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ANALYTICAL METHODS FOR A SINGULAR PERTURBATION PROBLEM. THE QUARTER PLANE

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Contents

Introduction

In order to determine the asymptotic solution of a singular perturbation problem in partial differential equations, frequent use is made of special devices, such as the introduction of local coordinates for those parts of the domain where the solution behaves capriciously. However, by choosing relatively simple differential operators and appropriate domains, it may be possible to determine the solution explicitly. Then, the solution may be given as an integral or as a series development in eigenfunctions and known asymptotic methods may be applied to obtain the asymptotic solution of the problem. In this report a singular perturbation problem will be investigated in this way, by using saddle point methods. Owing to the singular nature of the problem, gratuitous application of this method is not allowed. But with some modifications of the saddle point method an asymptotic expansion is derived, which holds uniformly for the domain considered, particulary in the so called boundary layer.

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1. A modification of the method of saddle points

A well-known method for finding the asymptotic approximation to a contour integral of the form

$$(1\text{-}1) \qquad f(z) = \int_C w(v)\, e^{-zu(v)}\, dv$$

is the method of saddle points, also called the method of steepest descent. Here $f(z)$ denotes the function to be approximated, the integration is taken along a curve C in the complex v plane along which the functions w and u are defined and z denotes a large parameter. In the application of the saddle point method a difficulty arises as soon as a singularity of w or u lies near a saddle point of u, i.e. the points in the v plane where $du/dv = 0$. For instance the function w or u may depend on parameters other than z and when one of these parameters assumes a value for which a pole of w coincides with a saddle point of u the method of steepest descent becomes invalid. Methods to cope with such cases have been worked out and discussed by van der Waerden [6] and Oberhettinger [5], among others. The former treats the case of a saddle point of order 2 in the neighbourhood of a pole of the first order, while the latter shows that van der Waerden's method can be extended to the case involving both pole and saddle point of arbitrary order. In the first case the asymptotic expansion consists of an error function as leading term and an asymptotic series of inverse powers of z and in the second case of a finite number (equal to the order of the pole) of terms involving Whittaker functions and again an asymptotic series of inverse powers of z.

To make things clear the results of [5] will be reproduced. Since the contour integral in (1) can be transformed into a sum of integrals of the Laplace transform type

$$(1\text{-}2) \qquad f(z) = \int_0^\infty e^{-zt}\, F(t)\, dt$$

by distorting C through the appropriate saddle points of u, only integrals of this kind has been considered by Oberhettinger. The function F(t) in (1-2) is assumed to be of the form

$$(1\text{-}3) \qquad F(t) = t^{\alpha}\, g(t), \qquad \operatorname{Re} \alpha > -1,$$

where the function g is analytic in the vicinity of $t = 0$, that is to say

$$(1\text{-}4) \qquad g(t) = \sum_{n=0}^{\infty} a_n t^n, \qquad a_n = \frac{1}{n!}\, g^{(n)}(0).$$

The series is convergent for $|t| < |t_0|$, where t_0 is that singular point of g(t) which is closest to the origin. With the condition

$$g(t) = O(\exp(at)) \qquad \text{for } t \to \infty,$$

where a is a real constant, a lemma due to Watson [7] yields the asymptotic approximation of f(z) for large $|z|$,

$$(1\text{-}5) \qquad f(z) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+1+\alpha)}{z^{n+1+\alpha}}$$

holding uniformly in any sector $|\arg z| \leq \theta < \pi/2$. It is possible to generalize the conditions on g(t), but, to fix the ideas, the above-mentioned analytic properties of g(t) in a neighbourhood of $t = 0$ will be considered. When the radius of convergence of g(t), $|t_0|$ in this case, is small, the coefficients a_n, being functions of t_0, tend to infinity as $t_0 \to 0$. The assumption of small values of $|t_0|$ may invalidate the asymptotic expansion, although the integral in (1-2) may exist for $t_0 = 0$. When the small radius of convergence of g(t) is due to a pole of order m at the point $t = t_0$ (t_0 is not on the positive real axis) the function g(t) may be written as

$$(1\text{-}6) \qquad g(t) = \sum_{n=1}^{m} b_{-n}(t-t_0)^{-n} + g^{*}(t),$$

where $g^{*}(t)$ is analytic for $|t| < |t_1|$. Consequently $g^{*}(t)$ can be represented by the series

$$(1\text{-}7) \qquad g^{*}(t) = \sum_{n=0}^{\infty} b_n t^n , \quad b_n = \frac{1}{n!} g^{*(n)}(0)$$

for $|t| < |t_1|$, with $|t_1|$ possibly larger then a positive number. Inserting the decomposition (1-6) of g(t) and the series (1-7) for $g^{*}(t)$ in (1-2), the expansion for f(z) becomes

$$(1\text{-}8) \qquad f(z) \sim \sum_{n=1}^{m} b_{-n} \int_0^{\infty} e^{-tz} t^{\alpha}(t-t_0)^{-n} \, dt + \sum_{n=0}^{\infty} b_n \frac{\Gamma(n+\alpha+1)}{z^{n+\alpha+1}}$$

as $|z| \to \infty$, where, in addition, the range of validity for arg z depends on $g^{*}(t)$. The integrals in (1-8) can be written as Whittaker functions. These integrals will tend to infinity as $t_0 \to 0$ in the case Re $\alpha-n \leq -1$ and at first sight expansion (1-8) becomes useless again. However, the integral in (1-2) may converge uniformly with respect to the parameter t_0 in a domain in the complex t plane containing the origin (but certainly not positive values of t) and thus may exist for $t_0 = 0$. Since the coefficients b_n in (1-7) are bounded for $t_0 = 0$, the finite sum in (1-8) does not tend to infinity as $t_0 \to 0$. So, the singularities of the pertinent terms will cancel. The expansion in (1-8) represents the integral (1-2) for large values of $|z|$ and gives for $t_0 = 0$, if the integral exists, its correct value.

The case $\alpha = -\frac{1}{2}$, m = 1 has been treated by van der Waerden [6] and the single term in the finite sum in (1-8) reduces to an errorfunction.

In the present paper we consider integrals of the type

$$(1\text{-}9) \qquad \int_{-\infty}^{\infty} e^{-zt^2} F(t) \, dt.$$

Oberhettinger's method is applicable in slightly different form when F(t) has a pole near the origin. In this case reduction to (1-2) is always possible, but when F(t) contains a factor

$$(t-t_0)^\nu, \qquad \nu \text{ not a positive integer,}$$

reduction to (1-2) is not advisable. We will point out that in this case the singularity cannot be split off in the same way as in the foregoing. In section 4 an alternative method will be used to solve this case properly.

2. The singular perturbation problem

The type of integrals of the preceding section is encountered when solving the following singular perturbation problem. We consider the partial differential equation in two independent variables x and y

$$\text{(2-1)} \qquad \varepsilon\Delta\, \Phi_\varepsilon(x,y) - \frac{\partial}{\partial y}\Phi_\varepsilon(x,y) = 0,$$

where ε is a small positive parameter and Δ is Laplace's operator. The function $\Phi_\varepsilon(x,y)$ satisfies the differential equation in the domain G: $x > 0$, $y > 0$. Along the boundaries of this domain $\Phi_\varepsilon(x,y)$ is subjected to the following conditions

$$\text{(2-2)} \qquad \Phi_\varepsilon(x,0) = 0 \quad , \quad \Phi_\varepsilon(0,y) = \phi(y).$$

When ε is very small, the boundary value problem described above is a so-called singular perturbation problem. The qualification "singular" is due to the fact that in the limit problem (ε=0), equation (2-1) simplifies to the first-order equation, the reduced equation,

$$\text{(2-3)} \qquad \frac{\partial}{\partial y}\Phi_\varepsilon(x,y) = 0$$

while solutions of the reduced equation do not satisfy both boundary conditions in (2-2). The characteristics of the reduced equation (2-3) are the lines x = constant. Hence the half line $x = 0$, $y \geq 0$ of the boundary of G coincides with a characteristic of (2-3).

As pointed out by Eckhaus and de Jager [2], these two features give rise to a so-called parabolic boundary layer along the positive y-axis.

In view of the first boundary condition in (2-2), outside the boundary layer

$$\Phi_\varepsilon(x,y) \sim 0$$

for $\varepsilon \to 0$, will be a good approximation, except in the boundary layer near $x = 0$, $y \geq 0$, where the second boundary condition in (2-2) is violated.

A general technique exists to solve problems of this kind. One of the most important papers on this subject is the paper of Eckhaus and de Jager [1]. Essential to this technique is the introduction of local coordinates in order to blow up the boundary layer and to indicate the boundare layer functions. Our approach is entirely different from the usual one. Due to the simplicity of the equation and the favourable shape of the domain G, an explicit analytic solution to (2-1) satisfying (2-2) can be constructed by elementary methods. An advantage of this approach is that the condition in the paper of Eckhaus and de Jager

$$\phi(0) = 0$$

can be omitted. We are even allowing a singular behaviour of $\phi(y)$ near the origin provided that the integral

$$\int_0^a \phi(t)dt$$

exists for some positive a.

The present problem has also been investigated by Grasman [3], who represents the solution $\Phi_\varepsilon(x,y)$ in terms of the Dirichlet Green's function for this problem and the boundary values. Our representation of the solution is in terms of the Laplace transform of $\phi(y)$ and can be deduced from Grasman's performing some substitutions. However, we propose a more elementary way and we use the methods outlined in section 1 in order to obtain an asymptotic approximation of the solution.

Before giving the solution of (2-1) it is useful to solve the problem for a special choice of $\phi(y)$ in order to gain an insight into some of the phenomenae, especially the parabolic boundary layer. We will first remove the first order derivative in equation (2-1) with the aid of the substitution

$$(2\text{-}4) \qquad \Psi(x,y) = \exp(-y/2\varepsilon)\ \Phi_\varepsilon(x,y).$$

Then the function $\Psi(x,y)$ has to satisfy the following boundary value problem

$$(2\text{-}5) \qquad \begin{aligned} &\Delta\Psi(x,y) - \omega^2\ \Psi(x,y) = 0 \\ &\Psi(x,0) = 0,\ \Psi(0,y) = e^{-\omega y}\phi(y)\ , \end{aligned}$$

where $\omega = {}^1/(2\varepsilon)$.

Looking for an elementary solution of this problem, we note that

$$(2\text{-}6) \qquad \Psi(x,y) = c\ \sin\lambda\theta\ K_\lambda(\omega r)$$

satisfies the differential equation in (2-5) for each value of c and λ. In this formula r and θ are the polar coordinates ($x=r\cos\theta$, $y=r\sin\theta$), λ and c are constants and $K_\lambda(\omega r)$ is a modified Bessel-function. If $\lambda = {}^1/2$ this function can be written as an exponential function and an elementary solution of the original problem (2-1) is

$$(2\text{-}7) \qquad \Phi_\varepsilon(x,y) = \sqrt{\frac{2}{r}}\ \sin\theta/2\ \ \exp\left(\frac{r(\sin\theta-1)}{2\varepsilon}\right)$$

in which case the boundary function ϕ has to be chosen as $y^{-1/2}$.

The argument of the exponential function in (2-5) is negative when $0 \leq \theta < {}^\pi/2$. So for these values of θ

$$\Phi_\varepsilon(x,y) = O(\varepsilon^n)$$

for $\varepsilon \to 0$, where n is an arbitrary positive number.

This estimate, however, is not uniform in θ. For certain values of r and θ the exponential function in (2-7) is not at all small. This happens when $r(1-\sin\theta)/(2\varepsilon) = O(1)$ for $\varepsilon \to 0$. The locus in the (x,y)-plane determined by the equation

$$\frac{r(1-\sin\theta)}{2\varepsilon} = c$$

where c is a positive constant, independent of r, θ and ε, is the parabola

$$y = \frac{x^2 - 4\varepsilon^2 c^2}{4\varepsilon c} \quad . \tag{2-8}$$

The axis of symmetry of this parabola is the line $x = 0$ and when ε tends to zero the "width" of the parabola is of order ε. Outside this parabola the exponential function is gaining influence and as a consequence the function $\Phi_\varepsilon(x,y)$ tends strongly to zero outside the parabola. The solution of (2-1) with a general boundary condition $\phi(y)$ will show a similar behaviour.

The inside of the parabola (2-8) is called the parabolic boundary layer. Although this is not a proper mathematical definition, which cannot easily be given, it gives a good idea of the concept. In this part of the domain G, along the line y = constant (> 0), in the positive x-direction the solution rapidly changes from the boundary value $\phi(y)$ to very small values, resembling outside the boundary layer the solution of the reduced equation (2-3), which has to be taken as

$$\Phi_\varepsilon(x,y) = 0.$$

3. The solution of the singular perturbation problem

In the preceding section the singular perturbation problem (2-1) is transformed (c.f. (2-4) and (2-5)) into the Helmholtz problem

$$\Delta\Psi(x,y) - \omega^2\,\Psi(x,y) = 0$$

$$(3\text{-}1)\qquad \Psi(x,0) = 0,\ \Psi(0,y) = e^{-\omega y}\phi(y)\ ,$$

where $\omega = 1/(2\varepsilon)$. The equation of Helmholtz in an angular region has comprehensively been studied by Lauwerier [4]. Considering his methods we try the tentative solution

$$(3\text{-}2)\qquad \Psi(x,y) = \int_{-\infty}^{\infty} \exp\{-\omega(x\,\mathrm{ch}u - iy\,\mathrm{sh}u)\}\ g(u)\,d\,\mathrm{sh}u$$

for $x > 0$. The function

$$f(y) = e^{-\omega y}\phi(y),$$

defined only for $y \geq 0$, may be continued for negative values of y such that $f(-y) = -f(y)$. The function g in (3-2) will be determined from the boundary value at $x = 0$

$$f(y) = \int_{-\infty}^{\infty} e^{i\omega y\,\mathrm{sh}u}\ g(u)\ d\ \mathrm{sh}u$$

giving

$$g(u) = \frac{\omega}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega y\,\mathrm{sh}u}\ f(y)dy$$

$$= \frac{\omega}{2\pi}\ \{\bar{\phi}(\omega(1+i\,\mathrm{sh}u)) - \bar{\phi}(\omega(1-i\,\mathrm{sh}u))\},$$

where $\bar{\phi}$ is the Laplace transform of ϕ, i.e.

$$\bar{\phi}(s) = \int_0^{\infty} e^{-st}\phi(t)dt.$$

The introduction of polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ yields

$$\Psi(x,y) = \int_{-\infty}^{\infty} e^{-\omega r\ \mathrm{ch}(u-i\theta)}\ g(u)\,d\ \mathrm{sh}u$$

and a shift of the path of integration into a line with Im $u = \theta$ gives

$$(3\text{-}4)\quad \Psi(x,y) = \frac{\omega}{2\pi}\int_{-\infty}^{\infty} e^{-\omega r\mathrm{ch}u}\mathrm{ch}(u+i\theta)\{\bar{\phi}(\omega(1+i\mathrm{sh}(u+i\theta)))-\bar{\phi}(\omega(1-i\mathrm{sh}(u+i\theta)))\}du.$$

The representation (3-4) of the solution of the boundary value problem (3-1) will be the starting point of the investigations on the asymptotic behaviour of $\Psi(x,y)$ for large values of ω.

The integral in (3-4) with ω as a large parameter can be evaluated by the method of saddle points. The real u axis is a steepest descent line for the exponential function in (3-4) and the saddle point is situated at $u = 0$. When the method of saddle points is applied a detailed knowledge of the function $\bar{\phi}$ is required.
Since there is a general lack of information on the Laplace transform of an arbitrary function $\phi(y)$, the research has to be limited to some special choices of $\phi(y)$. However, from appropriate selections insight in the problem may be gained and general results may be derived. As mentioned in the previous section, the singular perturbation problem (2-1) has been treated by Eckhaus and de Jager in [1], in which the condition $\phi(0) = 0$ was imposed on ϕ. Our first task will be to analyse the case $\phi(y) \equiv 1$, not only in order to give an extension of the results of the paper cited above, but also since it is such a nice problem. In section 5 the case $\phi(y) \equiv y^{\nu}$, with Re $\nu > -1$, will be investigated and the results may be applied to a more general boundary value ϕ.

4. The special case $\phi(y) \equiv 1$

In this case the Laplace transform of ϕ is $\bar{\phi}(s) = 1/s$. The solution of (3-1) becomes

$$\Psi(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega r \mathrm{ch} u} \mathrm{ch}(u+i\theta)\left\{\frac{1}{1+i\,\mathrm{sh}(u+i\theta)} - \frac{1}{1-i\,\mathrm{sh}(u+i\theta)}\right\} du$$

$$= \frac{1}{i\pi} \int_{-\infty}^{\infty} e^{-\omega r \mathrm{ch} u} \mathrm{th}(u+i\theta) du.$$

Writing

(4-1) $$\alpha = {}^{\pi}/_{2} - \theta$$

there follows

(4-2) $$\Psi(x,y) = \frac{1}{i} \int_{-\infty}^{\infty} e^{-\omega r \mathrm{ch} u} \, \frac{\mathrm{ch}(u-i\alpha)}{\mathrm{sh}(u-i\alpha)} \, du.$$

This integral is not of the same type as the one encountered in the first section, although some symptoms mentioned there can be recognised. Namely the saddle point is located at $u = 0$ and the integrand has a simple pole for $u = i\alpha$. As a consequence, when $\alpha \to 0$ (i.e. $\theta \to \pi/2$) the saddle point and the pole will coincide. (There are more poles but those are located far off the origin for the values of θ considered in our problem). Remark that the coincidence of saddle point and pole just happens for $\theta \sim \pi/2$. As outlined before, the boundary layer is situated along the positive y-axis; i.e. for $\theta \sim \pi/2$.

The integral in (4-2) can be written as an integral of the Laplace transform type (1-2). However, in order to handle the coalescence of pole and saddle point for this integral we propose a direct method, using (c.f. Lauwerier [4]- VI)

(4-3) $$F(r,\alpha) = \frac{1}{\pi i} \int_{-\infty}^{\infty} e^{-r \mathrm{ch} u} \, \frac{du}{\mathrm{sh}\frac{1}{2}(u-i\alpha)} = 2\, e^{-r\cos\alpha} \mathrm{erfc}(\sqrt{2r}\, \sin\tfrac{1}{2}\alpha),$$

where erfc is the complementary errorfunction defined by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^2} dt.$$

A proof of (4-3) is easily obtained by verifying that

$$\frac{\partial}{\partial r}\{e^{r\cos\alpha}\,F(r,\alpha)\} = -2\sqrt{\frac{2\pi}{r}}\,\sin\frac{\alpha}{2}\,e^{-r(1-\cos\alpha)} .$$

Now, we determine a constant A such that the function

$$\text{(4-4)} \qquad \frac{\operatorname{ch}(u-i\alpha)}{\operatorname{sh}(u-i\alpha)} - \frac{A}{\operatorname{sh}\frac{1}{2}(u-i\alpha)}$$

is a regular function at $u = i\alpha$. It turns out that A should be taken as $\frac{1}{2}$. With this choice of A the function in (4-4) will be denoted by $g(u)$. Next we expand this function $g(u)$ in the following way

$$\text{(4-5)} \qquad g(u) = i\,\operatorname{ch}\tfrac{1}{2}u \sum_{k=0}^{\infty} c_k(\operatorname{sh}\tfrac{1}{2}u)^k.$$

A simple computation shows

$$c_0 = \frac{\cos\alpha - \cos\frac{1}{2}\alpha}{\sin\alpha}, \qquad c_1 = \frac{2}{i}\,\frac{1-\cos^3\frac{1}{2}\alpha}{\sin^2\alpha} .$$

The coefficients c_k are bounded, as functions of α, for $0 \leq \alpha \leq \pi/2$.

The integral in (4-2) may now be written as

$$\text{(4-6)} \qquad \Psi(x,y) = F(\omega r,\alpha) + \frac{1}{\pi i}\int_{-\infty}^{\infty} e^{-\omega r\operatorname{ch}u}\,g(u)du,$$

where F is defined in (4-3) and g is the function in (4-4) with $A = {}^1/2$. Putting the expansion (4-5) of $g(u)$ in (4-6) and reverting the order of summation and integration we obtain the asymptotic expansion

$$\text{(4-7)} \qquad \Psi(x,y) \simeq F(\omega r,\alpha) + \frac{2}{\pi}\,e^{-\omega r}\sum_{k=0}^{\infty} c_{2k}\,\frac{\Gamma(k+\frac{1}{2})}{(2\omega r)^{k+\frac{1}{2}}} ,$$

as $\omega r \to \infty$, uniformly valid for $0 \leq \theta \leq \pi/2$. For the function $\Phi_\varepsilon(x,y)$ of the original problem (2-1) we have, using (2-4),

$$(4\text{-}8)\quad \Phi_\varepsilon(x,y) \simeq \operatorname{erfc}(\sin\tfrac{1}{2}\alpha\,\sqrt{2\omega r}) + \frac{2}{\pi}\,e^{-2\omega r\sin^2\frac{1}{2}\alpha}\sum_{k=0}^{\infty} c_{2k}\,\frac{\Gamma(k+\frac{1}{2})}{(2\omega r)^{k+\frac{1}{2}}}\,.$$

When $\alpha \to 0$ ($\theta \to \pi/2$) the asymptotic behaviour is completely determined by the errorfunction in (4-8), since all coefficients c_{2k} vanish for $\alpha = 0$ (namely, $g(u)$ is an odd function of u if $\alpha = 0$). The errorfunction is the boundary layer function for this problem and it plays an important role in the neighbourhood of the positive y-axis ($\theta = \pi/2$), that is, in the boundary layer.

It has to be pointed out that (4-8) furnishes an expansion for $\omega r \to \infty$; so in the region of the (x,y)-plane for which $r = O(\varepsilon)$, $\varepsilon \to 0$, this expansion is invalid.

5. The case $\phi(y) \equiv y^\nu$

In this section the asymptotic solution of the boundary value problem (3-2) with $\phi(y) = y^\nu$, Re $\nu > -1$, will be investigated. With this choice of $\phi(y)$ the solution $\Psi(x,y)$ of (3-1), given in (3-4), can be written as

$$(5\text{-}1)\quad \Psi(x,y) = \frac{\Gamma(\nu+1)}{2\pi\,\omega^\nu}\int_{-\infty}^{\infty} e^{-\omega r\operatorname{ch}u}\left\{\frac{1}{(1+i\operatorname{sh}(u+i\theta))^{\nu+1}} - \frac{1}{(1-i\operatorname{sh}(u+i\theta))^{\nu+1}}\right\}\operatorname{ch}(u+i\theta)\,du\,.$$

By the use of some identities between hyperbolic and circular functions the function Ψ can be represented in the form

$$\Psi(x,y) = \Psi_1(x,y) + \Psi_2(x,y)\;,$$

where

$$(5\text{-}2)\quad \Psi_1(x,y) = \frac{\Gamma(\nu+1)}{2\pi(2\omega)^\nu}\int_{-\infty}^{\infty} e^{-\omega r\operatorname{ch}u}\,\frac{\cos\frac{1}{2}(iu+\alpha)}{\{\sin\frac{1}{2}(iu+\alpha)\}^{2\nu+1}}\,du$$

and

$$(5\text{-}3)\quad \Psi_2(x,y) = \frac{\Gamma(\nu+1)}{2\pi(2\omega)^{\nu}} \int_{-\infty}^{\infty} e^{-\omega r \operatorname{ch} u} \frac{\sin\frac{1}{2}(iu+\alpha)}{\{\cos\frac{1}{2}(iu+\alpha)\}^{2\nu+1}}\, du$$

where, again, $\alpha = {}^{\pi}/2 - \theta$.

The singularities of the integrand of (5-2) are located at the zeros of the function $\sin\frac{1}{2}(iu+\alpha)$. When $2\nu+1$ is not a positive integer, the integrand has a branch point at these zeros. Only the zero at $u = -i\alpha$ is close to the saddle point $u = 0$ and for $\alpha = 0$ it coincides with the saddle point.

As for the singularities of the integrand of (5-3), for all values of α concidered in our problem $(0 \leq \alpha \leq {}^{\pi}/2)$ these points are not close to the origin. Therefore, the asymptotic expansion of $\Psi_2(x,y)$ can be determined by substitution of the expansion

$$\frac{\sin\frac{1}{2}(iu+\alpha)}{\{\cos\frac{1}{2}(iu+\alpha)\}^{2\nu+1}} = \operatorname{ch}\tfrac{1}{2}u \sum_{k=0}^{\infty} a_k (\operatorname{sh}\tfrac{1}{2}u)^k$$

and reverting the order of summation and integration. The asymptotic expansion is a series in inverse powers of $\sqrt{\omega r}$ and does not contain boundary layer terms. Therefore, Ψ_2 is not as interesting as Ψ_1, which does have a boundary layer behaviour near $\theta = {}^{\pi}/2$ $(\alpha=0)$.

We will now construct an asymptotic expansion of $\Psi_1(x,y)$, defined in (5-2) for large values of ωr, which holds uniformly in $0 \leq \alpha \leq {}^{\pi}/2$.

Since generally, $2\nu + 1$ is not a positive integer it is not possible to split off the singularity of the integrand. For this case we expand the integrand in the following way

$$(5\text{-}4)\quad \frac{\cos\frac{1}{2}(iu+\alpha)}{\{\sin\frac{1}{2}(iu+\alpha)\}^{2\nu+1}} = \frac{\operatorname{ch}\frac{1}{2}u}{(\sin\frac{1}{2}iu+\sin\frac{1}{2}\alpha)^{2\nu+1}} \sum_{k=0}^{\infty} b_k(\alpha)(\sin\tfrac{1}{2}iu)^k .$$

Each coefficient b_k is a bounded function of α in the interval $0 \leq \alpha \leq \pi/2$. It is found that

$$(5\text{-}5) \qquad b_0(\alpha) = \cos\tfrac{1}{2}\alpha \ , \quad b_1(\alpha) = \mathrm{tg}\tfrac{1}{4}\alpha \quad (2\nu \cos\tfrac{1}{2}\alpha - 1).$$

When $\alpha = 0$ (5-4) becomes

$$1 = \sum_{k=0}^{\infty} b_k(0)\ (\sin\tfrac{1}{2}iu)^k.$$

Therefore, all coefficients $b_k(\alpha)$ vanish when $\alpha \to 0$ except $b_0(\alpha)$.

Substitution of the series (5-4) in (5-2) gives the expansion for $\omega r \to \infty$

$$(5\text{-}6) \qquad \Psi_1(x,y) \sim (2r)^{\nu}\ \Gamma(\nu+1)\ e^{-\omega r} \sum_{k=0}^{\infty} \frac{b_k}{(2\sqrt{\omega r})^k}\ G_k \quad ,$$

where

$$(5\text{-}7) \qquad G_k = \frac{(2\sqrt{\omega r})^{k-2\nu}\ e^{\omega r}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\omega r \mathrm{ch} u}(\sin\frac{1}{2}iu)^k\ \mathrm{ch}\frac{1}{2}u}{(\sin\frac{1}{2}iu + \sin\frac{1}{2}\alpha)^{2\nu+1}}\ du \ .$$

It turns out that the functions G_k are connected with the parabolic cylinder functions.

To show this, we first simplify the integral in (5-7). Writing $t = 2\sqrt{\omega r}\ \mathrm{sh}\frac{1}{2}u$, we obtain

$$(5\text{-}8) \qquad G_k = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} (it)^k\ (it+z)^{-(2\nu+1)} dt \quad ,$$

where

$$(5\text{-}9) \qquad z = 2\sqrt{\omega r}\ \sin\tfrac{1}{2}\alpha \ .$$

Next we consider the case $k = 0$, viz.

$$G_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} (it+z)^{-(2\nu+1)} dt$$

$$= \frac{1}{\pi\Gamma(2\nu+1)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} \{\int_0^{\infty} e^{-(it+z)u}\, u^{2\nu} du\} dt \; .$$

Interchanging the order of integration and evaluating the inner integral we have

$$(5\text{-}10) \qquad G_0 = \sqrt{\frac{2}{\pi}}\ \frac{1}{\Gamma(2\nu+1)} \int_0^{\infty} e^{-uz-\frac{1}{2}u^2}\, u^{2\nu}\, du$$

$$= \sqrt{\frac{2}{\pi}}\ e^{\frac{1}{4}z^2}\, D_{-2\nu-1}(z) \quad ,$$

where $D_{-2\nu-1}(z)$ is a parabolic cylinder function. For $k > 0$, we replace the factor $(it)^k$ in the integral in (5-8) by

$$(it+z-z)^k = \sum_{m=0}^{k} \binom{k}{m} (it+z)^m (-z)^{k-m}$$

which gives

$$(5\text{-}11) \qquad G_k = \sqrt{\frac{2}{\pi}}\ e^{\frac{1}{4}z^2} \sum_{m=0}^{k} \binom{k}{m} (-z)^{k-m}\, D_{m-2\nu-1}(z) \; .$$

On using (5-11) and the recurrence formula for the functions $D_\mu(z)$, namely

$$D_{\mu+1}(z) - z\, D_\mu(z) + \mu\, D_{\mu-1}(z) = 0 \; ,$$

the functions G_k $(k = 0,1,2,\ldots)$ can be determined when $D_\mu(z)$ is known for $\mu = 2\nu-1$ and $\mu = -2\nu$. Furthermore, by partial integration of (5-8), a recurrence formula for the G_k with respect to k can be derived

$$G_{k+3} + z\, G_{k+2} + (k+1-2\nu)G_{k+1} + z(k+1)G_k = 0 \; .$$

In the asymptotic expansion (5-6), which is uniformly valid in $0 \leq \alpha \leq \pi/2$, the functions G_k are the boundary layer functions. The expansion is not an asymptotic expansion in the Poincaré sense but of a more general type introduced by Erdélyi [2]. It has to be interpreted as

$$(5\text{-}13)\quad \Psi_1(x,y) = (2r)^{\nu}\,\Gamma(\nu+1)e^{-\omega r}\{\sum_{k=0}^{N-1} \frac{b_k(\alpha)}{(2\sqrt{\omega r})^k}\, G_k + O((\sqrt{\omega r})^{-N})\}\ ,$$

$N = 1,2,3,\ldots$, for $\omega r \to \infty$, uniformly valid in $0 \leq \theta \leq \pi/2$.

By using the value of $D_\mu(0)$, given by

$$D_\mu(0) = \frac{\sqrt{\pi}\ 2^{\mu/2}}{\Gamma(\frac{1-\mu}{2})}\ ,$$

the asymptotic expansion can be evaluated for $\alpha = 0$ ($\theta = \pi/2$). It has already been shown that $b_k(0) = 0$ if $k \geq 1$ and $b_0(0) = 1$. Thus if $\alpha = 0$ (5-6) becomes

$$\Psi_1(x,y) \sim (2r)^{\nu}\,\Gamma(\nu+1)e^{-\omega r}\sqrt{\tfrac{2}{\pi}}\ D_{-2\nu-1}(0)$$

$$= e^{-\omega r}\ r^{\nu}.$$

Since $\theta = \pi/2$, this equals $e^{-\omega y}y^{\nu}$ which is the boundary function of problem (3-1) with $\phi(y) = y^{\nu}$.

6. Remarks on the general case

The solution $\Psi(x,y)$, given in (3-4), of the boundary value problem (3-1) has been expressed in terms of the Laplace transform of the boundary function $\phi(y)$. In section 4 we determined the asymptotic expansion of Ψ for the case $\phi(y) = 1$ in section 5 for the function $\phi(y) = y^{\nu}$.

The methods used for the different cases depend on the behaviour of the function $\bar{\Phi}(\omega(1 \pm i\,\mathrm{sh}(u+i\theta)))$ in the neighbourhood of the saddle point at $u = 0$. To handle the case for a general boundary function $\phi(y)$ we have to know the nature of the singularities of the function $\bar{\Phi}(s)$. It might be expected that if $\bar{\Phi}(s)$ has no singular points in the finite s-plane, the asymptotic expansion of $\Psi(x,y)$ does not contain boundary layer functions. This is, in general, not true, as can be seen from the following example.

Consider

$$\phi(y) = \begin{cases} 1 & 0 \leq y \leq 1 \\ 0 & y > 1 \,, \end{cases}$$

in which case $\bar{\Phi}(s)$ is given by $\dfrac{1 - e^{-s}}{s}$. The function $\bar{\Phi}(\omega(1+i\,\mathrm{sh}(u+i\theta))) - \bar{\Phi}(\omega(1-i\,\mathrm{sh}(u+i\theta)))$ has no singular points, but to treat this case the two parts of $\bar{\Phi}(s)$, viz. $1/s$ and e^{-s}/s have to be treated separately. The first part corresponds completely to the problem of section 4; the second one gives rise to the following integral

$$e^{-\omega} \int_{-\infty}^{\infty} e^{-\omega r\,\mathrm{ch}u \,-\, i\omega\,\mathrm{sh}(u+i\theta)} \, \frac{\mathrm{ch}(u+i\theta)du}{1 + i\,\mathrm{sh}(u+i\theta)} =$$

$$e^{-\omega} \int_{-\infty}^{\infty} \frac{e^{-\omega\rho\,\mathrm{ch}u} \,\mathrm{ch}(u+i\theta')}{1 + i\,\mathrm{sh}(u+i\theta')} \, du \,,$$

with $x = \rho \cos\theta'$, $y-1 = \rho \sin\theta'$. So, also the second part of $\bar{\Phi}(s)$ corresponds to the problem of section 4.

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