STICHTING
MATHEMATISCH CENTRUM
2e BOERHAAVESTRAAT 49
AMSTERDAM

AFDELING TOEGEPASTE WISKUNDE

TW 113

Mean convergence of orthogonal series and Lagrange interpolation

by

Richard Askey

September 1969

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam, The Netherlands.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications; it is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O) and the Central Organization for Applied Scientific Research in the Netherlands (T.N.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

Mean convergence of orthogonal series and Lagrange interpolation

Richard Askey *)

We will primarily be concerned with the convergence of Lagrange interpolation taken at the zeros of orthogonal polynomials. This is a very old problem, the first results being Stieltjes's results on mechanical quadrature. In the course of solving some of the problems that have been posed about L^p convergence, we will be lead to consider a number of other problems. Some of these we can solve, but most of them are now only conjectures or even just problems. The problems and conjectures are probably the most interesting part of this paper and I hope that others will find them interesting and solve some of them.

Let $d\alpha(x)$ be a nonnegative measure on $[-1,1]$ and let $p_n(x)$ be the sequence of polynomials orthonormal with respect to $d\alpha(x)$ and normalized by $p_n(1) > 0$. Let $x_{k,n}$ be the zeros of $p_n(x)$ ordered by $-1 < x_{n,n} < \ldots < x_{1,n} < 1$. For a continuous function $f(x)$ on $[-1,1]$, the Lagrange interpolation polynomial $L_n^f(x)$ is defined to be the unique polynomial of degree $n-1$ which satisfies

$$L_n^f(x_{k,n}) = f(x_{k,n}). \tag{1}$$

It was shown by Faber that $L_n^f(x)$ does not necessarily converge uniformly to $f(x)$. For $d\alpha(x) = (1-x^2)^{-\frac{1}{2}}dx$, Grünwald and Marcinkiewicz have shown the existence of a continuous function $f(x)$ for which $L_n^f(x)$ is everywhere divergent. See Szegö [37, chapters XIV and XV] for references to these results, as well as to all other results that are mentioned without a specific reference. As we remarked above, Stieltjes proved a convergence theorem for all continuous functions. He proved that

$$\lim_{n\to\infty} \int_{-1}^{1} [L_n^f(x) - f(x)]\, d\alpha(x) = 0. \tag{2}$$

*) Supported in part by a fellowship from the John Simon Guggenheim Memorial Foundation, in part by the Office of Naval Research under Contract N00014-67-A-0128-0012, and in part by the Stichting Mathematisch Centrum, Amsterdam.

There are various theorems related to or generalizing (2). The most satisfying one is due to Erdös and Turán [15]. They prove

$$\lim_{n\to\infty} \int_{-1}^{1} [L^f(x) - f(x)]^2 \, d\alpha(x) = 0. \tag{3}$$

Actually they prove more than this, since the points of interpolation are zeros of polynomials that are more general than orthogonal polynomials, but we will not be concerned with this generalization since we have nothing new to add to their results.

There is another direction in which (2) and (3) can be extended. This is to find a value of $p > 2$ for which

$$\lim_{n\to\infty} \int_{-1}^{1} |L_n^f(x) - f(x)|^p \, d\alpha(x) = 0. \tag{4}$$

for all continuous functions.
We will show that it is not possible to find a $p > 2$ for which (4) holds for all measures. For certain specific measures, it is possible to find some $p > 2$. In particular, for $d\alpha(x) = (1-x^2)^{-\frac{1}{2}}dx$ Marcinkiewicz [26] and Erdös - Feldheim have shown that (4) holds for all $p < \infty$. For $d\alpha(x) = (1-x^2)^{\frac{1}{2}}dx$, Feldheim has shown that (4) fails for some continuous function for $p = 4$. We will give the answer to (4) for $d\alpha(x) = (1-x)^{\alpha} (1+x)^{\beta} dx$. In particular, for $\alpha = \beta = \frac{1}{2}$ we will show that (4) holds for $p < 3$ and that it fails for $p > 3$.

Two closely related questions are the following.

$$\lim_{n\to\infty} \int_{-1}^{1} [L_n^f(x) - f(x)] \, dx = 0, \tag{5}$$

$$\lim_{n\to\infty} \int_{-1}^{1} |L_n^f(x) - f(x)|^p dx = 0, \tag{6}$$

where the interpolation is still done at the zeros of $p_n(x)$. Since the notation $L_n^f(x)$ does not specify the points of interpolation, we shall sometimes use the more complicated notation $L_n^f(x;d\alpha)$, $L_n^f(x;w(x))$, where

$w(x)dx = d\alpha(x)$, or $L_n^f(x;\alpha,\beta)$, where $w(x) = (1-x)^\alpha (1+x)^\beta$. For these problems, Szegö proved that (5) holds for $d\alpha(x) = (1-x)^\alpha (1+x)^\beta dx$, $\alpha, \beta > -1$, $\max(\alpha,\beta) \leqq \frac{3}{2}$, and Holló proved that (6) holds for $p = 1$ if $\max(\alpha,\beta) < \frac{3}{2}$, and for $p = 2$ if $\max(\alpha,\beta) < \frac{1}{2}$. A proof of Holló's result is given by Turán in [39]. In [14], Erdös conjectured that (6) holds for all $p < \infty$ if $\max(\alpha,\beta) \leqq -\frac{1}{2}$. We will show even more, that (4) holds for all $p < \infty$ if $\max(\alpha,\beta) \leqq -\frac{1}{2}$.

The general question that suggests itself is to find the values of p for which

$$(7) \qquad \lim_{n\to\infty} \int_{-1}^{1} |L_n^f(x;d\alpha) - f(x)|^p \, d\beta(x) = 0$$

for all continuous functions $f(x)$. In this generality I have no idea what the answer might be. But the following special case is a reasonable conjecture.

Conjecture 1. $\lim_{n\to\infty} \int_{-1}^{1} |L_n^f(x,\alpha,\beta) - f(x)|^p (1-x)^\gamma (1+x)^\delta \, dx = 0$ for all continuous functions if the following conditions hold,

(i) if $\max(\alpha,\beta) > -\frac{1}{2}$, then (7) holds for

$$(8) \qquad p < \min(4(\gamma+1)/(2\alpha+1),\ 4(\delta+1)/(2\beta+1)),$$

where a negative term on the right is ignored. If $\beta < -\frac{1}{2}$, say, then we require $\delta \geqq \beta$.

(ii) if $\max(\alpha,\beta) \leqq -\frac{1}{2}$, then (7) holds for all continuous functions if $\gamma \geqq \alpha$, $\delta \geqq \beta$ and $p < \infty$.

While this general conjecture is beyond what we can do at present, we can show it for a number of interesting special cases. In particular, if $\gamma = \alpha$, $\delta = \beta$ (i.e. the case (4)), we can show that the conjecture is true. In this case, as in many others, we can also show that these results are almost best possible.

We will give a proof of this result first, then give our method of forming counterexamples, and then go back and sketch a different proof. The first proof has the defect of using some extremely deep analysis for what should be a relatively easy theorem. We will give the second proof in the hopes that someone will be able to help improve it. Once this has been done the second proof will be much more elementary than the first, and will apply to a more general class of polynomials. As we will show, the problem in the case $d\alpha(x) = d\beta(x)$ essentially reduces to the problem of mean convergence of orthogonal series, and the numbers given by (8) occur in Pollard's work on mean convergence. This result of Pollard can be slightly simplified, so we will sketch a slightly revised proof.

We start with the case $d\beta(x) = d\alpha(x)$. Then we must show that

$$\| L_n^f \|_p = \left[\int_{-1}^{1} |L_n^f(x)|^p d\alpha(x)\right]^{\frac{1}{p}} \leqq A\left[\int_{-1}^{1} |f(x)|^p d\alpha(x)\right]^{\frac{1}{p}}, \quad n = 0,\ 1,\ \ldots,$$

This is sufficient since $L_n^f(x) = f(x)$ for polynomials of degree $n - 1$, and these polynomials are dense. We compute $\| L_n^f \|_p$ as follows.

By the converse of Hölder's inequality we have

$$\| L_n^f \|_p = \sup_{\| g \|_q = 1} \int_{-1}^{1} L_n^f(x)\ g(x)\ d\alpha(x), \qquad {}^1/p + {}^1/q = 1.$$

Since the Erdös - Turán theorem implies that (4) holds for $p \leqq 2$, we may assume $2 < p < \infty$, and so $1 < q < 2$. If we expand $g(x)$ in an orthogonal series of $p_n(x)$ we have

$$\int_{-1}^{1} L_n^f(x)\ g(x)\ d\alpha(x) = \int_{-1}^{1} L_n^f(x)\ S_{n-1}^g(x)\ d\alpha(x),$$

since $p_n(x)$ is orthogonal to all polynomials of lower degree.

$$S_n^g(x) = \sum_{k=0}^{n} a_k\ p_k(x),$$

$$a_k = \int_{-1}^{1} g(x)\ p_k(x)\ d\alpha(x).$$

We use the fundamental property of Gaussian quadrature that

$$\int_{-1}^{1} r(x)\, d\alpha(x) = \sum_{k=1}^{n} r(x_{k,n})\, \lambda_k, \quad r(x) \text{ a polynomial of degree } 2n-1,$$

where λ_k are the Christoffel numbers, which are nonnegative. This gives

$$\int_{-1}^{1} L_n^f(x)\, S_{n-1}^g(x)\, d\alpha(x) = \sum_{k=1}^{n} L_n^f(x_{k,n})\, S_{n-1}^g(x_{k,n})\, \lambda_k =$$

$$= \sum_{k=1}^{n} f(x_{k,n})\, S_{n-1}^g(x_{k,n})\, \lambda_k.$$

Apply Hölder's inequality to get

$$\sum_{k=1}^{n} f(x_{k,n})\, S_{n-1}^g(x_{k,n})\, \lambda_k \leqq \Big[\sum_{k=1}^{n} |f(x_{k,n})|^p\, \lambda_k\Big]^{\frac{1}{p}}$$

$$\Big[\sum_{k=1}^{n} [S_{n-1}^g(x_{k,n})|^q\, \lambda_k\Big]^{\frac{1}{q}}.$$

Using Stieltjes's result (2) we have

$$\Big[\sum_{k=1}^{n} |f(x_{k,n})|^p\, \lambda_k\Big]^{\frac{1}{p}} \leqq A\| f\|_p$$

so it is sufficient to estimate

$$\Big[\sum_{k=1}^{n} |S_{n-1}^g(x_{k,n})|^q\, \lambda_k\Big]^{\frac{1}{q}}. \tag{9}$$

If we could show that

$$\Big[\sum_{k=1}^{n} |S_{n-1}^g(x_{k,n})|^q\, \lambda_k\Big]^{\frac{1}{q}} \leqq A\| S_{n-1}^g\|_q \tag{10}$$

then we would have reduced the problem to one involving mean convergence of orthogonal series, and we will say more about this problem later. However, I can only prove (10) in a few cases at present, the most interesting being $d\alpha(x) = (1-x^2)^\alpha dx$, $\alpha \geqq -\frac{1}{2}$. So we must use a different estimate. The trouble with estimating (9) is that $S^g_{n-1}(x_{k,n})$ not only depends on n in the argument, but the function itself depends on n. We can do away with this dependence on n be introducing the maximal partial sum, $M^g(x)$, defined by

$$M^g(x) = \sup_{n = 0,\, 1,\, \dots} |S^g_n(x)|, \; -1 \leqq x \leqq 1.$$

Clearly we have

$$\Big[\sum_{k=1}^{n} |S^g_{n-1}(x_{k,n})|^q \lambda_k\Big]^{\frac{1}{q}} \leqq \Big|\sum_{k=1}^{n} \big[M^g(x_{k,n})|^q \lambda_k\big]^{\frac{1}{q}}$$

and then we may use Stieltjes's theorem again to get

$$\Big[\sum_{k=1}^{n} |M^g(x_{k,n})\big]^q \lambda_k\Big|^{\frac{1}{q}} \leqq A\| M^g\|_q.$$

Thus it is sufficient to show that

$$\| M^g\|_q \leqq A\| g\|_q \tag{11}$$

and this we can do for $d\alpha(x) = (1-x)^\alpha (1+x)^\beta dx$, α, $\beta > -1$, for some $q < 2$. B. Muckenhoupt suggested to me that M^g could be used in the above way. Inequalities like (11) are very deep and they have only recently been obtained for Fourier series by Hunt [21], using ideas of Carleson [12]. Gilbert [19] has shown how to combine their results with methods of Pollard for partial sums of orthogonal series to prove that

$$\Big[\int_{-1}^{1} |M^g(x)|^q (1-x)^\alpha (1+x)^\beta dx\Big]^{\frac{1}{q}} \leqq A\Big[\int_{-1}^{1} |g(x)|^q (1-x)^\alpha (1+x)^\beta dx\Big]^{\frac{1}{q}},$$

for α, $\beta \geqq -\frac{1}{2}$, $4(\alpha+1)/(2\alpha+3) < q < 4(\alpha+1)/(2\alpha+1)$ and the same inequalities with α replaced by β. Actually Gilbert only gives the details for $\alpha = \beta$, but the more general case follows from the same argument. For $\min(\alpha,\beta) < -\frac{1}{2}$, these ideas can be combined with recent work of Muckenhoupt [30] to obtain the same type of theorem, with $1 < q < \infty$ if $\max(\alpha,\beta) < -\frac{1}{2}$, and $4(\alpha+1)/(2\alpha+3) < q < 4(\alpha+1)/(2\alpha+1)$ if $-1 < \beta < -\frac{1}{2} \leqq \alpha$.

In summary we have proven the following theorem.

Theorem 1. Let $f(x)$ be a continuous function on $[-1,1]$ and let $1 < p < \infty$, $\alpha, \beta > -1$ be given. Then

$$\lim_{n\to\infty} \int_{-1}^{1} |L_n^f(x) - f(x)|^p (1-x)^\alpha (1+x)^\beta dx = 0$$

if

(i) $1 < p < \min(4(\alpha+1)/(2\alpha+1), 4(\beta+1)/(2\beta+1)$, $\alpha, \beta > -\frac{1}{2}$

(ii) $1 < p < 4(\alpha+1)/(2\alpha+1)$, $-1 < \beta \leqq -\frac{1}{2}$, $\alpha > -\frac{1}{2}$

(iii) $1 < p < \infty$ $\quad -1 < \beta$, $\alpha \leqq -\frac{1}{2}$.

There are other possible extensions of Theorem 1. First, it can almost surely be extended to Riemann integrable functions. The techniques used by Erdös and Turán can probably be used. There is also a possible extension to L^p functions. However L^p functions are only defined almost everywhere so one must use a two dimensional type convergence, averaging over translated Lagrange polynomials. The appropriate translate is probably the generalized tranlate given in [10], and so these theorems can only be proven for $\alpha, \beta \geq -\frac{1}{2}$ at present. See Marcinkiewicz and Zygmund [27] for the L^p result for interpolation associated with $\cos n\theta$ and $\sin n\theta$. Finally Theorem 1 can be combined with the L^1 and L^2 results of Holló - Turán to obtain some other cases of Conjecture 1. You use the M. Riesz interpolation theorem generalized to include the case of a change of measure. However this method can not hope to give us all of Conjecture 1, for among other reasons we have no way of getting results for $0 < p < 1$, which is unfortunately a fairly common case. For instance, $\gamma = -\frac{1}{2}$, $\alpha = 0$ leads to the conjecture that (7) holds for $p < \frac{1}{2}$.

Now we consider the problem of showing that the condition $p < 4(\alpha+1)/(2\alpha+1)$ can not be improved upon. There is a very simple argument which can be given to show that Theorem 1 fails for some continuous function if $\alpha > 4(\alpha+1)/(2\alpha+1)$.

In [37], Szegö proved the existence of a continuous function f(x) for which

$$(12) \qquad L_n^f(1) \geqq A\, n^{\alpha+\frac{1}{2}}, \quad \alpha > -\tfrac{1}{2},$$

where the interpolation is at the zeros of $P_n^{(\alpha,\beta)}(x)$, the orthogonal polynomials of $(1-x)^\alpha\,(1+x)^\beta$. It is easy to show that

$$(13) \qquad |\wp_n(x)| \leqq A\, n^{(2\alpha+2)/p} \left[\int_{-1}^{1} |\wp_n(x)]^p\,(1-x)^\alpha(1+x)^\beta dx\right|^{\frac{1}{p}},$$

$$\alpha \geqq \beta \geqq -\tfrac{1}{2}, \; p \geqq 1.$$

This can be done in an elementary way using the case p = 2 which is classical [37, Theorem 7.71.2] as in [38] or by an interpolation argument as in [7], now using the convolution structure for Jacobi polynomials which is given in [10]. The elementary method is easier, but the interpolation argument has the advantage that all of the machinary of interpolation theory can be used, and it may be possible to obtain inequalities like (13) for more general norms. In particular, the classical inequalities of Berstein and Markoff on derivatives can be thought of as inequalities on the same polynomials in a Lip 1 norm and an L^∞ norm. This can be used with (13), and it is likely these inequalities will also prove useful. In particular they should be obtained for Besov spaces with weights like $(1-x)^\alpha\,(1+x)^\beta$.

If we combine (13) and (12) we see that

$$(14) \qquad A\, n^{(2\alpha+2)/p} \left[\int_{-1}^{1} |L_n^f(x)|^p\,(1-x)^\alpha(1+x)^\beta dx\right]^{\frac{1}{p}} \geqq A\, n^{\alpha+\frac{1}{2}}$$

where A stands for some arbitrary positive constant, independent of f and n, which may vary even in the same formula. (14) shows that

$$(15) \qquad \left[\int_{-1}^{1} |L_n^f(x)|^p\,(1-x)^\alpha(1+x)^\beta dx\right]^{\frac{1}{p}} \geqq A\, n^{(2\alpha+1)/2\,-\,(2\alpha+2)/p}$$

and the right hand side goes to infinity if $p > 4(\alpha+1)/(2\alpha+1)$. If we use

(13) for $\alpha = \gamma, \beta = \delta$, we have

$$\left[\int_{-1}^{1} |L_n^f(x,\alpha,\beta)|^p (1-x)^\gamma (1+x)^\delta dx\right]^{\frac{1}{p}} \geqq A\, n^{(2\alpha+1)/_2 - (2\gamma+2)/_p}$$

and it is this inequality that suggested Conjecture 1.

A more general problem than (13) is to find the correct order of growth of $A(n)$ in

$$(16) \qquad \left[\int_{-1}^{1} |\varphi_n(x)|^q v(x)dx\right]^{\frac{1}{q}} \leqq A(n)\left[\int_{-1}^{1} |\varphi_n(x)]^p w(x)dx\right|^{\frac{1}{p}},$$

$$0 < p \leqq q \leqq \infty.$$

For $1 \leqq p \leqq q$ and $v(x) = (1-x)^\gamma (1+x)^\delta$, $w(x) = (1-x)^\alpha (1+x)^\beta$, I can give some results, but they are not needed here so I forgo them. Some special cases are given by Hille, Szegö and Tamarkin [20].

If we let α become large in (15) we see that the Erdös - Turán L^2 result cannot be improved. For given any $p > 2$, we can find α large enough so that $p > 4(\alpha+1)/(2\alpha+1)$, and then L^p convergence fails for this value of p and α. Prof. Turán asked the interesting question of finding a weight function for which L^p convergence fails for all $p > 2$. I am sure that this happens for the weight function of some of the Pollaczek polynomials. This function $w(x)$ vanishes so rapidly at $x = 1$ that $\int_{-1}^{1} \frac{|\log w(x)|}{(1-x^2)^{\frac{1}{2}}} dx$ diverges.

It should be possible to show that Theorem 1 fails for $p = 4(\alpha+1)/(2\alpha+1)$, $\alpha \geq \beta$, $\alpha > -\frac{1}{2}$ using the function Szegö used to prove (12). The technical details are complicated so I will not include it here.

Now to return to the convergence theorem. The only problem that arose was in trying to prove that

$$\left[\sum_{k=1}^{n} |S_{n-1}^g(x_{k,n})|^q \lambda_k\right]^{\frac{1}{q}} \leqq A\|g\|_q.$$

If we could show that

$$(17)\qquad \Big[\sum_{k=1}^{n} |S^{g}_{n-1}(x_{k,n})|^{q}\,\lambda_k\Big]^{\frac{1}{q}} \leqq A\Big[\int_{-1}^{1} |S^{g}_{n-1}(x)|^{q}\,d\alpha(x)\Big]^{\frac{1}{q}}$$

we would only have to show that

$$(18)\qquad \| S^{g}_{n}\|_{q} \leqq A\| g\|_{q}.$$

(17) is true for $q = 2$ and $q = \infty$ with $A = 1$ and is known for $1 \leqq q \leqq \infty$ if $d\alpha(x) = (1-x^2)^{-\frac{1}{2}}dx$. See [40, vol. II] for a simple proof. This result was first proven by Marcinkiewicz [27]. The proof Zygmund gives for this $d\alpha$ can be extended to handle $d\alpha(x) = (1-x^2)^{\alpha}\,dx$, $\alpha > -\frac{1}{2}$. However this proof uses the positivity of the Cesaro means of some order, and it is very unlikely that this positivity can be extended beyond Jacobi series, and even there it is still unknown for most values of (α,β). I suspect that the $(C,\alpha+\frac{3}{2})$ means of $\sum a_n P_n^{(\alpha,-\frac{1}{2})}(x) \sim f(x) \geqq 0$ are nonnegative. If so, then the $(C,\alpha+\beta+2)$ means of $\sum a_n P_n^{(\alpha,\beta)}(x) \sim f(x) \geqq 0$ should be nonnegative, $\alpha \geqq \beta \geqq -\frac{1}{2}$. Kogbetliantz [23] has proven this result for $\alpha = \beta > -\frac{1}{2}$ and Fejér has proven it for $\alpha = \beta = \frac{1}{2}$ and $\alpha = -\beta = \frac{1}{2}$. It is this last result of Fejér [16] that suggests this conjecture. This positivity follows for $x = 1$ by using Bateman's integral [6], and would follow for $-1 \leqq x \leqq 1$ if the positivity of the generalized translation operator had been proven. This however is still an open problem. See [5]. Also if this positivity had been proven then we could use the positivity of the $(C,2\alpha+2)$ means for $(\alpha,-\frac{1}{2})$, which would be enough to complete our proof for $\alpha \geqq \beta \geqq -\frac{1}{2}$. We will not give any details because (17) should not depend on such delicate theorems. It should be a general fact for most, if not all, measures, at least for $q > 2$ and probably for $1 \leqq q \leqq \infty$.

Thus the problem we are considering should reduce to showing that

$$\| S^{g}_{n}\|_{q} \leqq A\| g\|_{q}.$$

Let $p_n(x) = k_n x^n + \ldots$, $k_n > 0$, be the polynomials orthonormal with respect to $d\alpha(x)$. For $g(x)$ a function integrable with respect to $d\alpha(x)$

we define

$$a_n = \int_a^b g(x)\, p_n(x)\, d\alpha(x)$$

Then $S_n^g(x)$ is given by

$$S_n^g(x) = \sum_{k=0}^{n} a_k\, p_k(x) = \int_a^b g(y) \sum_{k=0}^{n} p_k(x)\, p_k(y)\, d\alpha(y)$$

$$= \frac{k_n}{k_{n+1}} \int_a^b \frac{g(y)}{x-y} \left[p_{n+1}(x)\, p_n(y) - p_n(x)\, p_{n+1}(y)\right] d\alpha(y).$$

If a and b are finite then $\frac{k_n}{k_{n+1}} \leqq C$. See [2]. I would like to thank G. Freud for bringing this to my attention. We are thus lead to consider $p_{n+1}(x)\, p_n(y) - p_n(x)\, p_{n+1}(y)$. We can try to handle each term separately, and this works if the polynomials are uniformly bounded and the measure does not grow too fast at any point. However this almost never happens, (for Jacobi polynomials it only works for $\alpha = \beta = -\frac{1}{2}$), so we must use some sort of cancellation. Pollard used a complicated procedure to obtain cancellation at $x = \pm 1$ at the same time. However this is not necessary, and an easier method works. We now consider $a = -1$, $b = 1$ and we may assume $0 \leqq x \leqq 1$, since the same type of argument will handle $-1 \leqq x < 0$. Then if $-1 \leqq y \leqq -\epsilon < 0$ the factor $(x-y)^{-1}$ is bounded and we no longer have a singular integral, except at possible singularities in $d\alpha(y)$. We now assume that

$$d\alpha(y) = w(y)\, dy = (1-y)^{\alpha}(1+y)^{\beta}\, t(y)\, dy,$$

$0 < A \leqq t(y) \leqq B < \infty$. Pollard also assumed that $\alpha, \beta \geqq -\frac{1}{2}$ and that $t'(y)$ was smooth. It is only necessary to assume that $\alpha, \beta > -1$ and $|t(x+h) - t(x)| \leqq Ah$, $-1 \leqq x, x+h \leqq 1$. The argument when $\alpha, \beta \geqq -\frac{1}{2}$ is not satisfied is similar to the one we will give, but it is slightly more complicated. It is given by Muckenhoupt in [30]. We will only consider $\alpha, \beta \geqq -\frac{1}{2}$ here. Then we have

$$(1-x^2)^{\frac{1}{4}} \left[w(x)\right]^{\frac{1}{2}} |p_n(x)| \leqq A, \quad -1 \leqq x \leqq 1. \tag{19}$$

For $t(x) = 1$ this is a well known fact about Jacobi polynomials [37, (7.32.5)] and the general case follows from Korous's theorem [37, Theorem 7.13]. The inequality (19) is an extremely useful inequality and it would be of real interest to prove it for more general weight functions. However it is not necessary for the uses we have, since it fails if $\alpha < -\frac{1}{2}$ for the Jacobi polynomials, and Muckenhoupt has proven the mean convergence theorem then.

To get back to our proof we can now consider each of the terms $p_{n+1}(x)\, p_n(y)$ and $p_n(x)\, p_{n+1}(y)$ seperately, and then we need to estimate

$$h(x) = \int_{-1}^{-\epsilon} |g(y)|\, (1-x)^{-\frac{\alpha}{2}-\frac{1}{4}} (1+y)^{-\frac{\beta}{2}-\frac{1}{4}+\beta} dy.$$

We have

$$\int_0^1 |h(x)|^p (1-x)^\alpha (1+x)^\beta dx \leqq A \int_0^1 (1-x)^{\alpha - p(\frac{\alpha}{2}+\frac{1}{4})} dx \left[\int_{-1}^{-\epsilon} |g(y)|\, (1+y)^{\frac{\beta}{2}-\frac{1}{4}} dy\right]^p$$

and applying Hölder's inequality we have

$$\int_0^1 |h(x)|^p\, (1-x)^\alpha (1+x)^\beta dx \leqq A \int_{-1}^1 |g(y)|^p\, (1-y)^\alpha (1+y)^\beta dy$$

if $p < 4(\alpha+1)/(2\alpha+1)$ and $p > 4(\beta+1)/(2\beta+3)$. These are just the conditions that Pollard needed in his proof, and they were shown to be best possible by Newman and Rudin [32].

The crux of the proof now comes. We consider

$$p_{n+1}(x)\, p_n(y) - p_n(x)\, p_{n+1}(y) \tag{20}$$

$$= \left[\frac{p_{n+1}(x)}{p_{n+1}(1)} - \frac{p_n(x)}{p_n(1)}\right] p_n(y)\, p_{n+1}(1) - p_n(x) \left[\frac{p_n(y)}{p_n(1)} - \frac{p_{n+1}(y)}{p_{n+1}(1)}\right] p_{n+1}(1).$$

The polynomial $\frac{p_n(x)}{p_n(1)} - \frac{p_{n+1}(x)}{p_{n+1}(1)} = c_n(1-x)\ \frac{q_n(x)}{q_n(1)}$ for some $c_n > 0$ where $q_n(x)$ are the polynomials orthonormal with respect to $(1-x)^{\alpha+1}\ (1+x)^{\beta}\ t(x) = (1-x)\ w(x)$. Then we also have

$$(1-x^2)^{\frac{1}{4}}(1-x)^{\frac{1}{2}}\ [w(x)]^{\frac{1}{2}}\ |q_n(x)| \leqq C.$$

To continue the proof we now need an estimate on the size of $\frac{p_{n+1}(1)}{p_n(1)}\ c_n$. Equating coefficients of y^{n+1} we see that

$$\frac{c_n\ p_{n+1}(1)}{p_n(1)} = \frac{k_{n+1}}{l_n}$$

where l_n is the highest coefficient of $q_n(x)$.

Szegö [37 Theorem 12.7.1] has shown that if we assume

$$\int_{-1}^{1} \frac{|\log w(x)|}{(1-x^2)^{\frac{1}{2}}}\ dx < \infty \tag{21}$$

then

$$\frac{k_n}{2^n} \to \pi^{-\frac{1}{2}} \exp\left[\frac{-1}{2\pi} \int_{-1}^{1} \frac{\log w(x)}{(1-x^2)^{\frac{1}{2}}}\ dx\right]. \tag{22}$$

In our case (21) is satisfied for $w(x)$ and $(1-x)\ w(x)$ so we have

$$\left|\frac{k_{n+1}}{l_n}\right| \leqq A$$

Thus the integrals we must estimate are bounded by

$$A_n(x) \int_{-\epsilon}^{1} \frac{g(y)(1-y)^{\frac{\alpha}{2}-\frac{1}{4}}\ (1-x)^{\frac{3}{4}-\frac{\alpha}{2}}\ B_n(y)\ dy}{x-y}$$

and

$$A_n(x)\int_{-\epsilon}^{1}\frac{g(y)(1-y)^{\frac{\alpha}{2}+\frac{3}{4}}(1-x)^{-\frac{1}{4}-\frac{\alpha}{2}}B_n(y)\,dy}{x-y}$$

where $A_n(x)$ and $B_n(y)$ are bounded functions in both x and n. Since we are interested in L^p norms we may ignore them since

$$\| g(y)\ B_n(y)\|_p \leqq A\| g(y)\|_p .$$

Now we have reduced our problem to estimating

$$(1-x)^{-\frac{\alpha}{2}+\frac{1}{4}\pm\frac{1}{2}}\int_{-\epsilon}^{1}\frac{g(y)(1-y)^{\frac{\alpha}{2}+\frac{1}{4}\mp\frac{1}{2}}dy}{x-y}$$

and such integrals are classical. They can be reduced to the classical M. Riesz transform and an absolutely convergent integral of Hardy type. The final theorem that comes out of all this is due to Muckenhoupt [30] for Jacobi polynomials with $\alpha,\ \beta > -1$, while special caes of it were obtained by Pollard [33], [34], [35] for $\alpha,\ \beta \geqq -\frac{1}{2}$.

Theorem 2. Let $w(x) = (1-x)^{\alpha}\ (1+x)^{\beta}\ t(x)$, $0 < c \leqq t(x) \leqq C < \infty$, $|t(x) - t(y)| \leqq A|x - y|$, and let $S_n^{(\alpha,\beta)}(x)$ be the partial sum of the orthogonal series $\sum a_n\ p_n(x)$, $p_n(x)$ orthogonal on $[-1,1]$ with respect to $w(x)$. Then

$$\lim_{n\to\infty}\int_{-1}^{1}|s_n^{(\alpha,\beta)}(x) - f(x)|^p\ (1-x)^a(1+x)^b dx = 0 \tag{23}$$

for all measurable f with $\int_{-1}^{1} |f(x)|^p\ (1-x)^a\ (1+x)^b\ dx$ finite if $1 < p < \infty$ and

$$\max(-1,\ (\tfrac{1}{4}+\tfrac{\alpha}{2})p - 1) < a < \min((1+\alpha)p - 1,\ (\tfrac{3}{4}+\tfrac{\alpha}{2})p - 1) \tag{24}$$

and the same inequalities are satisfied with a and α replaced by b and β.

In the Jacobi case, i.e. $t(x) = 1$, this theorem is best possible in the sense that there is an f satisfying the right integrability condition for which (23) fails if either of the inequalities in (26) does not hold.

Pollard [34] asked the very interesting question of trying to extend these results to more general measures and said that he did not have any conjectures about what the general theorem was. In [3] I made a conjecture, but it is not a very useful conjecture, since it is usually as hard to solve the Cesaro summability problem, which is the problem that I suspect has a strong relationship to the values of p for which we get mean convergence. (This is true in one dimension, but not in several where mean convergence problems are often extremely difficult).

There is a fairly general class of weight functions on $(-1,1)$ for which it is possible to make a reasonable conjecture.

Conjecture 2. Let $w(x) = (1-x)^{\alpha} \prod_{i=1}^{j} |x-x_i|^{\beta_i} (1+x)^{\gamma}$, $-1 < x_i < \ldots < x_1 < 1$. Then if $\alpha, \gamma \geqq -\frac{1}{2}$ and $\beta_i \geqq 0$ (for simplicity only), we have

$$\int_{-1}^{1} |f(x) - S_n^f(x)|^p \, w(x)\, dx \to 0, \quad 1 < p < \infty$$

for all $f \in L^p$ if

$$\frac{4(1+\alpha)}{2\alpha+3} < p < \frac{4(1+\alpha)}{2\alpha+1}$$

$$\frac{2(1+\beta_i)}{2+\beta_i} < p < \frac{2(1+\beta_i)}{\beta_i}, \quad i = 1, 2, \ldots, j,$$

$$\frac{4(1+\gamma)}{2\gamma+3} < p < \frac{4(1+\gamma)}{2\gamma+1} .$$

For $\alpha = \gamma$, $j = 1$, $x_1 = 0$ this can be proved using Theorem 2 and a quadratic transformation on Jacobi polynomials. If we could prove (19) then this conjecture would be easy to prove. This conjecture tells us what effect an isolated zero of the weight function has on mean convergence problems. The next case of interest would be to see what the effect of an interval of zeros would be. Thus we should solve the problem for $w(x) = 1$, $-1 \leqq x \leqq -a$, $a \leqq x \leqq 1$, $a > 0$, $w(x) = 0$, $-a < x < a$. I am sure this can be done using an idea of Achiezer [1] on how to calculate these orthogonal polynomials. Actually he handles a different measure but it is possible to extend his ideas in the case of a symmetric interval to quite general weight functions. The next case to treat after this is the case of a measure that only has point masses. On $[0,\infty)$, the Poisson measure and Charlier polynomials immediately suggest themselves. And there is one set of polynomials with a discrete measure whose only limit point is at $x = 0$ that may be possible to handle. See Carlitz [13] and Karlin and McGregor [22] as well as Maki [24] for further results on this type of polynomial. After this the problem becomes very hard and a purely singular measure concentrated on the Cantor set should be handled; but I have no idea at all how to attack this problem.

There is quite likely a strong connection between the p for which mean convergence holds for a measure $d\alpha(x)$ and the rate of growth of $A(n)$ in

$$\|\varphi_n\|_\infty \leqq A(n)\ \|\varphi_n\|_2, \tag{25}$$

or more generally in

$$\left[\int_{-1}^{1} |\varphi_n(x)|^q\, d\alpha(x)\right]^{\frac{1}{g}} \leqq A(n,p,g)\ \left[\int_{-1}^{1} |\varphi_n(x)|^p\, d\alpha(x)\right]^{\frac{1}{p}},\quad q > p.$$

If so this would be a useful result, since it is easier to work with (25) than with mean convergence theorems.

There is one other set of orthogonal polynomials for which mean convergence theorems have been obtained. These are the Laguerre polynomials, and their special cases, the Hermite polynomials. The results for Hermite series follow from the Laguerre series so we will only state the results for Laguerre series. The Laguerre polynomials will be denoted by $L_n^\alpha(x)$ and they are orthogonal on $(0,\infty)$ with respect to $w(x) = x^\alpha e^{-x}$, $\alpha > -1$. Pollard [34] showed that an inequality of the form

$$(26) \qquad \left[\int_0^\infty |S_n(x)|^p x^\alpha e^{-x}\,dx\right]^{\frac{1}{p}} \leqq A\left[\int_0^\infty |f(x)|^p x^\alpha e^{-x}\,dx\right]^{\frac{1}{p}}$$

could hold for all f with the right hard side finite only for $p = 2$.

Wainger and I proved [8] that

$$(27) \qquad \left[\int_0^\infty |S_n(x)\, x^{\frac{\alpha}{2}} e^{-\frac{x}{2}}|^p\,dx\right]^{\frac{1}{p}} \leqq A\left[\int_0^\infty |f(x)\, x^{\frac{\alpha}{2}} e^{-\frac{x}{2}}|^p\,dx\right]^{\frac{1}{p}}$$

for all f with the right hand side finite for $\frac{4}{3} < p < 4$, and that (27) fails to hold with A independent n for $1 \leqq p \leqq \frac{4}{3}$ and $p \geqq 4$. By analogy with the results for Fourier series, and even Jacobi series, there should be a theorem which holds for all p, $1 < p < \infty$, of the sort

$$(28) \qquad \left[\int_0^\infty |S_n(x)\, u(x)|^p\,dx\right]^{\frac{1}{p}} \leqq A\left[\int_0^\infty |f(x)\, u(x)|^p\,dx\right]^{\frac{1}{p}}.$$

It is surprising that this is not true. Muckenhoupt [28] has shown that if (28) holds for some p, $1 \leqq p \leqq \frac{4}{3}$ or $p \geqq 4$ then $u(x) = 0$ almost everywhere.

Muckenhoupt [29] has obtained some theorems for $1 < p < \infty$ when the problem is changed slightly. These theorems are complicated and we will only state one of them to give the reader an idea of the type of result that can be obtained.

Theorem 3. Let $1 < p < \infty$, $\alpha > -1$, $u(x) = e^{-\frac{x}{2}} x^{\frac{\alpha}{2}} (\frac{x}{1+x})^a (1+x)^b$, $v(x) = e^{-\frac{x}{2}} x^{\frac{\alpha}{2}} (\frac{x}{1+x})^A (1+x)^B (1+\log^+ x)^\beta$, where $\beta = 1$ if $b = B$ and p is $\frac{4}{3}$ or 4 and $\beta = 0$ otherwise. Assume that

$$a > -\frac{1}{p} + \max(-\frac{\alpha}{2}, \tfrac{1}{4}),\ A < 1 - \frac{1}{p} - \max(-\frac{\alpha}{2}, \tfrac{1}{4}),\ A \leqq a.$$

Then

$$\int_0^\infty |S_n(x)\ u(x)|^p\ dx \leqq A \int_0^\infty |f(x)\ v(x)|^p\ dx$$

if

$$\begin{array}{ll}
b < \frac{3}{4} - \frac{1}{p}, & 1 < p \leqq 4, \\
b \leqq \frac{7}{12} - \frac{1}{3p}, & p > 4, \\
B \geqq -\frac{1}{4} - \frac{1}{3p}, & 1 < p < \frac{4}{3}, \\
B > \frac{1}{4} - \frac{1}{p}, & \frac{4}{3} \leqq p, \\
b \leqq B + \frac{1}{2} - \frac{2}{3p}, & 1 < p < \frac{4}{3}, \\
b \leqq B, & \frac{4}{3} \leqq p \leqq 4, \\
b \leqq B - \frac{1}{6} + \frac{2}{3p}, & 4 < p < \infty
\end{array}$$

and if we have equality in one of the last three conditions, we do not have equality in the second or third condition.

These conditions are essentially best possible, except possibly for the cases when $\beta = 1$. For technical reasons, the lack of suitable asymptotic estimates, the proof in [8] was only for the cases $\alpha \geqq 0$ and $\alpha = -\frac{1}{2}$. Muckenhoupt [29] showed how to obtain these results for $\alpha > -1$ by obtaining the proper estimates and then in [31] he showed how the asymptotic formulas could be obtained by recurrence relation from the known estimates for $\alpha \geqq 0$ of Erdélyi.

The lack of nice theorems for $p = \frac{4}{3}$ and $p = 4$ suggest that there are only fairly weak results to be obtained for Lagrange interpolation at the zeros of the Laguerre or Hermite polynomials. Turán raised this question in [39] and I too would like to see some results on this question. However I am afraid that they will be weaker than one might have suspected.

For Jacobi series and even for a large class of Sturm - Liouville expansions it is possible to prove theorems that are much deeper than mean convergence theorems. It is possible to set up a mapping between Jacobi series and cosine series that is bounded in L^p, $1 < p < \infty$, and so obtain many multiplier theorems for Jacobi series directly from the corresponding results for cosine series. See [4], [9], and [18]. However not all multiplier theorems can be obtained in this fashion, and some, especially those dealing with fractional integration and smoothness conditions, must be obtained directly from the generalized translation operators. For ultraspherical series some of these theorems have recently been obtained by Löfström and Peetre [24] and Berens, Butzer and Pawalke [11] and the boundedness of the generalized translation operator for Jacobi series was demonstrated by Askey and Wainger [10]. There are also some applications given there and in Ganser [17].

In another direction Schindler [36] has proven some mean convergence and bounded mapping theorems for Mehler transforms. These are integral transforms with $P_{-\frac{1}{2}+it}(\cosh x)$ as kernel. Due to the complexity of the asymptotic formulas of these functions, this is a harder result to prove than the corresponding theorems for Jacobi series. It is unlikely that we can prove these mapping theorems for a wide class of orthogonal series (and they fail for some p, $1 < p < \infty$ for $P_n^{(\alpha,\beta)}(x)$, $\alpha < -\frac{1}{2}$), so there is still a need to handle the mean convergence theorems directly.

[1] N.I. Achieser, Uber eine Eigenschaft der elliptischen Polynome, Comm. de la Sociêtê Math. de Kharkoff (4) 9(1934), 3-8.

[2] G. Alexitz, Konvergenz Probleme der Orthogonalreihen, Akadêmiai Kiadó, Budapest VEB Deutscher Verlag der Wissenschafter, Berlin, 1960.

[3] R. Askey, Norm inequalities for some orthogonal series, Bull. Amer. Math. Soc. 73(1966), 808-823.

[4] R. Askey, A transplantation theorem for Jacobi series, Ill. Jour. Math. 13(1969).

[5] R. Askey, Orthogonal polynomials and positivity, to appear in SIAM Symposium on Special Functions, also Mathematisch Centrum technical report TW112.

[6] R. Askey and J. Fitch, Integral representations for Jacobi polynomials and some applicationsm 26(1969), 411-437.

[7] R. Askey and I.I. Hirschman Jr., Mean summability for ultraspherical polynomials, Math. Scand. 12(1963), 167-177.

[8] R. Askey and S. Wainger, Mean convergence of expansions in Laguerre and Hermite series, Amer. Jour. Math. 87(1965), 695-708.

[9] R. Askey and S. Wainger, Tranplantation theorem for ultraspherical series, Ill. Jour. Math. 10(1966), 322-344.

[10] R. Askey and S. Wainger, A convolution structure for Jacobi series, Amer. Jour. Math. 91(1969).

[11] H. Berens, P.L. Butzer and S. Pawalke, Limitierungsverfahren von Reihen mehrdimensionaler Kugelfunktionen und deren Saturationverhalten, Publ. of R.I.M.S., Kyoto, ser.A, 4(1968), 201-268.

[12] L. Carleson, On convergence and growth of partial sums of Fourier series, Acta Math. 116(1966), 135-157.

[13] L. Carlitz, On some polynomials of Tricomi, Boll. Unione Math. Ital. (3) 13(1958), 58-64.

[14] P. Erdös, Problems and results on the convergence and divergence properties of the Lagrange interpolation polynomials and some extremal problems, Mathematica (Cluj) 19(33) (1968), 65-73.

[15] P. Erdös and P. Turán, On interpolation. I. Quadrature and mean convergence in the Lagrange interpolation, Ann. Math. (2) 38(1937), 142-155.

[16] L. Fejer, Neue Eigenschaften der Mittel werte bei den Fourierreihen, J.L.M.S. 8(1933), 53-62.

[17] C. Ganser, Modulus of continuity conditions for Jacobi series, Jour. Math. Anal. Appl. 27(1969), 575-600.

[18] J.E. Gilbert, Maximal theorems for some orthogonal series, I. Trans. Amer. Math. Soc.

[19] J.E. Gilbert, On convergence and maximal theorems for some orthogonal series, to appear.

[20] E. Hille, G. Szegö and J.D. Tamarkin, On some generalizations of a theorem of A. Markoff, Duke Math. Jour. 3(1937), 729-739.

[21] R. Hunt, On the convergence of Fourier series, in Orthogonal Expansions and their Continuous Analogues, edited by D.T. Haimo, Southern Illinois Univ. Press, Carbondale and Edwardsville, 1968.

[22] S. Karlin and J. McGregor, Many server queueing processes with Poisson input and exponential service times. Pacific Jour. Math. vol. 8(1958), 87-118.

[23] E. Kogbetliantz, Recherches sur la sommabilité des séries ultrasphériques par la méthode des moyennes arithmétiques, Jour. de Math. (9) 3(1924), 107-187.

[24] J. Löfström and J. Peetre, Approximation theorems connected with generalized translations, Math. Ann. 18(1969), 255-268.

[25] D. Maki, On constructing distribution functions; a bounded denumerable spectrum with n limit points, Pacific Jour. Math. 22(1967), 431-451.

[26] J. Marcinkiewicz, Sur l'interpolation (I), Studia Math. 6(1936), 1-17.

[27] J. Marcinkiewicz and A. Zygmund, Mean values of trigonometric polynomials, Fund. Math. 28(1937), 131-166.

[28] B. Muckenhoupt, Mean convergence of Hermite and Laguerre series, I, Trans. Amer. Math. Soc.

[29] B. Muckenhoupt, Mean convergence of Hermite and Laguerre series, II, Trans. Amer. Math. Soc.

[30] B. Muckenhoupt, Mean convergence of Jacobi series, Proc. Amer. Math. Soc.

[31] B. Muckenhoupt, Asymptotic forms for Laguerre polynomials, Proc. Amer. Math. Soc.

[32] J. Newman and W. Rudin, Mean convergence of orthogonal series, Proc. Amer. Math. Soc. 3(1952), 219-222.

[33] H. Pollard, The mean convergence of orthogonal series of polynomials, Proc. Nat. Acad. Sci., U.S.A. 32(1946), 8-10.

[34] H. Pollard, The mean convergence of orthogonal series, II, Trans. Amer. Math. Soc. 63(1948), 355-367.

[35] H. Pollard, The mean convergence of orthogonal series, III, Duke Math. Jour. 16(1949), 189-191.

[36] S. Schindler, Some tranplantation theorems for the generalized Mehler transform and related asymptotic expansions, Ph. D. Dissertation, U. of Wisconsin, Madison, 1969.

[37] G. Szegö, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ., Third edition, 1967.

[38] A.F. Timan, Theory of Approximation of Functions of a Real Variable, (English Translation) Pergamon Press, Oxford, 1963.

[39] P. Turán, On some problems in the theory of the mechanical quadrature, Mathematica (Cluj) 8(31) (1966), 181-192.

[40] A. Zygmund, Trigonometric Series, vol. I and vol. II, Cambridge Univ. Press, Cambridge, 1959.