frac{\partial Y_1}{\partial\eta} - \varepsilon^{\mu-1/3}\,R_0(x,x,-\omega^2x,-\omega x,\xi,\eta)\} + {}$$
$$- m^2\varepsilon^{\mu-1/3}\,R_0(\omega^2x^2,\omega x^2,-\omega^2x^2,-\omega x^2,\xi,\eta);$$

$$R_i(p_1,p_2,p_3,p_4,\xi,\eta) = \psi'(0)\exp\{\varepsilon^{(3\mu-1)}(-\frac{1}{2}\,\xi\eta - \frac{1}{12}\,\eta^3)\}\,\cdot$$
$$\cdot\Big[\int_0^\infty p_1\,\frac{Ai'(x)}{Ai(x)}\,G_i(x - m\omega\xi\varepsilon^{2\mu-2/3})e^{m^2\omega^2\eta x\varepsilon^{\mu-1/3}}\,dx + {}$$
$$+ \int_0^\infty p_2\,\frac{Ai'(x)}{Ai(x)}\,G_i(x - m\omega^2\xi\varepsilon^{2\mu-2/3})e^{m^2\omega\eta x\varepsilon^{\mu-1/3}}\,dx + {}$$
$$- \int_0^\infty p_3\,\frac{Ai'(\omega x)}{Ai(\omega x)}\,G_i(\omega x - m\omega\xi\varepsilon^{2\mu-2/3})e^{m^2\eta x\varepsilon^{\mu-1/3}}\,dx + {}$$
$$- \int_0^\infty p_4\,\frac{Ai'(\omega^2 x)}{Ai(\omega^2 x)}\,G_i(\omega^2 x - m\omega^2\xi\varepsilon^{2\mu-2/3})e^{m^2\eta x\varepsilon^{\mu-1/3}}\,dx\Big],$$

$i = 0,\ 1;\ G_0(z) = Ai(z),\ G_1(z) = Ai'(z).$

We divide the domain A_1 into two parts

a. $0 \leq \xi \leq K\varepsilon^{2/3-2\mu}$, $0 \leq |\eta| \leq K\varepsilon^{1/3-\mu}$;

b. $0 \leq \xi \leq K$, $0 \leq |\eta| \leq K$, with exception of part a.

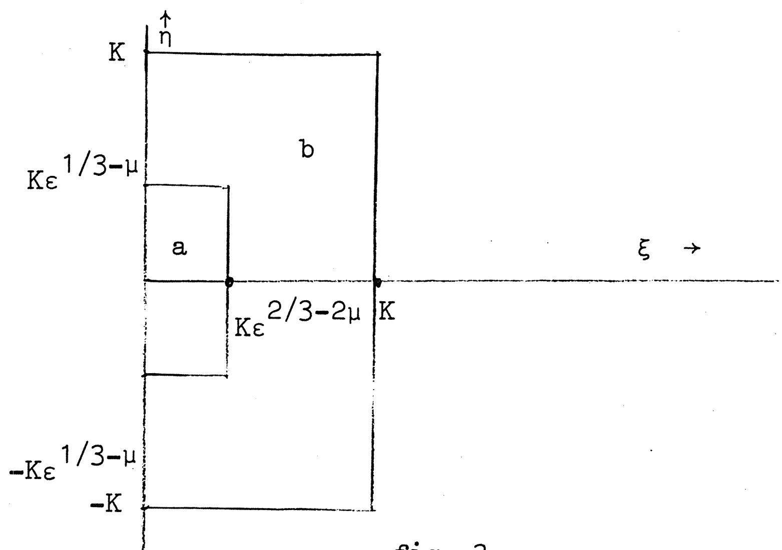

fig. 3

In part a:

The terms $\frac{\partial Y_1}{\partial \xi}$, $\frac{\partial Y_1}{\partial \eta}$ and $\frac{\partial^2 Y_1}{\partial \eta^2}$ contain integrals, which are bounded in this domain. As an example we take the first integral of $\frac{\partial Y_1}{\partial \xi}$:

$$\underline{I} = \int_0^\infty \frac{Ai'(x)}{Ai(x)} Ai(x - m\omega\xi\varepsilon^{2\mu-2/3}) e^{m^2 x\omega^2 \eta\varepsilon^{\mu-1/3}} dx.$$

For $0 \leq \xi\varepsilon^{2\mu-2/3} \leq K$ and $0 \leq |\eta|\varepsilon^{\mu-1/3} \leq K$ there holds:
given any arbitrarily small number $\sigma > 0$, there exists a number $R(\sigma)$, such that

$$\left|\int_R^\infty \frac{Ai'(x)}{Ai(x)} Ai(x - m\omega\xi\varepsilon^{2\mu-2/3}) e^{m^2 x\omega^2 \eta\varepsilon^{\mu-1/3}} dx\right| \leq \sigma, \text{ if } R \geq R_0(\sigma).$$

So indeed $\underline{I}$ is bounded and from (c.1) and (c.2) it follows that

$$L[Z_{A_1}(\xi,\eta)] = O(\varepsilon^{8\mu-5/3}) \quad \text{in a.}$$

In part b:

Here holds $\xi\varepsilon^{2\mu-2/3} >> 1$ and $\eta\varepsilon^{\mu-1/3} >> 1$.

Application of the saddle-point method (as in appendix a) gives:

$$\frac{\partial Y_1}{\partial \xi} = O(\varepsilon^{4\mu-4/3}),\ \frac{\partial Y_1}{\partial \eta} = O(\varepsilon^{4\mu-4/3}),\ \frac{\partial^2 Y_1}{\partial \eta^2} = O(\varepsilon^{7\mu-7/3}).$$

It is easily seen that $L[Z_{A_1}(\xi,\eta)] = O(\varepsilon^{9\mu-2})$ in b.

In the parts a and b together ($= A_1$) is $L[Z_{A_1}(\xi,\eta)] = O(\varepsilon^{9\mu-2})$.

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