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On absolute convergence of Jacobi series

by

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On absolute convergence of Jacobi series

1. Introduction

This report answers a question concerning the expansion of functions in an absolutely convergent series of Jacobi polynomials. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are the polynomials which are orthogonal on the interval $[-1,1]$ with respect to the weight function

$$(1-x)^\alpha \ (1+x)^\beta \ , \qquad (\alpha > -1, \ \beta > -1).$$

They satisfy the relation

$$(1.1) \quad (1-x)^\alpha \ (1+x)^\beta \ P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n \ n!} \left(\frac{d}{dx}\right)^n \{(1-x)^{n+\alpha} \ (1+x)^{n+\beta}\},$$

(see Szegö [5], section 4.3), usually called Rodrigues' formula. The orthogonality property is given by

$$(1.2) \quad \int_{-1}^{1} P_n^{(\alpha,\beta)}(x) \ P_m^{(\alpha,\beta)}(x) \ (1-x)^\alpha \ (1+x)^\beta \ dx = h_n(\alpha,\beta) \ \delta_{m,n}$$

with

$$(1.3) \qquad h_n(\alpha,\beta) = \frac{2^{\alpha+\beta+1} \ \Gamma(n+\alpha+1) \ \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n! \ \Gamma(n+\alpha+\beta+1)} \ ,$$

$\delta_{m,n} = 0$ if $m \neq n$ and $\delta_{m,n} = 1$ if $m = n$.

With a function $f(x)$ we can associate a series

$$(1.4) \qquad f(x) \sim \sum_{k=0}^{\infty} a_k \ P_k^{(\alpha,\beta)}(x) \ ,$$

where

$$(1.5) \quad a_k = (h_k \ (\alpha,\beta))^{-1} \int_{-1}^{1} f(x) \ P_k^{(\alpha,\beta)}(x) \ (1-x)^\alpha \ (1+x)^\beta \ dx \ ,$$

provided that the integral in (1.5) exists for all k.
The coefficients a_k are then called the Fourier coefficients associated with $f(x)$.
We now define the class of functions $A(\alpha,\beta)$.

Definition

A function f(x) is said to be in the class $A(\alpha,\beta)$ if $f(x) = \sum_{k=0}^{\infty} a_k P_k^{(\alpha,\beta)}(x)$ and its Fourier coefficients a_k, defined by (1.5), have the property that the series $\sum_{k=0}^{\infty} |a_k|\ |P_k^{(\alpha,\beta)}(x)|$ converges uniformly on the interval $-1 \leq x \leq 1$.

It is a well-known fact, (see Szegö [5], section 7.32), that the Jacobi polynomials reach their maximum in absolute value on the interval $[-1,1]$ at x = 1, provided that $\alpha \geq \beta$ and $\alpha \geq -\frac{1}{2}$. Since we have

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{k!\ \Gamma(\alpha+1)} = O(k^\alpha) \ ,$$

it follows that a necessary and sufficient condition for $f(x) \in A(\alpha,\beta)$ $(\alpha \geq \beta,\ \alpha \geq -\frac{1}{2})$ is

$$(1.6) \qquad \sum_{k=0}^{\infty} |a_k|\ k^\alpha < \infty.$$

The question studied here is the following: for which values of γ and δ does the relation

$$(A) \qquad f(x) \in A(\alpha,\beta) \ \text{ implies } f(x) \in A(\gamma,\delta)$$

hold if $\alpha \geq \beta$ and $\alpha \geq -\frac{1}{2}$.

In the following it will always be assumed that $\alpha \geq \max\ (\beta, -\frac{1}{2}),\ \beta > -1$.

2. Theorems

There is a unique way of expressing the polynomials $P_k^{(\alpha,\beta)}(x)$ in terms of the polynomials $P_j^{(\gamma,\delta)}(x)$, $j = 1,2,\dots,k$,

$$(2.1) \qquad P_k^{(\alpha,\beta)}(x) = \sum_{j=0}^{k} c_{jk}(\alpha,\beta;\gamma,\delta)\ P_j^{(\gamma,\delta)}(x).$$

The coefficients $c_{jk}(\alpha,\beta;\gamma,\delta)$ are defined to be 0 if $j > k$. Rivlin and Wilson [4] have proved the following theorem.

Theorem 1. If $\gamma \geq \delta$ and $\gamma \geq -\frac{1}{2}$ and the expression (2.1) is such that $c_{jk}(\alpha,\beta;\gamma,\delta) \geq 0$ for all j and k, then relation (A) holds.

Proof. We take a function $f(x) \in A(\alpha,\beta)$. Thus

$$\sum_{k=0}^{\infty} |a_k| \; P_k^{(\alpha,\beta)}(1) < \infty,$$

where the coefficients a_k are given by (1.5).
We now consider the expansion

$$\sum_{j=0}^{\infty} b_j \; P_j^{(\gamma,\delta)}(x) \; .$$

Then

$$\begin{aligned} b_j &= (h_j(\gamma,\delta))^{-1} \int_{-1}^{1} f(x) \; P_j^{(\gamma,\delta)}(x) \; (1-x)^{\gamma} \; (1+x)^{\delta} \; dx \\ &= (h_j(\gamma,\delta))^{-1} \int_{-1}^{1} \{ \sum_{k=0}^{\infty} a_k \; P_k^{(\alpha,\beta)}(x) \} \; P_j^{(\gamma,\delta)}(x) \; (1-x)^{\gamma} \; (1+x)^{\delta} \; dx \\ &= \sum_{k=0}^{\infty} a_k \; \{ (h_j(\gamma,\delta))^{-1} \int_{-1}^{1} P_k^{(\alpha,\beta)}(x) \; P_j^{(\gamma,\delta)}(x) \; (1-x)^{\gamma} \; (1+x)^{\delta} \; dx \} \\ &= \sum_{k=j}^{\infty} a_k \; c_{jk} \; (\alpha,\beta;\gamma,\delta) \; . \end{aligned}$$

The term-by-term integration is justified by the uniform convergence. Since $\gamma \geq \delta$ and $\gamma \geq -\frac{1}{2}$ we know that

$$\max_{-1 \leq x \leq 1} |P_j^{(\gamma,\delta)}(x)| = P_j^{(\gamma,\delta)}(1), \quad j = 0,1,2,\ldots \; .$$

Thus it remains to show that the sequence

$$F_m = \sum_{j=0}^{m} |b_j| \; P_j^{(\gamma,\delta)}(1)$$

is bounded.
Using the fact that $c_{jk}(\alpha,\beta;\gamma,\delta) \geq 0$ for all j and k we obtain

$$F_m = \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \; | \sum_{k=j}^{\infty} a_k \; c_{jk} \; (\alpha,\beta;\gamma,\delta)|$$

$$\leq \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \sum_{k=j}^{\infty} |a_k| \; c_{jk}(\alpha,\beta;\gamma,\delta)$$

$$\leq \sum_{k=0}^{\infty} |a_k| \sum_{j=0}^{m} c_{jk}(\alpha,\beta;\gamma,\delta) \; P_j^{(\gamma,\delta)}(1)$$

$$\leq \sum_{k=0}^{\infty} |a_k| \; P_k^{(\alpha,\beta)}(1) < \infty.$$

It is known, (see Askey [1]), that the positivity condition for $c_{jk}(\alpha,\beta;\gamma,\delta)$ is satisfied in the following cases: (see fig. 1)

(i) $\beta = \delta$ and $\alpha > \gamma$, $\gamma \geq \delta$.

(ii) $\alpha = \beta$, $\gamma = \delta$ and $\alpha > \gamma$,

(iii) $\alpha = \gamma$, $\beta = \delta - n$, n positive integer, $\gamma \geq \delta$.

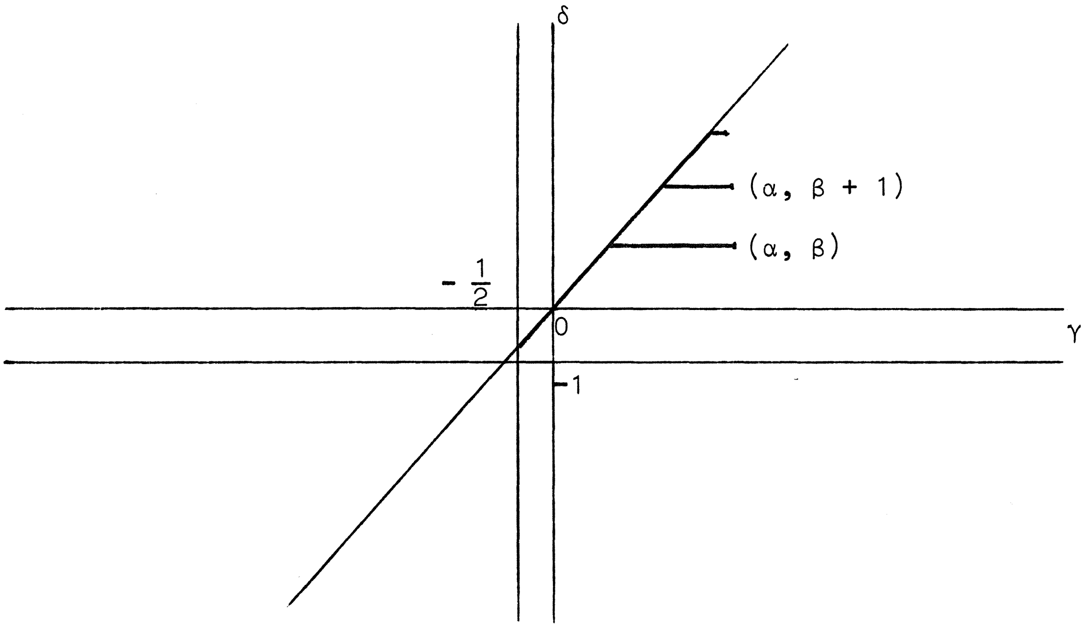

fig. 1.

We shall prove now, that relation (A) holds in the following cases:

(i) $\alpha = \gamma$, $\beta < \delta$, $\gamma \geq \delta$,

(ii) $\alpha = \gamma + \mu$, $\beta = \delta + \mu$, $\mu > 0$, $\gamma \geq \max(\delta, -\frac{1}{2})$, $\delta > -1$.

Theorem 2. If $\gamma = \alpha$ and $\delta = \beta + \mu$, where $\mu > 0$ and $\gamma \geq \delta$, then relation (A) holds.

Proof. Following the proof of theorem 1, it remains to show that the sequence

$$F_m = \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \; \Big|\sum_{k=j}^{\infty} a_k \, c_{jk}(\alpha,\beta;\gamma,\delta)\Big|$$

is bounded.

We now have

$$F_m \leq \sum_{j=0}^{m} P_j^{(\gamma,\delta)}(1) \sum_{k=j}^{\infty} |a_k| \; |c_{jk}(\alpha,\beta;\gamma,\delta)|$$

$$\leq \sum_{k=0}^{\infty} |a_k| \sum_{j=0}^{m} |c_{jk}(\alpha,\beta;\gamma,\delta)| \; P_j^{(\gamma,\delta)}(1).$$

As

$$P_k^{(\alpha,\beta)}(x) = \sum_{j=0}^{k} c_{jk}(\alpha,\beta;\alpha,\beta+\mu) \; P_j^{(\alpha,\beta+\mu)}(x),$$

it follows from the identity

$$P_k^{(\alpha,\beta)}(x) = (-1)^n \, P_n^{(\beta,\alpha)}(-x), \quad \text{(see Szegö [5], section 4.1)},$$

that

$$P_k^{(\beta,\alpha)}(x) = \sum_{j=0}^{k} (-1)^{k-j} \; c_{jk}(\alpha,\beta;\alpha,\beta+\mu) \; P_j^{(\beta+\mu,\alpha)}(x).$$

In section 9.4 of Szegö [5] the following relation is derived

$$P_k^{(\beta,\alpha)}(x) =$$

$$= \frac{\Gamma(k+\alpha+1)}{\Gamma(-\mu)\Gamma(k+\alpha+\beta+1)} \sum_{j=0}^{k} \frac{\Gamma(k+j+\alpha+\beta+1)\Gamma(k-j-\mu)\Gamma(j+\alpha+\beta+\mu+1)(2j+\alpha+\beta+\mu+1)}{\Gamma(k+j+\alpha+\beta+\mu+2)\Gamma(k-j+1)\Gamma(j+\alpha+1)} \, P_j^{(\beta+\mu,\alpha)}(x).$$

Hence

$$F_m \leq \sum_{k=j}^{\infty} |a_k| \sum_{j=0}^{k} \Big|\frac{\Gamma(k+\alpha+1)\Gamma(k+j+\alpha+\beta+1)\Gamma(k-j-\mu)\Gamma(j+\alpha+\beta+\mu+1)(2j+\alpha+\beta+\mu+1)}{\Gamma(-\mu)\Gamma(k+\alpha+\beta+1)\Gamma(k+j+\alpha+\beta+\mu+2)\Gamma(k-j+1)\Gamma(j+\alpha+1)}\Big| P_j^{(\alpha,\beta+\mu)}(1).$$

Because $\frac{\Gamma(k+\alpha)}{\Gamma(k)} = O(k^{\alpha})$ it is possible to estimate the order of magnitude of F_m.

$$F_m \le c \sum_{k=0}^{\infty} |a_k|\, k^{-\beta} \sum_{j=0}^{k} (k+j)^{-\mu-1} (k-j)^{-\mu-1} j^{\alpha+\beta+\mu+1}$$

$$\le c \sum_{k=0}^{\infty} |a_k|\, k^{-\beta-\mu-1} \Big(\sum_{j=0}^{[k/2]} k^{-\mu-1} j^{\alpha+\beta+\mu+1} + \sum_{j=[k/2]+1}^{k} k^{\alpha+\beta+\mu+1} (k-j)^{-\mu-1} \Big)$$

$$\le c \sum_{k=0}^{\infty} |a_k|\, k^{\alpha} < \infty.$$

Theorem 3. If $\gamma = \alpha - \mu$ and $\delta = \beta - \mu$, where $\mu > 0$ and $\gamma \ge \max(\delta, -\frac{1}{2})$, $\delta > -1$, then relation (A) holds.

Proof. It suffices to show that

$$\sum_{j=0}^{k} |c_{jk}(\alpha,\beta;\alpha-\mu,\beta-\mu)|\; P_j^{(\alpha-\mu,\beta-\mu)}(1) = O(k^{\alpha}).$$

Substituting the values of $c_{jk}(\alpha,\beta;\alpha-\mu,\beta-\mu)$ we obtain

$$\sum_{j=0}^{k} P_j^{(\alpha-\mu,\beta-\mu)}(1)(h_j(\alpha-\mu,\beta-\mu))^{-1} \Big|\int_{-1}^{1} P_k^{(\alpha,\beta)}(x) P_j^{(\alpha-\mu,\beta-\mu)}(x)(1-x)^{\alpha-\mu}(1+x)^{\beta-\mu}dx\Big|$$

$$= \sum_{j=0}^{k} \frac{\Gamma(j+\alpha+\beta-2\mu+1)(2j+\alpha+\beta-2\mu+1)}{\Gamma(\alpha-\mu+1)\Gamma(j+\beta-\mu+1)}.$$

$$\Big|\int_0^{\pi} P_k^{(\alpha,\beta)}(\cos\theta) P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta)(\sin\tfrac{\theta}{2})^{2\alpha-2\mu+1}(\cos\tfrac{\theta}{2})^{2\beta-2\mu+1}d\theta\Big|.$$

We will take the liberty of omitting lower order terms in k when they are inessential.

We shall only consider the integration in the interval $[0,\frac{\pi}{2}]$. The interval $[\frac{\pi}{2}, \pi]$ can be handled by the same argument.

It suffices to show that

$$\sum_{j=0}^{k} j^{\alpha-\mu+1}.$$

$$\Big|\int_0^{\frac{\pi}{2}} (\sin\tfrac{\theta}{2})^{2\alpha-2\mu+1}(\cos\tfrac{\theta}{2})^{2\beta-2\mu+1} P_k^{(\alpha,\beta)}(\cos\theta) P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta)d\theta\Big| = O(k^{\alpha}).$$

We shall need some estimates for Jacobi polynomials and Bessel functions:

(2.2) $|P_n^{(\alpha,\beta)}(\cos\theta)| \le An^{\alpha}$, $0 \le \theta \le \frac{\pi}{2}$,

(2.3) $|P_n^{(\alpha,\beta)}(\cos\theta)| \le An^{-1/2}\,\theta^{-\alpha-1/2}$, $0 \le \theta \le \frac{\pi}{2}$, (Szegö [5], (7.32.6)),

(2.4) $|J_{\alpha}(x)| \le Ax^{\alpha}$, $0 \le x \le 1$, (Szegö [5], (1.71.10)),

(2.5) $|J_{\alpha}(x)| \le Ax^{-1/2}$, $x \ge 1$, (Szegö [5], (1.71.11)),

(2.6) $J_{\alpha}(x) = (\frac{2}{\pi x})^{1/2}\cos(x - \alpha\frac{\pi}{2} - \frac{\pi}{4}) + O(x^{-3/2})$, (Szegö [5], (1.71.7)).

We shall also need the Sonine integral

(2.7) $$\int_0^{\infty} \frac{J_{\mu}(at)J_{\nu}(bt)}{b^{\nu}t^{\mu-\nu-1}}\,dt = \frac{(a^2-b^2)^{\mu-\nu-1}}{2^{\mu-\nu-1}a^{\mu}\Gamma(\mu-\nu)}$$, $(a > b)$, (Watson [6], section 13.46)

and Hilb's formula

(2.8) $$(\sin\tfrac{\theta}{2})^{\alpha}(\cos\tfrac{\theta}{2})^{\beta}P_n^{(\alpha,\beta)}(\cos\theta) = N^{-\alpha}\,\frac{\Gamma(n+\alpha+1)}{n!}\,(\frac{\theta}{\sin\theta})^{1/2}\,J_{\alpha}(N\theta) + \begin{cases}\theta^{1/2}O(n^{-3/2}) & \text{if } cn^{-1} \le \theta \le \pi-\varepsilon,\\ \theta^{\alpha+2}O(n^{\alpha}) & \text{if } 0 < \theta < cn^{-1},\end{cases}$$

where $N = n + \frac{\alpha+\beta+1}{2}$; c and ε are fixed positive numbers, (Szegö [5], (8.21.17)).

We follow the same method as used in the paper of Askey and Wainger [2] and therefore we want to replace

$$2^{1/2}\,(\sin\tfrac{\theta}{2})^{\alpha-\mu+1/2}\,(\cos\tfrac{\theta}{2})^{\beta-\mu+1/2}\,P_j^{(\alpha-\mu,\beta-\mu)}(\cos\theta)$$

by $\theta^{1/2}\,J_{\alpha-\mu}(J\theta)$, $J = j + \frac{\alpha+\beta-2\mu+1}{2}$,

using Hilb's formula (2.8).

We must then consider

$$I = \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{\frac{\pi}{2}} (\sin \tfrac{\theta}{2})^{\alpha-\mu+1/2} (\cos \tfrac{\theta}{2})^{\beta-\mu+1/2} P_k^{(\alpha,\beta)}(\cos \theta) \right.$$

$$\{2^{1/2}(\sin \tfrac{\theta}{2})^{\alpha-\mu+1/2}(\cos \tfrac{\theta}{2})^{\beta-\mu+1/2} P_j^{(\alpha-\mu,\beta-\mu)}(\cos \theta) -$$

$$\left. - \frac{J^{-\alpha+\mu}\Gamma(j+\alpha-\mu+1)}{\Gamma(j+1)} \theta^{1/2} J_{\alpha-\mu}(J\theta)\} \, d\theta \right| .$$

Setting $I = I_1 + I_2$, where in I_1 the range of integration is $[\frac{1}{k}, \frac{\pi}{2}]$ and in I_2 the range of integration is $[0, \frac{1}{k}]$ and using some of the estimates mentioned above we get

$$I_1 = O(\sum_{j=0}^{k} j^{\alpha-\mu+1} \int_{1/k}^{\frac{\pi}{2}} k^{-1/2} \theta^{-\alpha-1/2} \theta \, j^{-3/2} \theta^{\alpha-\mu+1/2} \, d\theta)$$

$$= O(k^{\alpha-\mu} \int_{1/k}^{\frac{\pi}{2}} \theta^{1-\mu} \, d\theta)$$

$$= O(k^{\alpha-\mu}(c + k^{\mu-2} + \delta_{\mu,2} \log k))$$

$$= O(k^{\alpha}).$$

$$I_2 = O(\sum_{j=0}^{k} j^{\alpha-\mu+1} \int_0^{1/k} k^{\alpha} \theta \, k^{-3/2} \theta^{\alpha-\mu+1/2} \, d\theta)$$

$$= O(k^{2\alpha-\mu+1/2} \int_0^{1/k} \theta^{\alpha-\mu+3/2} \, d\theta)$$

$$= O(k^{\alpha-2}).$$

The process of replacing the other Jacobi polynomial by the appropriate Bessel function is similar.

Thus we are led to investigate

$$L = \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{\frac{\pi}{2}} (\sin\tfrac{\theta}{2})^{-\mu} (\cos\tfrac{\theta}{2})^{-\mu}\, \theta\, J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right|$$

where $K = k + \frac{\alpha+\beta+1}{2}$.

We want to replace $(\sin\frac{\theta}{2})^{-\mu}$ $(\cos\frac{\theta}{2})^{-\mu}$ by $\theta^{-\mu}$. It is easily seen that $(\sin\frac{\theta}{2})^{-\mu}$ $(\cos\frac{\theta}{2})^{-\mu} = (\frac{\theta}{2})^{-\mu}$ $G(\theta)$, where $G(0) = 1$, $G(\theta)$ is bounded and $1-G(\theta) = O(\theta^2)$. Thus we must consider

$$E = \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{\frac{\pi}{2}} \theta^{1-\mu}\, (1-G(\theta))\, J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right| .$$

We split up E in $E = E_1 + E_2$, where in E_1 the range of integration is $[0,\frac{1}{k}]$ and in E_2 $[\frac{1}{k},\frac{\pi}{2}]$.

Applying some of the estimates mentioned above we get

$$\begin{aligned} E_1 &= \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{1/k} \theta^{1-\mu}(1-G(\theta))\, J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right| \\ &= O\Big(\sum_{j=1}^{k} j^{\alpha-\mu+1}\, j^{\alpha-\mu}\, k^{\alpha} \int_0^{1/k} \theta^{2\alpha-\mu+3-\mu} d\theta\Big) \\ &= O(k^{\alpha-2}). \end{aligned}$$

Using the asymptotic formula for Bessel functions and the error term we obtain for $\mu < 1$

$$\begin{aligned} E_2 &= \sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_{1/k}^{\pi/2} \theta^{1-\mu}(1-G(\theta))\, J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right| \\ &= O\Big(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} \left| \int_{1/k}^{\pi/2} \theta^{-\mu}(1-G(\theta))\, e^{i(J\pm K)\theta}\, d\theta \right|\Big) \\ &\quad + O\Big(k^{-3/2} \sum_{j=0}^{k} j^{\alpha-\mu-1/2} \int_{1/k}^{\pi/2} \theta^{-\mu}\, d\theta\Big) \end{aligned}$$

$$= O(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} \frac{1}{K\underline{+}J}) + O(k^{\alpha-\mu-1} + k^{\alpha-2})$$

$$= O(k^{\alpha-\mu}) + O(k^{-1/2} \sum_{j=0}^{[k/2]} \frac{j^{\alpha-\mu+1/2}}{k-j} + k^{-1/2} \sum_{j=[\frac{k}{2}]+1}^{k} \frac{j^{\alpha-\mu+1/2}}{k-j})$$

$$= O(k^{\alpha-\mu}) + O(k^{\alpha-\mu}) + O(k^{\alpha-\mu} \log k)$$

$$= O(k^{\alpha}).$$

The case $\mu \geq 1$ is easily handled.

$$E_2 = O(\sum_{j=1}^{k} j^{\alpha-\mu+1} \left| \int_{1/k}^{\pi/2} \theta^{3-\mu} j^{-1/2} k^{-1/2} \theta^{-1} d\theta \right|)$$

$$= \begin{cases} O(k^{\alpha-\mu+1} (c+k^{\mu-3})) & \mu \neq 3 \\ O(k^{\alpha-2} \log k) & \mu = 3 \end{cases}$$

$$= O(k^{\alpha}).$$

Finally we want to replace the range of integration $[0,\frac{\pi}{2}]$ by $[0,\infty)$. Therefore we must investigate

$$\sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_{\pi/2}^{\infty} \theta^{1-\mu} J_{\alpha-\mu}(J\theta) J_{\alpha}(K\theta) d\theta \right| = A_1 + A_2$$

by using (2.6). A_1 contains the main terms and A_2 all the error terms.

$$A_1 = O(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} \left| \int_{\pi/2}^{\infty} \theta^{-\mu} e^{i(K\underline{+}J)\theta} d\theta \right|)$$

$$= O(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu+1/2} (k\underline{+}j)^{-1})$$

$$= O(k^{\alpha-\mu} \log k).$$

$$A_2 = O(k^{-1/2} \sum_{j=0}^{k} j^{\alpha-\mu-1/2} \int_{\pi/2}^{\infty} \theta^{-\mu-1}\, d\theta) = O(k^{\alpha-\mu}).$$

Up to an error term that we have estimated, we may write for L

$$\sum_{j=0}^{k} j^{\alpha-\mu+1} \left| \int_0^{\infty} \theta^{1-\mu} J_{\alpha-\mu}(J\theta)\, J_\alpha(K\theta)\, d\theta \right|.$$

Then, using Sonine's integral (2.7), this leads to

$$\sum_{j=0}^{k} j^{\alpha-\mu+1} \frac{2^{1-\mu} J^{\alpha-\mu} (K^2-J^2)^{\mu-1}}{K^\alpha\, \Gamma(\mu)}$$

$$= O(k^{-\alpha} \sum_{j=0}^{k} j^{2\alpha-2\mu+1} (k+j)^{\mu-1} (k-j)^{\mu-1})$$

$$= O(k^{-\alpha+\mu-1} \{ \sum_{j=0}^{[k/2]} j^{2\alpha-2\mu+1} (k-j)^{\mu-1} + \sum_{j=[\frac{k}{2}]+1}^{k} j^{2\alpha-2\mu+1} (k-j)^{\mu-1} \})$$

$$= O(k^\alpha).$$

Combining all the estimates we have shown that

$$\sum_{j=0}^{k} |c_{jk}(\alpha,\beta;\alpha-\mu,\beta-\mu)|\; P_j^{(\alpha-\mu,\beta-\mu)}(1) = O(k^\alpha)$$

which proves theorem 3.

3. Results

Combination of the theorems 1, 2 and 3 leads to the following result. (see fig. 2).

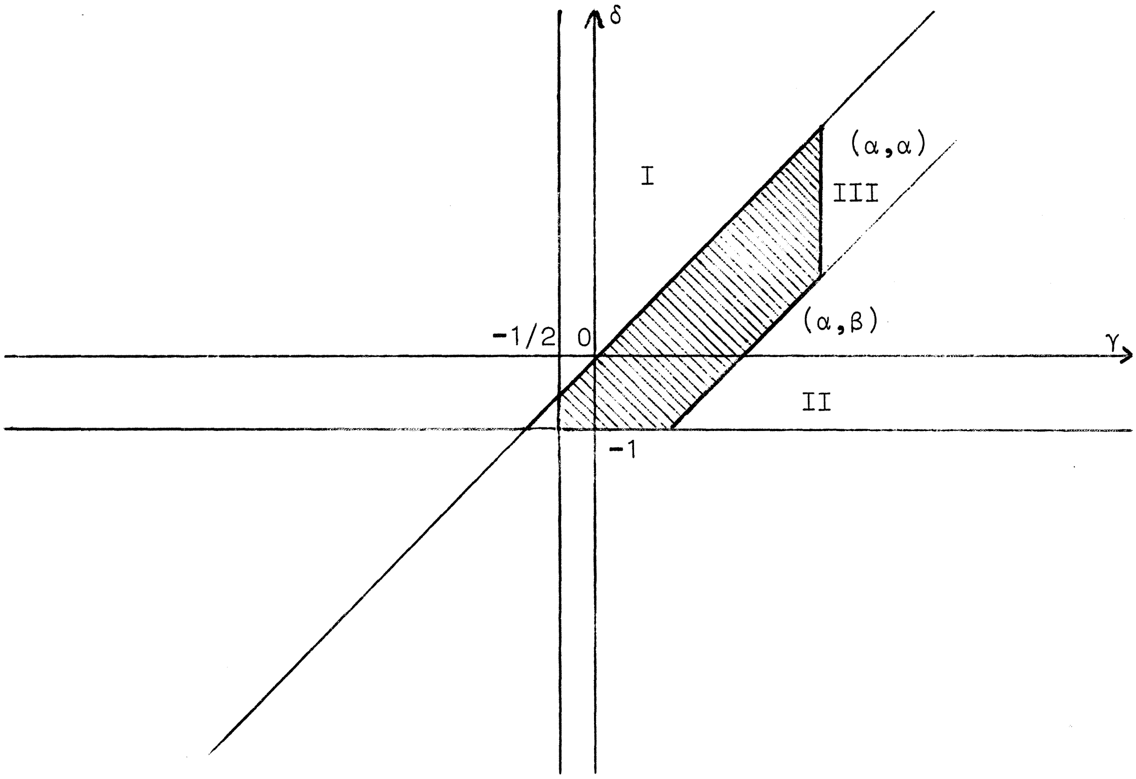

fig. 2

For all (γ,δ) in the shaded region it has been proved that relation (A) holds. We shall show now by means of examples that the complete region where relation (A) is valid has been obtained ($\gamma \geq -\frac{1}{2}$).
The first function we study is $(1+x)^{\mu}$, $\mu > 0$.
The Fourier coefficients become

$$a_n = h_n(\alpha,\beta))^{-1} \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\ (1-x)^{\alpha}\ (1+x)^{\beta+\mu}\ dx.$$

Using Rodrigues' formula (1.1) and integration by parts, it follows that

$$a_n = \frac{(-1)^n}{2^n n!h_n(\alpha,\beta)} \int_{-1}^{1} (1+x)^{\mu}\ (\frac{d}{dx})^n\ \{(1-x)^{n+\alpha}\ (1+x)^{n+\beta}\}\ dx$$

$$= \frac{\Gamma(\mu+1)}{2^n n!h_n(\alpha,\beta)\Gamma(\mu-n+1)} \int_{-1}^{1} (1-x)^{n+\alpha}\ (1+x)^{\beta+\mu}\ dx$$

(3.1) $a_n =$

$$= (-1)^{n+1} \frac{2^\mu}{\pi} \Gamma(\mu+1) \sin \mu\pi \; \Gamma(\beta+\mu+1)(2n+\alpha+\beta+1) \frac{\Gamma(n+\alpha+\beta+1)\Gamma(n-\mu)}{\Gamma(n+\alpha+\beta+\mu+2)\Gamma(n+\beta+1)}$$

Thus $|a_n| = O(n^{-\beta-2\mu-1})$.

It follows that $(1+x)^\mu \in A(\alpha,\beta)$ if $\alpha-\beta < 2\mu$.
From (3.1) it is easily derived that the function $(1+x)^\mu$ with $\frac{\alpha-\beta}{2} < \mu < \frac{\gamma-\delta}{2}$ and μ not an integer is an example of a function, which belongs to $A(\alpha,\beta)$ but not to $A(\gamma,\delta)$.
Thus we have found a function for which relation (A) fails in region II of fig. 2.
In the same way we can calculate the Fourier coefficients of the function $(1-x)^\mu$ and we obtain

$$|a_n| = O(n^{-\alpha-2\mu-1}).$$

It follows that $(1-x)^\mu \in A(\alpha,\beta)$ if $\mu > 0$.
But if $\delta > \gamma$ the maximum in absolute value of the Jacobi polynomials is at $x = -1$ and $P_n^{(\gamma,\delta)}(-1) = O(n^\delta)$. If $\delta > \gamma$ the function $(1-x)^\mu$ with $0 < \mu < \frac{\delta-\gamma}{2}$ and μ not an integer is an example of a function, which is a member of $A(\alpha,\beta)$ but not of $A(\gamma,\delta)$. Relation (A) is not valid in region I of fig. 2.

In order to decide whether relation (A) holds in region III we study the function $|x|^\mu$. We investigate the Fourier coefficients.

$$a_x = (h_n(\alpha,\beta))^{-1} \int_{-1}^{1} |x|^\mu \, P_n^{(\alpha,\beta)}(x) \, (1-x)^\alpha (1+x)^\beta \, dx$$

$$= (h_n(\alpha,\beta))^{-1} \{ \int_0^1 x^\mu P_n^{(\alpha,\beta)}(x)(1-x)^\alpha (1+x)^\beta dx +$$

$$+ (-1)^n \int_0^1 x^\mu P_n^{(\beta,\alpha)}(x)(1-x)^\beta (1+x)^\alpha dx \} .$$

If Re μ > n-1 we can use Rodrigues' formula and integration by parts. We obtain

$$(3.2)\ a_n = \frac{(2n+\alpha+\beta+1)\Gamma(\mu+1)\Gamma(n+\alpha+\beta+1)}{2^{n+\alpha+\beta+1}\Gamma(n+\beta+1)\Gamma(\alpha+\mu+2)}\ {}_2F_1\ (\mu-n+1,-\beta-n;\alpha+\mu+2;-1)\ +$$

$$(-1)^n\ \frac{(2n+\alpha+\beta+1)\Gamma(\mu+1)\Gamma(n+\alpha+\beta+1)}{2^{n+\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(\beta+\mu+2)}\ {}_2F_1\ (\mu-n+1,-\alpha-n;\beta+\mu+2;-1).$$

The hypergeometric function ${}_2F_1$ (a,b;c;-1) is an absolutely convergent series if Re(a+b-c) < 0 which means here $-\alpha-\beta-2n-1 < 0$. This is always satisfied. In this case ${}_2F_1$(a,b;c;-1) is an analytic function of the parameters a, b and c. As for Re μ > n-1 a_n is given by (3.2), it follows by analytic continuation that (3.2) holds for all μ with Re μ > -1.
Using the simple relation

$${}_2F_1(a,b;c;z) = (1-z)^{-b}\,{}_2F_1(b,c-a;c;\tfrac{z}{z-1}) = (1-z)^{-b}\,{}_2F_1(c-a,b;c;\tfrac{z}{z-1})$$

(see Luke [3], section 3.8 (4)), a_n can be written in the following way:

$$a_n = \frac{(2n+\alpha+\beta+1)\Gamma(\mu+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+1}\Gamma(n+\beta+1)\Gamma(\alpha+\mu+2)}\ {}_2F_1\ (\alpha+n+1,-\beta-n;\alpha+\mu+2;\tfrac{1}{2})$$

$$+\ (-1)^n\ \frac{(2n+\alpha+\beta+1)\Gamma(\mu+1)\Gamma(n+\alpha+\beta+1)}{2^{\beta+1}\Gamma(n+\alpha+1)\Gamma(\beta+\mu+2)}\ {}_2F_1\ (\beta+n+1,-\alpha-n;\beta+\mu+2;\tfrac{1}{2}).$$

The asymptotic expansion of the hypergeometric function in this case for large n has been given by Watson [7].
The leading term is

$${}_2F_1(a+n,b-n;c;\tfrac{1-z}{2}) \sim \frac{2^{a+b-1}\Gamma(1-b+n)\Gamma(c)(1+e^{-\zeta})^{c-a-b-1/2}}{(n\pi)^{1/2}\Gamma(c-b+n)(1-e^{-\zeta})^{c-1/2}}$$

$$\times\ \{e^{(n-b)\zeta} + \exp[\pm\ i\pi(c-\tfrac{1}{2})]\ e^{-(n+a)\zeta}\}\ .$$

where ζ is defined by $z = \cosh\zeta$ and Re $\zeta \geq 0$, $-\pi \leq$ Im $\zeta \leq \pi$. The upper (lower) sign is taken if Im z > (<) 0. In the case in which z-1 is real and negative it is supposed that z attains its value by a limiting process which then determines if $\arg(z-1)$ is π or $-\pi$. The discontinuity in the formula is only apparent; if z crosses the real axis between ± 1, account has to be taken of the discontinuity in the value of Im ζ. Therefore

$$(3.3)\quad |a_n| = O\left(\frac{n^{\alpha+1}\Gamma(n+\beta+1)}{n^{1/2}\Gamma(n+\alpha+\beta+\mu+2)} + \frac{n^{\beta+1}\Gamma(n+\alpha+1)}{n^{1/2}\Gamma(n+\alpha+\beta+\mu+2)}\right) = O(n^{-\mu-1/2}).$$

Thus in the case that $\mu > \alpha + \frac{1}{2}$ the function $|x|^\mu$ belongs to $A(\alpha,\beta)$.

In the ultraspherical case $(\alpha = \beta)$ the Fourier coefficients can easily be calculated. We have to consider

$$a_n = (h_n(\alpha,\alpha))^{-1} \int_{-1}^{1} |x|^\mu \, P_n^{(\alpha,\alpha)}(x)\,(1-x^2)^\alpha\, dx.$$

Because $|x|^\mu$ is an even function the Fourier coefficients vanish for odd n. Application of a well-known formula for ultraspherical polynomials (see Szegö [5], (4.1.5)) yields

$$(3.4)\quad a_{2n} = \frac{2n!\,\Gamma(2n+\alpha+1)}{h_{2n}(\alpha,\alpha)(2n)!\,\Gamma(n+\alpha+1)} \int_0^1 P_n^{(\alpha,-1/2)}(y)(1-y)^\alpha(1+y)^{\frac{\mu-1}{2}}\, dy$$

$$= \frac{(-1)^n(4n+2\alpha+1)\Gamma(2n+2\alpha+1)\Gamma(\mu+1)\sin\frac{\mu}{2}\pi\;\;\Gamma(n-\frac{\mu}{2})}{2^{2\alpha+\mu+1}\Gamma(2n+\alpha+1)\Gamma(n+\alpha+\frac{\mu}{2}+\frac{3}{2})\pi^{1/2}}.$$

From (3.3) and (3.4) it follows that if $\gamma > \alpha$ the function $|x|^\mu$ with $a + \frac{1}{2} < \mu < \gamma + \frac{1}{2}$ and μ not an even integer is an example of a function which belongs to $A(\alpha,\beta)$ but not to $A(\gamma,\gamma)$. Combined with theorem 2 this leads to the conclusion that relation (A) cannot be true in region III of fig. 2.

Thus the shaded region in fig. 2 is the complete region $(\gamma \geq -1/2)$ where relation (A) holds.

By using $P_n^{(\alpha,\beta)}(x) = (-1)^n\, P^{(\beta,\alpha)}(-x)$ similar results can be obtained when $\alpha < \beta$.

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