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APPROXIMATION PROCESSES FOR FOURIER-JACOBI EXPANSIONS

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Abstract

This report deals with summability methods for Fourier-Jacobi series. When a function f is expanded in terms of Jacobi polynomials, every summability method for this Jacobi series may be looked upon as an approximation process for the function f. The main object of this report is to investigate the order of approximation of these approximation processes and to characterize the functions which allow a certain order of approximation. Many of these approximation processes exhibit the phenomenon of saturation, which is equivalent to the existence of an optimal order of approximation (the saturation order). For the summability methods treated in this paper the saturation order and the saturation class, that is the class of functions which can be approximated with the optimal order, are derived. The characterization of the classes of functions is accomplished by means of the theory of intermediate spaces due to Peetre [20]. Another basic tool in this work is the convolution structure for Jacobi series introduced by Askey and Wainger [3].

1. Introduction

1.1. In this paper we are concerned with approximation theorems in some Banach spaces of complex-valued functions on the interval $[0,\pi]$. By C we denote the space of continuous functions, L^∞ is written for the essentially bounded functions and the L^p spaces are introduced with respect to the weight function

$$\rho^{(\alpha,\beta)}(\theta) = (\sin \frac{\theta}{2})^{2\alpha+1} (\cos \frac{\theta}{2})^{2\beta+1}. \tag{1.1}$$

In this paper we shall always assume that $\alpha \geq \beta \geq -\frac{1}{2}$. We call M the space of all regular finite Borel measures on $[0,\pi]$. The spaces C, L^p $(1 \leq p \leq \infty)$ and M are Banach spaces if they are endowed with the following norms

$$||f||_c = \sup_{0\leq\theta\leq\pi} |f(\cos\theta)|,$$

$$||f||_p = [\int_0^\pi |f(\cos\theta)|^p \rho^{(\alpha,\beta)}(\theta)d\theta]^{1/p}, \qquad 1 \leq p < \infty,$$

$$||f||_\infty = \operatorname{ess\,sup}_{0\leq\theta\leq\pi} |f(\cos\theta)|,$$

$$||\mu||_M = \int_0^\pi |d\mu(\cos\theta)|.$$

With elements of these Banach spaces we can associate an expansion in terms of Jacobi polynomials. If $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of order (α,β) and degree n (see Szegö [25]), the functions

$$R_n^{(\alpha,\beta)}(\cos\theta) = \frac{P_n^{(\alpha,\beta)}(\cos\theta)}{P_n^{(\alpha,\beta)}(1)}$$

satisfy

$$\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\, R_m^{(\alpha,\beta)}(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)d\theta = \delta_{n,m}[\omega_m^{(\alpha,\beta)}]^{-1}, \tag{1.2}$$

where

$$\delta_{n,m} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases} \qquad \text{and}$$

(1.3) $$\omega_n^{(\alpha,\beta)} = \frac{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+\alpha+1)}{\Gamma(n+\beta+1)\Gamma(n+1)\Gamma(\alpha+1)\Gamma(\alpha+1)} = O(n^{2\alpha+1}).$$

With f belonging to one of the spaces C or L^p ($1\leq p\leq\infty$) we associate the Fourier-Jacobi expansion

(1.4) $$f(\cos\theta) \sim \sum_{n=0}^{\infty} f^{\wedge}(n)\, \omega_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta),$$

where

(1.5) $$f^{\wedge}(n) = \int_0^{\pi} f(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)d\theta,\ n = 0, 1, \ldots.$$

The generalized translation operator T_ϕ, introduced by Askey and Wainger [3], maps a function f with (1.4) into

$$T_\phi f(\cos\theta) \sim \sum_{n=0}^{\infty} f^{\wedge}(n)\, \omega_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta)\, R_n^{(\alpha,\beta)}(\cos\phi)$$

and is shown to be a positive operator (see Gasper [14]) satisfying

(1.6) $$||T_\phi f|| \leq ||f|| \qquad \text{in C and } L^p\ (1\leq p\leq\infty),$$

and

(1.7) $$\lim_{\phi\to 0^+} ||T_\phi f - f|| = 0 \qquad \text{in C and } L^p\ (1\leq p<\infty).$$

Following Askey and Wainger [3] we define for f_1, $f_2 \in L^1$ the convolution $f_1 * f_2$ by

(1.8) $$(f_1 * f_2)(\cos\theta) = \int_0^{\pi} T_\phi f_1(\cos\theta)\, f_2(\cos\phi)\, \rho^{(\alpha,\beta)}(\phi)d\phi.$$

This convolution has the following properties:

<u>1.2. Lemma</u>. Let $f_1, f_2, f_3 \in L^1$. Then $f_1 * f_2 \in L^1$ and

(i) $f_1 * f_2 = f_2 * f_1$,

(ii) $f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3$,

(iii) $(f_1 * f_2)^\wedge(n) = f_1^\wedge(n)\, f_2^\wedge(n)$,

(iv) If $f \in L^p$ $(1 \le p \le \infty)$, then $f_1 * g \in L^p$ and

$$||f_1 * g||_p \le ||f_1||_1\, ||g||_p.$$

<u>1.3</u>. With a measure $\mu \in M$ we associate the Jacobi-Stieltjes expansion

$$d\mu(\cos\theta) \sim \sum_{n=0}^{\infty} \mu^{\vee}(n)\, \omega_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta), \tag{1.9}$$

where

$$\mu^{\vee}(n) = \int_0^{\pi} R_n^{(\alpha,\beta)}(\cos\theta)\, d\mu(\cos\theta), \qquad n = 0, 1, \ldots. \tag{1.10}$$

By F. Riesz' representation theorem the space M is the dual space of C. We use this fact to give an implicit definition of the convolution of measures. Suppose $\mu, \nu \in M$ and $f \in C$. Then the map

$$f \to \int_0^{\pi}\int_0^{\pi} T_\phi f(\cos\theta)\, d\mu(\cos\theta)\, d\nu(\cos\phi)$$

defines a bounded linear functional on C and thus there exists a unique measure $\mu * \nu$ such that

$$\int_0^{\pi} f(\cos\theta)\, d(\mu*\nu)(\cos\theta) = \int_0^{\pi}\int_0^{\pi} T_\phi f(\cos\theta)\, d\mu(\cos\theta)\, d\nu(\cos\phi).$$

The following properties are easily verified:

<u>1.4. Lemma</u>. Let $\mu_1, \mu_2, \mu_3 \in M$. Then $\mu_1 * \mu_2 \in M$ and

(i) $\mu_1 * \mu_2 = \mu_2 * \mu_1$,

(ii) $\mu_1 * (\mu_2 * \mu_3) = (\mu_1 * \mu_2) * \mu_3$,

(iii) $(\mu_1 * \mu_2)^\vee(n) = \mu_1^\vee(n)\ \mu_2^\vee(n)$,

(iv) $||\mu_1 * \mu_2||_M \leq ||\mu_1||_M\ ||\mu_2||_M$.

1.5. There is an obvious embedding of L^1 into M, namely $f \to m_f$, where $dm_f = f(\cos\theta)\ \rho^{(\alpha,\beta)}(\theta)d\theta$. The space L^1 consists of all the measures which are absolutely continuous with respect to $\rho^{(\alpha,\beta)}(\theta)d\theta$. The convolution defined for L^1 coincides with the convolution defined for M restricted to the absolutely continuous measures. In fact

$$d(m_{f_1} * m_{f_2})(\cos\theta) = \{\int_0^\pi T_\phi f_1(\cos\theta) f_2(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi\}\ \rho^{(\alpha,\beta)}(\theta)d\theta.$$

If $f \in L^p$ $(1 \leq p \leq \infty)$ and $\mu \in M$ it follows that $f * \mu \in L^p$ with

$$(f*\mu)(\cos\theta) = \int_0^\pi T_\phi f(\cos\theta)d\mu(\cos\phi) \tag{1.11}$$

and

$$||f*\mu||_p \leq ||f||_p\ ||\mu||_M\ . \tag{1.12}$$

1.6. The generalized translation and the convolution structure play an essential role in the approximation theory for the spaces C and L^p $(1 \leq p < \infty)$ treated in this paper. In general the degree of approximation improves with the increasing smoothness of the function and the well-known theorems of Jackson and Bernstein relate in fact the smoothness and the degree of approximation to each other. Theorems of this type can also be proved, if the smoothness is defined with respect to the generalized translation. Furthermore, if special summability methods for the summation of Jacobi series are used, it is also possible to derive theorems of the Jackson and the Bernstein type. For many of these approximation processes, however, there exists an optimal order of approximation, which means that we do not get better approximation than this order, even if we presuppose still greater smoothness of the function. The saturation problem consists of characterizing the class of functions,which allow approximation of optimal order. The optimal and non-optimal approximation in Banach spaces have been considered in Butzer and Berens [9] for semi-groups of operators and in Berens [6] for a more general class of operators. Many of the summability methods for Jacobi series happen to satisfy the conditions mentioned in Berens' paper, so that the general theory can be applied. In section 2 we shall give a short survey of the results on approximation processes in Banach spaces. Section 3 deals with some general results on summability methods for Fourier-Jacobi expansions. Then, in section 4 we shall prove a Jackson and a Bernstein type theorem, where the Lipschitz classes are defined with respect to the generalized translation. The next section is devoted to the study of a number of more or less classical summability methods. In the last section we give a characterization of the classes of functions which occur in the preceding sections.

1.7. Notation. We shall use the o and O symbol in the usual meaning. If we write $f(\theta) \approx g(\theta)$, $\theta \to 0^+$, we mean that there exist two positive constants C_1 and C_2 such that

$$C_1\, g(\theta) \le f(\theta) \le C_2\, g(\theta), \qquad \theta \to 0^+.$$

If we write $f(\theta) \simeq g(\theta)$, $\theta \to 0^+$, we mean that

$$\lim_{\theta\to 0^+} \frac{f(\theta)}{g(\theta)} = 1.$$

If we write for two Banach spaces X, Y that $X \cong Y$, we mean that the spaces are equal and have equivalent norms.

2. Some general results on approximation processes in Banach spaces

In this section we state some of the results on approximation processes in Banach spaces treated in Berens [6] and Butzer-Scherer [13]. The K-method of interpolation developed by Peetre [20] in the theory of intermediate spaces plays an important role in this field. Another fundamental concept is the relative completion introduced by Gagliardo (see Aronszajn-Gagliardo [1]). In order to keep this section as short as possible we avoid the J-method of interpolation, which entails that some of the theorems cannot be stated in their most general form. Also some of the theorems mentioned here are in fact combinations of several theorem from the book of Butzer-Scherer [13].

2.1. Intermediate spaces of K-interpolation

Let X be a Banach space and let Y be a Banach subspace of X with the property that for all $f \in Y$,

$$||f||_X \leq ||f||_Y. \tag{2.1}$$

Then we call Y a normalized Banach subspace of X.

If for $0 < t < \infty$ and for every $f \in X$ we consider the function norm

$$K(t,f) = K(t,f;X,Y) = \inf_{f=f_1+f_2} (||f_1||_X + t||f_2||_Y), \quad (f_1 \in X,\ f_2 \in Y), \tag{2.2}$$

then we denote by $(X,Y)_{\theta,q;K}$ the set of all elements $f \in X$ for which the norm

$$||f||_{\theta,q;K} = \begin{cases} \left(\sum\limits_{n=1}^{\infty} [n^\theta K(n^{-1},f)]^q \frac{1}{n}\right)^{1/q}, & 0 < \theta < 1,\ 1 \leq q < \infty, \\ \sup\limits_{n=1,2,\dots} (n^\theta K(n^{-1},f)), & 0 < \theta \leq 1,\ q = \infty, \end{cases} \tag{2.3}$$

is finite. The spaces $(X,Y)_{\theta,q;K}$ have the following properties:

a) For each pair (θ,q), $0 < \theta < 1$, $1 \le q < \infty$ and $0 < \theta \le 1$, $q = \infty$, the space $(X,Y)_{\theta,q;K}$ is a normalized Banach subspace of X under the norm (2.3) with the inclusion $Y \subset (X,Y)_{\theta,q;K} \subset X$. The spaces $(X,Y)_{\theta,q;K}$ are called intermediate spaces of X and Y.

b) $(X,Y)_{\theta,q_1;K} \subset (X,Y)_{\theta,q_2;K}$, $0 < \theta < 1$, $1 \le q_1 \le q_2 \le \infty$.

c) $(X,Y)_{\theta_1,q_1;K} \subset (X,Y)_{\theta_2,q_2;K}$, $0 < \theta_2 < \theta_1 < 1$, $1 \le q_1$, $q_2 \le \infty$.

d) <u>Reiteration property</u>. Let the relation

(2.4) $$(X,Y)_{\theta_i,1;K} \subset X_{\theta_i} \subset (X,Y)_{\theta_i,\infty;K}$$

be valid for the spaces X_{θ_1} and X_{θ_2}, $0 < \theta_1 < \theta_2 < 1$ and for X_{θ_2}, if $0 < \theta_2 < 1$ and $X_{\theta_1} = X$. Then for $0 < \theta' < 1$, $1 \le q \le \infty$ and $\theta = (1-\theta')\,\theta_1 + \theta'\theta_2$

$$(X_{\theta_1},X_{\theta_2})_{\theta',q;K} \cong (X,Y)_{\theta,q;K}\,.$$

2.2. Spaces of best approximation

Let X be a Banach space and let P_n $(n=1,2,\ldots)$ be subspaces such that

$$\{0\} = P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_n \subset \ldots \subset X.$$

If we define the degree of best approximation of $f \in X$ by elements of P_n by

$$E(P_n,f) = \inf_{p_n \in P_n} ||f-p_n||_X\,,$$

then we denote by $X^K_{\theta,q}$ the set of all elements $f \in X$ for which the norm

$$(2.5) \qquad ||f||^K_{\theta,q} = \begin{cases} ||f||_X + \{\sum_{n=1}^{\infty} [n^\theta E(P_n,f)]^q \frac{1}{n}\}^{1/q}, & \theta > 0,\ 1 \le q < \infty, \\ ||f||_X + \sup_{n=1,2,\dots} n^\theta E(P_n,f), & \theta > 0,\ q = \infty, \end{cases}$$

is finite. The spaces $X^K_{\theta,q}$ have the following properties:

a) For each pair (θ,q), $0 < \theta < 1$, $1 \le q < \infty$ and $0 < \theta \le 1$, $q = \infty$, the space $X^K_{\theta,q}$ is a normalized Banach subspace of X under the norm (2.5). The spaces $X^K_{\theta,q}$ are called spaces of best approximation.

b) $X^K_{\theta,q_1} \subset X^K_{\theta,q_2}$, $\qquad (\theta>0,\ 1\le q_1 \le q_2 \le \infty)$.

c) $X^K_{\theta_1,q_1} \subset X^K_{\theta_2,q_2}$, $\qquad (\theta_1>\theta_2,\ 1\le q_1,\ q_2 \le \infty)$.

<u>2.3. Definition</u>. A space Y, $P_n \subset Y \subset X$, $n = 1, 2, \dots$, belongs to

a) the class $D^K_\theta(X)$, $\theta \ge 0$, if for $n = 1, 2, \dots$ and for $f \in Y$ the relation

$$n^\theta E(P_n,f) \le C||f||_Y$$

holds;

b) the class $D^J_\theta(X)$, $\theta \ge 0$, if for $p_n \in P_n$, $n = 1, 2, \dots$, the relation

$$||p_n||_Y \le Cn^\theta ||p_n||_X$$

holds;

c) the class $D_\theta(X)$ if it belongs to the classes $D^K_\theta(X)$ and $D^J_\theta(X)$.

<u>2.4. Lemma</u>. For $\theta > 0$ the space Y, $P_n \subset Y \subset X$, $n = 1, 2, \dots$, belongs to

a) the class $D_\theta^K(X)$ if and only if

$$Y \subset X^K_{\theta,\infty} ;$$

b) the class $D_\theta^J(X)$ if and only if

$$X^K_{\theta,1} \subset Y;$$

c) the class $D_\theta(X)$ if and only if

$$X^K_{\theta,1} \subset Y \subset X^K_{\theta,\infty} .$$

2.5. Theorem. Let X_i be spaces of the classes $D_{\theta_i}(X)$, $i = 1, 2$, and $\theta_2 > \theta_1 \geq 0$. Then for $0 < \theta' < 1$ and $1 \leq q \leq \infty$

$$(X_1,X_2)_{\theta',q;K} \cong X^K_{\theta,q} ,$$

where

$$\theta = (1-\theta')\theta_1 + \theta'\theta_2 .$$

2.6. Spaces of S-approximation

A family $S = \{S_\rho;\rho>0\}$ of commutative operators mapping a Banach space X into itself and satisfying the properties

$$(2.6) \quad \begin{cases} \text{(i)} \quad ||S_\rho f||_X \leq M||f||_X, \text{ uniformly in } \rho > 0, \ f \in X, \\ \text{(ii)} \quad \lim_{\rho\to\infty} ||S_\rho f-f||_X = 0, \qquad f \in X, \\ \text{(iii)} \quad S_\rho S_\tau = S_\tau S_\rho , \qquad \rho, \tau > 0, \end{cases}$$

is called an approximation process for the identity operator I as $\rho \to \infty$.

The expression

(2.7) $$\omega_S(\rho,f) = \sup_{\sigma\geq\rho} ||S_\sigma f-f||_X$$

is called the modulus of S-approximation of f.

The space of S-approximation $X_{\lambda,q;S}$ consists of all the elements $f \in X$, for which the norm

(2.8) $$||f||_{\lambda,q;S} = \begin{cases} ||f||_X + \{\sum_{n=1}^{\infty} [n^\lambda \omega_S(n,f)]^q \frac{1}{n}\}^{1/q}, & \lambda>0,\ 1\leq q<\infty; \\ ||f||_X + \sup_{n=1,2,\dots} n^\lambda \omega_S(n,f), & \lambda>0,\ q=\infty, \end{cases}$$

is finite. The spaces $X_{\lambda,q;S}$ have the following properties:

a) For each pair (λ,q), $\lambda > 0$ and $1 \leq q \leq \infty$, the space $X_{\lambda,q;S}$ is a normalized Banach subspace of X under the norm (2.8).

b) $X_{\lambda,q_1;S} \subset X_{\lambda,q_2;S}$, $\quad (\lambda>0,\ 1\leq q_1\leq q_2\leq\infty)$.

c) $X_{\lambda_1,q_1;S} \subset X_{\lambda_2,q_2;S}$, $\quad (\lambda_1>\lambda_2>0,\ 1\leq q_1,\ q_2\leq\infty)$.

2.7. Saturation. Let $\phi(u)$ be a positive non-increasing function on $0 < u < \infty$ with $\lim_{u\to\infty} \phi(u) = 0$. Let $S = \{S_\rho;\ \rho>0\}$ be an approximation process for the identity operator such that $\omega_S(\rho,f) = o(\phi(\rho))$, $(\rho\to\infty)$, implies that f belongs to a certain 'trivial' subspace of X and such that $\omega_S(\rho,f) \approx \phi(\rho)$, $(\rho\to\infty)$, for at least one 'non-trivial' element of X. If F(X,S) is the collection of elements $f \in X$ satisfying:

a) $\omega_S(\rho,f) = O(\phi(\rho))$ implies $f \in F(X,S)$;

b) $f \in F(X,S)$ implies $\omega_S(\rho,f) = O(\phi(\rho))$;

then the approximation process is said to be saturated with the order $\phi(\rho)$ and with the saturation class (Favard class) F(X,S).

2.8. The relative completion of a Banach subspace

Let X be a Banach space and let Y be a normalized Banach subspace. The relative completion of Y with respect to X, written $\tilde{Y}^X$, is the space of all elements $f \in X$ for which there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset Y$ with $||f_n||_Y \leq R$ uniformly in n and $\lim_{n\to\infty} ||f_n - f||_X = 0$. Equivalently, if we write $\overline{A}^X$ for the closure of the set A in the space X, then

$$\tilde{Y}^X = \bigcup_{R>0} \overline{S_Y(R)}^X, \text{ where } S_Y(R) = \{f \in Y: ||f||_Y \leq R\}.$$

The space $\tilde{Y}^X$ is a normalized Banach subspace of X under the norm

$$||f||_{\tilde{Y}^X} = \inf\{R : f \in \overline{S_Y(R)}^X\}.$$

From the definition of $\tilde{Y}^X$ and the norm $||\ ||_{\tilde{Y}^X}$ the following may be concluded. If $f \in \tilde{Y}^X$, then there exists a sequence $\{f_n\}_{n=0}^{\infty} \subset Y$ with $||f_n||_Y = ||f||_{\tilde{Y}^X}$ for all n = 1, 2, ..., and f_n converges to f in the X norm for $n \to \infty$.

We list some of the properties of the relative completion:

a) (The relative completion of $\tilde{Y}^X$ with respect to X) $\cong \tilde{Y}^X$.

b) $\tilde{Y}^X \cong (X,Y)_{1,\infty;K}$.

c) If Y is reflective, then $Y = \tilde{Y}^X$.

Let B be a closed linear operator mapping the Banach space X into itself with the domain D(B) which is dense in X. Under the norm

$$(2.9) \qquad ||f||_{D(B)} = ||f||_X + ||Bf||_X$$

D(B) is a normalized Banach subspace of X.

We now state the following saturation theorem.

2.9. Theorem. We assume the approximation process S on X satisfies (2.6) and is connected with a closed linear operator B in such a way, that the range $S_\rho[X]$ of S_ρ in X belongs to D(B) for all $\rho > 0$ and that there exists a number $\gamma_0 > 0$ such that for all $f \in D(B)$

$$(2.10) \qquad \lim_{\rho\to\infty} ||\rho^{\gamma_0} \{S_\rho f-f\} - Bf||_X = 0.$$

Then the process S on X is saturated with the order $\rho^{-\gamma_0}$ and the saturation class $F(X,S) = \{f \in X: \omega_S(\rho,f) = O(\rho^{-\gamma_0}), \rho \to \infty\}$ is equal to $\widetilde{D(B)}^X$. The 'trivial' subspace mentioned in section 2.7 is the null space N(B) of the operator B. i.e. $N(B) = \{f \in D(B): Bf = 0\}$.

In the case of non-optimal approximation we have

2.10. Theorem. Let the process S satisfy the conditions of theorem 2.9 and in addition let the following relation be valid

$$(2.11) \qquad ||BS_\rho f||_X \leq N\rho^{\gamma_0}||f||_X, \quad (\rho>0;\ f\in X),$$

where N is a constant ≥ 1. Then the spaces of S-approximation $X_{\lambda,q;S}$, $0 < \lambda < \gamma_0$, $1 \leq q \leq \infty$, coincide with the intermediate spaces $(X,D(B))_{\lambda/\gamma_0,q;K}$ with equivalent norms.

3. Summability methods for Fourier-Jacobi expansions

The purpose of this section is to introduce summability methods for the Fourier-Jacobi expansion of a function and to show that they are approximation processes for the identity operator satisfying (2.6). The main tool we use is the convolution defined by (1.8).

In the rest of this paper X will always denote one of the spaces C or L^p $(1\leq p<\infty)$.

3.1. A first question that arises is whether the partial sums $S_N f$ of the Fourier-Jacobi expansion (1.4) of a function $f \in X$,

$$S_N f(\cos\theta) = \sum_{n=0}^{N} f^{\wedge}(n)\ \omega_n^{(\alpha,\beta)}\ R_n^{(\alpha,\beta)}(\cos\theta),$$

supply an approximation process for the identity operator (section 2.6). It is well-known that the answer is 'no' for X = C, a consequence of the Banach-Steinhaus theorem and the fact that $\lim_{N\to\infty} ||S_N|| = \infty$, see Rau [23], and is 'yes' for $X = L^2$ by the Riesz-Fischer theorem. The norm convergence of $S_N f$ in the L^p spaces has been treated by Pollard [21]. He showed that there is norm convergence if

$$\frac{4(\alpha+1)}{2\alpha+3} < p < \frac{4(\alpha+1)}{2\alpha+1} \qquad \text{and} \qquad \frac{4(\beta+1)}{2\beta+3} < p < \frac{4(\beta+1)}{2\beta+1}$$

and there is no norm convergence, if p is outside one of these ranges.

In order to introduce summability methods we give the following definition.

3.2. Definition. Let $K_\lambda(\cos\theta) \in L^1$ $(\lambda>0)$ satisfy the properties

(a) $$\int_0^\pi K_\lambda(\cos\theta)\ \rho^{(\alpha,\beta)}(\theta)d\theta = 1;$$

(b) $K_\lambda(\cos\theta) \geq 0,\ 0 \leq \theta \leq \pi;$

(c) $\lim_{\lambda\to\infty} K_\lambda^{\wedge}(n) = 1,\ n = 0, 1, \ldots .$

Then we call $K_\lambda(\cos\theta)$ a positive summability kernel or positive kernel. If instead of b we merely have

$$\text{(b')} \qquad \int_0^\pi |K_\lambda(\cos\theta)|\ \rho^{(\alpha,\beta)}(\theta)d\theta \le N, \qquad \text{uniformly in } \lambda,$$

with a constant $N \geq 1$, we call $K_\lambda(\cos\theta)$ a quasi-positive summability kernel or simply a kernel.

Every positive kernel is quasi-positive with $N = 1$, as follows from a. The convolution of $f \in X$ with a kernel has the following properties.

<u>3.3. Theorem</u>. If $f \in X$ and K_λ $(\lambda>0)$ is a kernel satisfying a, b' and c, then

$$(3.1) \qquad ||K_\lambda * f||_X \le N||f||_X, \qquad \text{uniformly in } \lambda,$$

$$(3.2) \qquad \lim_{\lambda\to\infty} ||K_\lambda * f - f||_X = 0.$$

<u>Proof</u>. Relation (3.1) is a direct consequence of condition b' and lemma 1.2 (iv). Relation (3.2) follows by application of Helly's theorem (see Szegö [25], theorem 1.6), using (3.1) and the fact that (3.2) holds for a dense set, the polynomials, as follows from c.

Theorem 3.3 justifies the name summability kernel. Moreover, it implies that the family of convolution operators K_λ $(\lambda>0)$, satisfying a, b' and c supplies an approximation process for the identity operator I as $\lambda \to \infty$ (section 2.6).

The condition c can be replaced by

$$\text{(c')} \qquad \lim_{\lambda\to\infty} \int_h^\pi |K_\lambda(\cos\theta)|\ \rho^{(\alpha,\beta)}(\theta)d\theta = 0, \quad \text{for each } h,\ 0 < h < \pi.$$

In this case we have

3.4. Theorem. If $f \in X$ and K_λ satisfies the conditions a, b' and c', then (3.1) and (3.2) are valid.

Proof. Relation (3.1) is a direct consequence of condition b' and lemma 1.2 (iv). In order to derive (3.2) we consider

$$||K_\lambda * f - f||_X = ||\int_0^\pi K_\lambda(\cos\phi)(T_\phi f(\cos\theta) - f(\cos\theta))\rho^{(\alpha,\beta)}(\phi)d\phi||_X$$

$$\leq \int_0^\pi |K_\lambda(\cos\phi)| \; ||T_\phi f(\cos\theta) - f(\cos\theta)||_X \; \rho^{(\alpha,\beta)}(\phi)d\phi,$$

where we have used the Hölder-Minkowski inequality (see [11] prop. 0.1.7.). We break up the range of integration into the parts $[0,h]$ and $[h,\pi]$. If we choose $h < \delta$, it follows from (1.7) and b' that

$$\int_0^h |K_\lambda(\cos\phi)| \; ||T_\phi f - f||_X \; \rho^{(\alpha,\beta)}(\phi)d\phi < \varepsilon N.$$

On the other hand, using (1.6) and c', we obtain

$$\int_h^\pi |K_\lambda(\cos\phi)| \; ||T_\phi f - f||_X \; \rho^{(\alpha,\beta)}(\phi)d\phi$$

$$\leq 2||f||_X \int_h^\pi |K_\lambda(\cos\phi)| \; \rho^{(\alpha,\beta)}(\phi)d\phi < \varepsilon, \text{ if } \lambda > \lambda_1(\varepsilon).$$

This proves relation (3.2).

Remark. Relation (3.2) implies c, which shows that a, b' and c' imply a, b' and c.

In section 5 we shall investigate the optimal and non-optimal approximation of a number of approximation processes, most of which can be interpreted as the convolution of $f \in X$ with a kernel K_λ. In these cases we verify the conditions a, b (or b') and c (or c'). Moreover, in order to apply the general theory of section 2, we must show that relations of the form (2.10) and (2.11) hold. For all the convolution operators we investigate, the operator B that occurs in (2.10) is of the factor sequence type as is defined in 3.5.

3.5. Definition. Let $\psi(x)$ be an arbitrary real or complex-valued function defined in $[0,\infty)$. The operator B_ψ, which maps $f \in X$ with the Fourier-Jacobi expansion (1.4) into $g \in X$, where

$$(3.3) \qquad g(\cos\theta) = B_\psi f(\cos\theta) \sim \sum_{n=0}^{\infty} f^\wedge(n)\psi(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta),$$

is called an operator of the factor sequence type with factors $\psi(n)$, $n = 0, 1, \ldots$.

3.6. Theorem. Let B_ψ be an operator of the factor sequence type with factors $\psi(n)$, $n = 0, 1, \ldots$. Then B_ψ is a closed, linear operator with domain

$$D(B_\psi) = \{f \in X : \exists g \in X,\ g(\cos\theta) \sim \sum_{n=0}^{\infty} f^\wedge(n)\psi(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta)\}$$

and range in X. The domain $D(B_\psi)$ is a normalized Banach subspace of X under the norm

$$||f||_{D(B_\psi)} = ||f||_X + ||B_\psi f||_X .$$

Proof. The proof is the same as the proof for Fourier series (Butzer-Scherer [13], lemma 4.1.1). We assume $\{f_i\}_{i=0}^{\infty}$ is a sequence in $D(B_\psi)$ with $\lim_{i\to\infty} f_i = f$ and $\lim_{i\to\infty} B_\psi f_i = g$ in X. It follows that $\lim_{i\to\infty} f_i^\wedge(n) = f^\wedge(n)$ $(n=0,1,\ldots)$ and $\lim_{i\to\infty} \psi(n)\, f_i^\wedge(n) = g^\wedge(n)$, which implies that $\psi(n)\, f^\wedge(n) = g^\wedge(n)$ $(n=0,1,\ldots)$. This means that $f \in D(B_\psi)$ and $B_\psi f = g$ or B_ψ is closed. The linearity of the operator B_ψ is obvious. The last assertion of the theorem is a consequence of section 2.8.

For operators of the factor sequence type B_ψ it is possible to give a characterization of the relative completion of $D(B_\psi)$ with respect to X (see section 2.8) in terms of the Fourier-Jacobi coefficients.

3.7. Theorem. If B_ψ is an operator of the factor sequence type with factors $\psi(n)$, $n = 0, 1, \ldots$, then the following statements are equivalent:

i) $f \in \widetilde{D(B_\psi)}^X$,

ii) $$f \in H(X,\psi(n)) = \begin{cases} f \in C & : \exists g \in L^\infty,\ \psi(n)f^\wedge(n) = g^\wedge(n), \\ f \in L^1 & : \exists \mu \in M,\ \psi(n)f^\wedge(n) = \mu^\vee(n), \\ f \in L^p\ (1<p<\infty): & \exists g \in L^p,\ \psi(n)f^\wedge(n) = g^\wedge(n). \end{cases}$$

Proof. We prove the theorem in the case $X = L^1$. The other cases are similar.

i → ii. If $f \in \widetilde{D(B_\psi)}^X$ there exists a sequence $\{f_i\}_{i=1}^\infty \in D(B_\psi)$ with

$$||f_i||_{D(B_\psi)} = ||f||_{\widetilde{D(B_\psi)}^X} \text{ for all } i = 1, 2, \ldots \text{ and } \lim_{i\to\infty} f_i = f \text{ in } L^1.$$

Then the sequence $\{B_\psi f_i\}_{i=1}^\infty$ is uniformly bounded in L^1. By the weak*-compactness of a closed sphere in M we may conclude that there exists a subsequence i_j of the positive integers and a measure $\mu \in M$ such that for each continuous function g

$$\lim_{j\to\infty} \int_0^\pi g(\cos\theta)\, B_\psi f_{i_j}(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)d\theta = \int_0^\pi g(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)d\mu(\cos\theta).$$

If for g we take the Jacobi polynomials $R_n^{(\alpha,\beta)}(\cos\theta)$ we obtain

$$\lim_{j\to\infty} \psi(n)\, f_{i_j}^\wedge(n) = \mu^\vee(n)$$

or by the fact that $\lim_{j\to\infty} f_j = f$

$$\psi(n)\, f^\wedge(n) = \mu^\vee(n).$$

ii → i. If we assume that there exists a measure $\mu \in M$ with $\psi(n)\, f^\wedge(n) = \mu^\vee(n)$, then we consider the sequence

$$f_N = V_N * f, \qquad N = 0, 1, \ldots .$$

Here the functions V_N (cos θ) (N=0,1,...) are polynomials of degree $\leq$ 2N, which form a positive summability kernel. Such kernels exist (e.g. the de la Vallée-Poussin kernel, see section 5.20).
By lemma 1.2 (iii) it follows that f_N is a polynomial of degree $\leq$ 2N and thus $\{f_N\}_{N=0}^{\infty} \in D(B_\psi)$. Moreover,

$$||B_\psi f_N||_1 = ||V_N * \mu||_1 \leq ||\mu||_M \qquad \text{uniformly in N.}$$

On the other hand $f_N \to f$ as $N \to \infty$, which shows that $f \in \widetilde{D(B_\psi)}^X$.

<u>3.8. Lemma</u>. There exists a measure $\mu \in M$ with

$$\left(\frac{n}{n+\alpha+\beta+1}\right)^\lambda = \overset{\vee}{\mu}(n),$$

when λ is an arbitrary real number.

<u>Proof</u>. For n = 1, 2, ... we have

$$\left(\frac{n}{n+\alpha+\beta+1}\right)^\lambda = \left(\frac{1}{1+(\alpha+\beta+1)/n}\right)^\lambda$$

$$= 1 + \frac{t_1(\lambda)}{n} + \frac{t_2(\lambda)}{n^2} + \ldots + f_\lambda^{(p)}(\xi_n)\,\frac{(\alpha+\beta+1)^p}{p!\,n^p}, \quad \left(0<\xi_n< \frac{\alpha+\beta+1}{n}\right),$$

because the function $f_\lambda(t) = (1+t)^{-\lambda}$ has an arbitrary number if derivatives for $t > -1$. At the right-hand side, the first term 1 forms the Jacobi-Stieltjes coefficients of a finite measure, the Dirac measure. In [5], section 2 we have shown that the terms n^{-k} (k>0) are the Fourier-Jacobi coefficients of an L^1 function. If we choose $p > 2\alpha+2$, the last expression forms the coefficients of an absolutely convergent series. Thus, together they form the coefficients of a measure $\mu \in M$.

<u>3.9. Corollary</u>. The operators of the factor sequence type $B_{[n(n+\alpha+\beta+1)]^\lambda}$ and $B_{n^{2\lambda}}$ (λ real) have the same domain.

<u>Proof</u>. This is an immediate consequence of lemma 3.8 and (1.12).

4. Best approximation by polynomials

In this section we deal with the connection between the modulus of continuity of a function f, defined with respect to the generalized translation, and the degree of approximation of f by polynomials in cos θ. We obtain theorems of the Jackson and the Bernstein type.

4.1. Definition. We define the modulus of continuity of $f \in X$ by

$$(4.1) \qquad \omega(\phi,f) = \sup_{0<\psi<\phi} ||T_\psi f-f||_X \,.$$

Here T_ψ is written for the generalized translation operator, introduced in section 1.1.

The modulus of continuity $\omega(\phi,f)$ is a positive, increasing function, which by (1.7) has the property $\omega(\phi,f) \to 0$ if $\phi \to 0^+$.

4.2. In order to derive another important of $\omega(\phi,f)$, we need more knowledge of the generalized translation. It is a well-known fact (Szegö [25]) that the Jacobi polynomials $R_n^{(\alpha,\beta)}(\cos\theta)$ satisfy the differential equation

$$(4.2) \quad P(\tfrac{d}{d\theta})R_n^{(\alpha,\beta)}(\cos\theta) = -\{\rho^{(\alpha,\beta)}(\theta)\}^{-1}\frac{d}{d\theta}\{\rho^{(\alpha,\beta)}(\theta)\frac{d}{d\theta}R_n^{(\alpha,\beta)}(\cos\theta)\}$$

$$= n(n+\alpha+\beta+1)\, R_n^{(\alpha,\beta)}(\cos\theta).$$

We shall write A for the operator of the factor sequence type (definition 3.5) with factors $\psi(n) = n(n+\alpha+\beta+1)$, $n = 0, 1, \ldots$. For each $f \in D(A)$ the following equation is satisfied

$$(4.3) \qquad P(\tfrac{d}{d\phi})\, T_\phi f = T_\phi Af.$$

The differential equation will play an essential role in deriving the approximation properties of the process T_ϕ $(\phi\to 0)$. Generalized translations connected with an equation of the form (4.3) are investigated by Löfström and Peetre [17]. Following them we introduce the function

$$(4.4) \qquad h(\phi,\tau) = \begin{cases} -\displaystyle\int_\tau^\phi \{\rho^{(\alpha,\beta)}(\lambda)\}^{-1}\, d\lambda, & 0 < \tau < \phi, \\ 0, & \text{otherwise.} \end{cases}$$

As $\rho^{(\alpha,\beta)}(\tau) \approx \tau^{2\alpha+1}$, $\tau \to 0^+$, it follows that

$$\rho^{(\alpha,\beta)}(\tau)\, h(\phi,\tau) \to 0, \qquad \tau \to 0^+.$$

Thus by (4.3) we have for $f \in D(A)$

$$\begin{aligned} \int_0^\phi h(\phi,\tau)\, T_\tau Af\, \rho^{(\alpha,\beta)}(\tau)d\tau &= \int_0^\phi h(\phi,\tau)\, P(\tfrac{d}{d\tau})\, T_\tau f\, \rho^{(\alpha,\beta)}(\tau)d\tau \\ (4.5) \qquad &= [-\rho^{(\alpha,\beta)}(\tau)h(\phi,\tau)\, \tfrac{d}{d\tau}\, T_\tau f]_0^\phi + \int_0^\phi \tfrac{d}{d\tau}\, T_\tau f\, d\tau \\ &= T_\phi f - f. \end{aligned}$$

The integration is meant in the sense of Bochner [8] (see Hille-Phillips [15]). It is easy to verify that

$$(4.6) \qquad C_1(\phi) = \int_0^\phi h(\phi,\tau)\, \rho^{(\alpha,\beta)}(\tau)d\tau = O(\phi^2), \; \phi \to 0^+.$$

Recalling definition 4.1, we obtain for $f \in D(A)$

$$\begin{aligned} \omega(\phi,f) &= \sup_{0<\psi<\phi} ||T_\psi f - f||_X \\ (4.7) \qquad & \\ &\le C\phi^2\, ||Af||_X, \qquad 0 < \phi < \frac{\pi}{2}. \end{aligned}$$

In the case $\frac{\pi}{2} < \phi \le \pi$ we use a computation similar to that in the paper of Butzer-Johnen [10] and attributed there to Chernoff-Ragozin.

$$T_\phi f - f = \int_0^\phi h(\phi,\tau)\ T_\tau Af\ \rho^{(\alpha,\beta)}(\tau)d\tau$$

$$= \int_0^\phi \{\rho^{(\alpha,\beta)}(\tau)\}^{-1} \int_0^\tau T_\sigma Af\ \rho^{(\alpha,\beta)}(\sigma)d\sigma\ d\tau$$

$$= \int_0^{\pi/2} \{\rho^{(\alpha,\beta)}(\tau)\}^{-1} \int_0^\tau T_\sigma Af\ \rho^{(\alpha,\beta)}(\sigma)d\sigma\ d\tau +$$

$$\int_{\pi/2}^\phi \{\rho^{(\alpha,\beta)}(\tau)\}^{-1} \int_0^\tau T_\sigma Af\ \rho^{(\alpha,\beta)}(\sigma)d\sigma\ d\tau = I_1 + I_2.$$

The Fourier-Jacobi expansion of Af shows that

$$\int_0^\pi T_\sigma Af\ \rho^{(\alpha,\beta)}(\sigma)d\sigma = 0.$$

Hence,

$$I_2 = -\int_{\pi/2}^\phi \{\rho^{(\alpha,\beta)}(\tau)\}^{-1} \int_\tau^\pi T_\sigma Af\ \rho^{(\alpha,\beta)}(\sigma)d\sigma\ d\tau,$$

which leads to

$$||T_\phi f-f||_X \leq \phi^2 \sup_{\frac{\pi}{2}\leq\psi\leq\pi} \psi^{-2}\Big(\int_0^{\pi/2}\{\rho^{(\alpha,\beta)}(\tau)\}^{-1} \int_0^\tau \rho^{(\alpha,\beta)}(\sigma)d\sigma\ d\tau$$

$$+ \int_{\pi/2}^\psi \{\rho^{(\alpha,\beta)}(\tau)\}^{-1} \int_\tau^\pi \rho^{(\alpha,\beta)}(\sigma)d\sigma\ d\tau\Big)\ ||Af||_X,$$

for $\frac{\pi}{2} \leq \phi \leq \pi$, and thus (4.7) is valid for $0 < \phi \leq \pi$.

We want to express the modulus of continuity $\omega(\phi,f)$ in terms of the K function norm (2.2) with the spaces X and D(A).

<u>4.3. Lemma</u>. For $f \in X$ and $0 < \phi \leq \pi$

(4.8) $$K(\phi^2,f;X,D(A)) \approx \min(1,\phi^2)\ ||f||_X + \omega(\phi,f).$$

<u>Proof</u>. Let $f = f_0+f_1$ with $f_0 \in X$, $f_1 \in D(A)$. By (1.6) and (4.7) we have

$$\omega(\phi,f_0) \leq 2 \, ||f_0||_X,$$

$$\omega(\phi,f_1) \leq C \, \phi^2 \, ||Af_1||_X.$$

Thus,

$$\omega(\phi,f) \leq \omega(\phi,f_0) + \omega(\phi,f_1)$$

$$\leq C(||f_0||_X + \phi^2||Af_1||_X).$$

Since

$$\min(1,\phi^2) \, ||f||_X \leq ||f_0||_X + \phi^2||f_1||_X,$$

we obtain

$$\min(1,\phi^2) \, ||f||_X + \omega(\phi,f) \leq C(||f_0||_X + \phi^2||f_1||_{D(A)}).$$

Taking the minimum of the right-hand side over all the representations $f = f_0+f_1$, we deduce by the definition of the K funtion norm (section 2.1)

$$\min(1,\phi^2) \, ||f||_X + \omega(\phi,f) \leq CK(\phi^2,f;X,(D(A)).$$

To prove the converse of this inequality we use

(4.9) $$f_{1,\phi} = [C_1(\phi)]^{-1} \int_0^\phi h(\phi,\tau) \, (T_\tau f) \, \rho^{(\alpha,\beta)}(\tau)d\tau,$$

(4.10) $$f_{0,\phi} = f - f_{1,\phi}.$$

Then, by (4.5) and the closeness of the operator A (theorem 3.6), it follows that $f_{1,\phi} \in D(A)$ and

(4.11) $$T_\phi f - f = C_1(\phi)\, Af_{1,\phi}.$$

Hence,

(4.12) $$||Af_{1,\phi}||_X \leq (C_1(\phi))^{-1}\, ||T_\phi f-f||_X.$$

By (1.6) and (4.9) it follows that

(4.13) $$||f_{1,\phi}||_X \leq ||f||_X.$$

On the other hand we have

$$f_{0,\phi} = -\,(C_1(\phi))^{-1} \int_0^\phi h(\phi,\tau)\; (T_\tau f-f)\; \rho^{(\alpha,\beta)}(\tau)d\tau,$$

which leads to

(4.14) $$||f_{0,\phi}||_X \leq \omega(\phi,f).$$

Using (4.6), (4.12) and (4.13) we conclude

$$\begin{aligned} ||f_{1,\phi}||_{D(A)} &= ||f_{1,\phi}||_X + ||Af_{1,\phi}||_X \\ &\leq ||f||_X + C\phi^{-2}\, \omega(\phi,f),\ 0 < \phi \leq \tfrac{\pi}{2}\,. \end{aligned}$$

Hence, by (4.14)

$$\begin{aligned} K(\phi^2,f;X,D(A)) &\leq ||f_{0,\phi}||_X + \phi^2||f_{1,\phi}||_{D(A)} \\ &\leq ||f_{0,\phi}||_X + \phi^2||f||_X + C\,\omega(\phi,f) \\ &\leq \phi^2||f||_X + C\,\omega(\phi,f). \end{aligned}$$

Noticing that $K(\phi^2,f;X,D(A)) \leq ||f||_X$, we derive

(4.15) $$K(\phi^2,f;X,D(A)) \leq C(\min(1,\phi^2)\ ||f||_X + \omega(\phi,f)),\ 0 < \phi \leq \frac{\pi}{2}\ .$$

If $\frac{\pi}{2} < \phi \leq \pi$, we observe that

$$K(\phi^2,f;X,D(A)) \leq 4\ K((\tfrac{\phi}{2})^2,f;X,D(A))$$

and we apply (4.15) to the right-hand side of this inequality, noticing the monotonicity of $\omega(\phi,f)$. Thus we conclude that (4.15) holds for $0 < \phi \leq \pi$. This completes the proof of lemma 4.3.

From (4.8) and the corresponding property for the K function norm we obtain

4.4. Corollary. For $f \in X$, $0 < \phi \leq \pi$ and $\lambda > 0$

(4.16) $$\omega(\lambda\phi,f) \leq C\max(1,\lambda^2)\ [\phi^2||f||_X + \omega(\phi,f)].$$

If one defines the best approximation of $f \in X$ by elements of P_n, the (n+1) dimensional subspace of polynomials of degree $\leq n$ in $\cos\theta$, by

$$E(P_n,f) = \inf_{p_n \in P_n} ||f-p_n||_X\ ,$$

we prove the following theorem of the Jackson type.

4.5. Theorem. There exists a constant C such that for each $f \in X$

(4.17) $$E(P_n,f) \leq C[n^{-2}\ ||f||_X + \omega(n^{-1},f)].$$

Proof. We use the kernel

$$L_{n,r}(\theta) = \lambda_{n,r}^{-1}\ (\frac{\sin\ n\ \theta/2}{\sin\ \theta/2})^{2r},$$

where r is a positive integer and

$$\lambda_{n,r} = \int_0^{\pi} \left(\frac{\sin n\,\theta/2}{\sin\,\theta/2}\right)^{2r} \rho^{(\alpha,\beta)}(\theta)d\theta$$

$$\leq C\, n^{2r-2\alpha-2}, \qquad \text{if } 2r > 2\alpha + 2.$$

As this kernel clearly satisfies the conditions a, b and c' with $\lambda = n$ (sections 3.2 and 3.3), the convolution with the kernel $L_{n,r}$ leads to an approximation process by theorem 3.4. Moreover, this kernel has the useful property (Lorentz [18], p. 57)

$$\int_0^{\pi} \theta^{\gamma}\, L_{n,r}(\theta)\, \rho^{(\alpha,\beta)}(\theta)d\theta \leq C\, n^{-\gamma}, \quad \text{if } 2r > 2\alpha + \gamma + 2.$$

The kernel $K_{n,r}(\theta) = L_{n',r}(\theta)$, $n' = [\frac{n}{r}] + 1$ is a positive polynomial in $\cos\theta$ of degree n with

$$(4.18) \qquad \int_0^{\pi} K_{n,r}(\theta)\, \rho^{(\alpha,\beta)}(\theta)d\theta = 1, \qquad 2r > 2\alpha + 2,$$

and for $\gamma > 0$

$$(4.19) \qquad \int_0^{\pi} \theta^{\gamma}\, K_{n,r}(\theta)\, \rho^{(\alpha,\beta)}(\theta)d\theta \leq C\, n^{-\gamma}, \qquad 2r > 2\alpha + \gamma + 2.$$

We shall now prove theorem 4.5. By the Hölder-Minkowski inequality we have

$$E(P_n,f) \leq ||K_{n,r}*f-f||_X$$

$$= ||\int_0^{\pi} K_{n,r}(\phi)(T_{\phi}f-f)\, \rho^{(\alpha,\beta)}(\phi)d\phi||_X$$

$$\leq \int_0^{\pi} K_{n,r}(\phi)\, \omega(\phi,f)\, \rho^{(\alpha,\beta)}(\phi)d\phi.$$

By (4.16) we obtain

$$E(P_n,f) \leq C \max[1,n^2 \int_0^{\pi} K_{n,r}(\phi)\, \phi^2 \rho^{(\alpha,\beta)}(\phi)d\phi][n^{-2}||f||_X + \omega(n^{-1},f)],$$

which, by (4.19) with $2r > 2\alpha + 4$, leads to

$$E(P_n,f) \leq C[n^{-2}||f||_X + \omega(n^{-1},f)].$$

<u>4.6. Theorem</u>. If $f \in D(A^k)$, there exists a constant $c(k)$ independent of f such that

$$(4.20) \qquad E(P_n,f) \leq Cn^{-2k}[n^{-2}(||f||_X + ||A^k f||_X) + \omega(n^{-1},A^k f)].$$

<u>Proof</u>. If we write

$$J_{n,r}\, f(\cos\theta) = K_{n,r}*f)\ (\cos\theta),$$

we define the operator $T^k_{n,r}$ by

$$T^k_{n,r} = -\ (I-J_{n,r})^{k+1} + I.$$

Clearly $T^k_{n,r}\, f$ is a polynomial of degree n and by (4.17)

$$||f-T^k_{n,r}f||_X = ||(I-J_{n,r})^k f - J_{n,r}(I-J_{n,r})^k f||_X$$

$$\leq c[||(I-J_{n,r})^k f||_X n^{-2} + \omega(n^{-1},(I-J_{n,r})^k f)].$$

From the Fourier-Jacobi expansion it follows that

$$A(I-J_{n,r})^k f = (I-J_{n,r})^k Af,$$

so that by (4.17) we obtain

$$||f-T^k_{n,r}f||_X \leq C[n^{-2}||(I-J_{n,r})^k f||_X + n^{-2}||(I-J_{n,r})^k Af||_X].$$

Continuing this process we obtain

$$E(P_n,f) \leq c(k)n^{-2k}[n^{-2}(||f||_X + \sum_{i=1}^{k} ||A^i f||_X) + \omega(n^{-1},A^k f)].$$

Since $Af = g_2 * A^2 f$, where

$$g_2(\cos\theta) = \omega_0^{(\alpha,\beta)} + \sum_{n=1}^{\infty} [n(n+\alpha+\beta+1)]^{-2}\, \omega_n^{(\alpha,\beta)}\, R_n^{(\alpha,\beta)}(\cos\theta),$$

and $g_2 \in L^1$ (see [5], section 4), we have by lemma 1.2 (iv)

$$||Af||_X \leq c||A^2 f||_X.$$

Hence,

$$\sum_{i=1}^{k} ||A^i f||_X \leq c||A^k f||_X$$

and (4.20) is proved.

For the operator A, Stein [24] has derived an inequality of the Bernstein type. There exists a constant C such that for each $p \in P_n$

$$(4.21) \qquad ||Ap_n||_p \leq Cn^2||p_n||_p, \qquad 1 \leq p \leq \infty,$$

and therefore for each positive integer k

$$(4.22) \qquad ||A^k p_n||_p \leq C^k n^{2k}||p_n||_p, \qquad 1 \leq p \leq \infty.$$

From definition 2.3 and the formulas (4.20) and (4.22) we conclude that the Banach space $D(A^k)$ belongs to the class $D_{2k}(X)$.
Application of theorem 2.5 yields

<u>4.7. Theorem</u>. For $0 < \gamma < 1$ and $1 \leq q \leq \infty$

$$(D(A^k), D(A^{k+1}))_{\gamma,q;K} \simeq X^K_{2k+2\gamma,q} .$$

<u>4.8. Definition</u>. The Lipschitz space $\mathrm{Lip}(\gamma,X)$, $0 < \gamma \leq 2$, is the set of all elements $f \in X$ such that

$$\sup_{0<\phi\leq\pi} \phi^{-\gamma}\,\omega(\phi,f) < \infty.$$

It is an immediate consequence of (4.8), that the spaces $\mathrm{Lip}(\gamma,X)$, $0 < \gamma \leq 2$, coicide with the intermediate spaces $(X,D(A))_{\gamma/2,\infty;K}$ (section 2.1) with equivalent norms. Moreover, the theorems 4.6 and 4.7 lead to

4.9. Corollary. Let $f \in X$. Then a necessary and sufficient condition for f to belong to $D(A^k)$ with $A^k f \in \mathrm{Lip}(\gamma,X)$, k natural number, $0 < \gamma < 2$, is $E(P_n,f) \leq Cn^{-2k-\gamma}$.

To a certain extend we have followed in this section the paper of Butzer-Johnen [10], who obtain similar theorem for spherical harmonic expansions. Ragozin [22] derives Jackson estimates for polynomial approximation of continuous functions on projective spaces.

5. Saturation and non-optimal approximation for some summability methods

5.1. Generalized translation

The first approximation process we consider is the family of operators $\{T_\phi, \phi>0\}$, which by (1.6) and (1.7) satisfies the conditions (2.6) with $\lambda = \phi^{-1}$. In this case the modulus of T approximation $\omega_T(\phi,f)$ coincides with the modulus of continuity $\omega(\phi,f)$. It has to be noted, that this approximation process cannot be ontained by the convolution of f with a kernel $\in L^1$ (see section 3), but it is generated by the convolution of f with a family of singular measures $\in$ M. We prove the following saturation theorem.

5.2. Theorem

a) If $f \in X$ satisfies the relation $\omega(\phi,f) = o(\phi^2)$, $\phi \to 0^+$, then $T_\phi f = f$, that is, f is a constant.

b) For $f \in X$ the following statements are equivalent:

i) $\omega(\phi,f) = O(\phi^2)$, $\phi \to 0^+$,

ii) $f \in \widetilde{D(A)}^X$, (section 2.8).

Proof

a) Let us assume $\omega(\phi,f) = o(\phi^2)$, $\phi \to 0^+$. If we define f by (4.9) and $f_{0,\phi}$ by (4.10), it follows from (4.14) that $f_{0,\phi} \to 0$ and $f_{1,\phi} \to f$, if $\phi \to 0^+$. Moreover, by (4.12) and the closeness of the operator A we conclude that

$$\lim_{\phi\to 0^+} Af_{1,\phi} = Af = 0.$$

Thus, by (4.5), $T_\phi f = f$ or f is a constant.

b) ii $\to$ i. If $f \in \widetilde{D(A)}^X$, then there exists a sequence $\{f_n\}_{n=1}^{\infty} \in D(A)$ with $||f_n||_{D(A)} = ||f||_{\widetilde{D(A)}^X}$, such that $f_n \to f$ in X. For each $f_n \in D(A)$ we have by (4.7)

$$||f_n||_X + \phi^{-2}||T_\phi f_n - f_n||_X \leq C||f_n||_{D(A)}.$$

Thus

$$||f||_X + \phi^{-2}||T_\phi f-f||_X \leq C||f||_{\widetilde{D(A)}^X} ,$$

showing that $\omega(\phi,f) = O(\phi^2)$, $\phi \to 0^+$.

i $\to$ ii. Let f satisfy $\omega(\phi,f) = O(\phi^2)$, $\phi \to 0^+$. Then the functions $f_{1,\phi}$ defined by (4.9) have the following properties:

1) $f_{1,\phi} \to f$ in X, if $\phi \to 0^+$.

2) $f_{1,\phi} \in D(A)$, $\phi > 0$.

3) Application of (4.12) and (4.13) yields

$$||f_{1,\phi}||_{D(A)} = ||f_{1,\phi}||_X + ||Af_{1,\phi}||_X \leq ||f||_X + C\phi^{-2}||T_\phi f-f||_X < N.$$

These three properties imply that $f \in \widetilde{D(A)}^X$.

In the case of non-optimal approximation we compare the spaces of T-approximation $X_{\theta,q;T}$ (see section 2.6) with the intermediate spaces $(X,D(A))_{\theta,q;K}$ (see section 2.1). We obtain

<u>5.3. Theorem</u>. The following statements are equivalent for $0 < \theta < 1$, $1 \leq q < \infty$ and $\theta = 1$, $q = \infty$:

i) $f \in (X,D(A))_{\theta,q;K}$,

ii) $f \in X_{2\theta,q;T}$.

<u>Proof</u>. The theorem is an immediate consequence of (4.8).

<u>5.4. The symmetric moving average operator</u>

We now study the approximation process $\{M_\phi,\phi>0\}$. This is a generalization of the symmetric moving average operator, which has been used by Lebesgue for the summation of Fourier series (Zygmund [27], p. 71, p. 321). We define for $f \in X$

$$(5.1)\qquad M_\phi f(\cos\theta) = \frac{1}{\Omega(\phi)}\int_0^\phi T_\psi f(\cos\theta)\,\rho^{(\alpha,\beta)}(\psi)d\psi,\quad (\phi>0),$$

where

$$(5.2)\qquad \Omega(\phi) = \int_0^\phi \rho^{(\alpha,\beta)}(\psi)d\psi.$$

This average operator can be represented as the convolution of f with a positive summability kernel. The positivity (definition 3.2.b) is an immediate consequence of (5.1) and the positivity of the generalized translation operator T_ϕ. The conditions 3.2.a and 3.2.c (with $\phi = \lambda^{-1}$) are also satisfied, as follows from the Fourier-Jacobi coefficients of the kernel $M_\phi(\cos\theta)$

$$(5.3)\qquad \hat{M}_\phi(n) = \frac{1}{\Omega(\phi)}\int_0^\phi R_n^{(\alpha,\beta)}(\cos\psi)\,\rho^{(\alpha,\beta)}(\psi)d\psi,\quad n = 0,\ 1,\ \ldots\ .$$

By theorem 3.3 we may conclude, that the family of operators $\{M_\phi, \phi>0\}$ is an approximation process for the identity I as defined in section 2.6.

It is not hard to verify that for $f \in D(A)$ the function $M_\phi f(\cos\theta)$ is a solution of the equation

$$(5.4)\qquad Q(\tfrac{d}{d\phi})\ M_\phi f = M_\phi Af,$$

where

$$Q(\tfrac{d}{d\phi}) = -\ \{\sigma(\phi)\}^{-1}\ \frac{d}{d\phi}\ \{\sigma(\phi)\ \frac{d}{d\phi}\}$$

with

$$\sigma(\phi) = \frac{\{\Omega(\phi)\}^2}{\rho^{(\alpha,\beta)}(\phi)}\ .$$

The differential equation (4.3) for $T_\phi f(\cos\theta)$ is of the same form as equation (5.4). Therefore, all the results obtained in section 4 for T_ϕ can be carried over to the operator M_ϕ. Also, we state the following theorems concerning saturation and non-optimal approximation of the process $\{M_\phi, \phi>0\}$. The proofs are similar to the proofs of the theorems 5.2 and 5.3.

5.5. Theorem

a) If $f \in X$ satisfies $\omega_M(\phi,f) = o(\phi^2)$, $\phi \to 0^+$, then $M_\phi f = f$.

b) For $f \in X$ the following statements are equivalent:

i) $\omega_M(\phi,f) = O(\phi^2)$, $\phi \to 0^+$,

ii) $f \in \widetilde{D(A)}^X$.

5.6. Theorem. The following statements are equivalent for $0 < \theta < 1$, $1 \le q \le \infty$ and $\theta = 1$, $q = \infty$:

i) $f \in (X,D(A))_{\theta,q;K}$,

ii) $f \in X_{2\theta,q;M}$.

The theorems 5.2 and 5.5 state that the approximation processes $\{T_\phi, \phi>0\}$ and $\{M_\phi, \phi>0\}$ are saturated with the order ϕ^2. The saturation class is the relative completion of the domain of the operator A, which by theorem 3.7 leads to

5.7. Corollary. The processes $\{T_\phi, \phi>0\}$ and $\{M_\phi, \phi>0\}$ are saturated with the order ϕ^2. The saturation class is

$$H(X,n(n+\alpha+\beta+1)) = \begin{cases} f \in C & : \exists g \in L^\infty,\ n(n+\alpha+\beta+1)\ f^\wedge(n) = g^\wedge(n), \\ f \in L^1 & : \exists \mu \in M,\ n(n+\alpha+\beta+1)\ f^\wedge(n) = \mu^\vee(n), \\ f \in L^p\ (1<p<\infty) & : \exists g \in L^p,\ n(n+\alpha+\beta+1)\ f^\wedge(n) = g^\wedge(n). \end{cases}$$

5.8. The Weierstrass approximation process

The Weierstrass kernel is defined by

$$(5.5)\qquad W_t(\cos\theta) = \sum_{n=0}^{\infty} e^{-n(n+\alpha+\beta+1)t}\ \omega_n^{(\alpha,\beta)}\ R_n^{(\alpha,\beta)}(\cos\theta), \quad (t>0).$$

It satisfies the generalized heat equation

$$(5.6)\qquad \frac{\partial}{\partial t} W_t(\cos\theta) = AW_t(\cos\theta).$$

We verify the conditions for a summability kernel (definition 3.2). The conditions a and c with $t = \lambda^{-1}$ are obviously satisfied. The positivity (5.5), condition b, is a consequence of the positivity of the generalized translation by the following argument due to Bochner [8].

If an operator of the factor sequence type with factors $c_n (n=0,1,\ldots)$ is positive, then the operator of the factor sequence type with the factors e^{tc_n} $(t\geq 0,\ n=0,1,\ldots)$ is also positive. Taking $c_n = R_n^{(\alpha,\beta)}(\cos\phi)$ and multiplying by e^{-t} we obtain, that the operator with the factors

$$\mu_n = e^{-t(1-R_n^{(\alpha,\beta)}(\cos\phi))}, \qquad n = 0,\ 1,\ \ldots,$$

is positive. If we now replace t by $t(2\alpha+2)\ (1-\cos\phi)^{-1}$ and we let $\phi \to 0^+$, we conclude that (5.5) is positive.

By theorem 3.3 it follows, that the sequence $\{W_t, t>0\}$ is an approximation process for the identity I. Since $W_t f(\cos\theta) = (W_t * f)\ (\cos\theta)$ is a solution of equation (5.6) for all $f \in X$ we have

$$\int_0^t AW_\tau f\, d\tau = \int_0^t \frac{\partial}{\partial\tau} W_\tau f\, d\tau = W_t f - f,$$

where we used integration in the sense of Bochner (see Hille-Phillips [15]). Hence, since $AW_t f = W_t Af$ for $f \in D(A)$,

$$(5.7)\qquad \left|\left|\frac{W_t f-f}{t} - Af\right|\right|_X = \left|\left|\frac{1}{t}\int_0^t AW_\tau f\, d\tau - Af\right|\right|_X$$

$$\leq \sup_{0<\tau<t} ||W_\tau Af - Af||_X = o(1).$$

Thus, by (5.7) and the fact that for every $f \in X$ we have $W_t f \in D(A)$, $t > 0$, we may apply theorem 2.9 to conclude that the process $\{W_t, t>0\}$ is saturated with order t and the saturation class is equal to $\widetilde{D(A)}^X$ or by corollary 5.7 to the set $H(X, n(n+\alpha+\beta+1))$.

To deal with the non-optimal approximation, we use the estimate

$$K(t,f;X,D(A)) \approx \min(1,t)\ ||f||_X + \omega_W(t,f) \qquad (t>0,\ f\in X)$$

due to Peetre [20]. In fact this approximation process is generated by a semi-group of operators with the operator A being the infinitesimal generator. For an extensive treatment of the theory of semi-groups of operators we refer to Butzer-Berens [9]. Summarizing the results we have

5.9. Theorem. The process $\{W_t, t>0\}$ is saturated with the order t. The saturation class is $H(X,n(n+\alpha+\beta+1))$. Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \leq q \leq \infty$ or $\theta = 1$, $q = \infty$:

i) $\quad f \in (X,D(A))_{\theta,q;K}$,

ii) $\quad f \in X_{\theta,q;W}$.

5.10. The generalized Weierstrass approximation process

The generalized Weierstrass kernel has been introduced by Bochner [8] for ultraspherical expansions. It is defined by

$$(5.8) \qquad W_t^\sigma(\cos\theta) = \sum_{n=0}^{\infty} e^{-[n(n+\alpha+\beta+1)]^{\sigma/2}t}\ \omega_n^{(\alpha,\beta)}\ R_n^{(\alpha,\beta)}(\cos\theta)$$

$$(0<\sigma\leq 2,\ t>0).$$

The conditions a and c of definition 3.2 are clearly satisfied. The positivity of (5.8) can be deduced from the positivity of (5.5) by Bochner's method of subordinators (see Bochner [8], p. 46). From theorem 3.3 it follows, that the sequence $\{W_t^\sigma, t>0\}$ is an approximation process for the identity I. The function $W_t^\sigma(\cos\theta)$ satisfies the equation

$$(5.9) \qquad \frac{\partial W_t^\sigma(\cos\theta)}{\partial t} = -\, D_\sigma W_t^\sigma(\cos\theta),$$

where D_σ denotes the fractional differential operator, defined in [5].

Since this approximation process is generated by a semi-group of operators, an argument similar to section 5.8 leads to the following theorem.

5.11. Theorem. The process $\{W_t^\sigma, t>0\}$ is saturated with the order t. The saturation class is $H(X,[n(n+\alpha+\beta+1)]^{\sigma/2})$. Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \leq q \leq \infty$ or $\theta = 1$, $q = \infty$:

i) $f \in (X,D(D_\sigma))_{\theta,q;K}$,

ii) $f \in X_{\theta,q;W^\sigma}$.

Later on we shall be able to characterize the spaces $(X,D(D_\sigma))_{\theta,q;K}$, $0 < \theta < 1$, $1 \leq q \leq \infty$ as intermediate spaces of X and D(A) (see theorem 6.6).

5.12. The Abel-Poisson approximation process

The Abel-Poisson kernel is defined by

$$(5.10) \qquad A_r(\cos\theta) = \sum_{n=0}^{\infty} r^n \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta) \qquad (0\leq r<1).$$

We check the conditions for a summability kernel (definition 3.2). The conditions a and c with $\lambda = (1-r)^{-1}$ are obviously satisfied. The positivity of (5.10) has been shown by Bailey [4], p. 102 by explicit calculation. Therefore, by theorem 3.3 the sequence $\{A_r, 0\leq r<1\}$ is an approximation process for the identity I.

We denote by B_{-n} the operator of the factor sequence type with factors $\psi(n) = -n$. If we write $A_r f(\cos\theta) = (A_r * f)(\cos\theta)$, then for each $f \in D(B_{-n})$ the following integral

$$U_r(\cos\theta) = \int_r^1 A_\rho B_{-n} f \frac{d\rho}{\rho}$$

exists in the sense of Bochner (see Hille-Phillips [15]). Taking the Fourier-Jacobi coefficients we obtain

$$U_r^{\wedge}(n) = -\int_r^1 \rho^n\, n\, f^{\wedge}(n)\, \frac{d\rho}{\rho}$$

$$= (r^n - 1)\, f^{\wedge}(n)$$

$$= (A_r f - f)^{\wedge}(n).$$

By the unicity of Fourier-Jacobi expansions we have for each $f \in D(B_{-n})$ the relation

$$(5.11) \qquad A_r f - f = \int_r^1 A_\rho\, B_{-n} f\, \frac{d\rho}{\rho}$$

and therefore

$$\frac{A_r f - f}{1-r} - B_{-n} f = \frac{1}{1-r}\int_r^1 [A_\rho B_{-n} f - B_{-n} f]\, \frac{d\rho}{\rho} - \left(1 + \frac{\log r}{1-r}\right) B_{-n} f.$$

Thus, for $r \to 1^-$,

$$\left|\left|\frac{A_r f - f}{1-r} - B_{-n} f\right|\right|_X \leq \frac{1}{r} \sup_{r \leq \rho < 1} ||A_\rho B_{-n} f - B_{-n} f||_X + \left|1 + \frac{\log r}{1-r}\right|\, ||B_{-n} f|| = o(1),$$

which implies for $f \in D(B_{-n})$

$$\lim_{r \to 1^-} \frac{A_r f - f}{r-1} = B_{-n} f \qquad \text{in } X.$$

Since $A_r f$ $(0 \leq r < 1)$ belongs to $D(B_{-n})$ for all $f \in X$, we may conclude by theorem 2.9, that the process $\{A_r,\ 0 \leq r < 1\}$ is saturated with the order $1-r$ and that the saturation class is equal to $\widetilde{D(B_{-n})}^X$ or by theorem 3.7 to the set $H(X,-n)$. Furthermore, by corollary 3.9, the set $H(X,-n)$ coincides with the set $\widetilde{D(D_1)}^X = H(X, [n(n+\alpha+\beta+1)]^{1/2})$.

With the substitution $r = e^{-t}$, the approximation process A_r can be interpreted as a semi-group of operators. The non-optimal approximation can be treated using the estimate

$$K(t,f;X,D(B_{-n})) \approx \min(1,t)\, ||f||_X + \omega_A(t,f), \quad (t>0,\ f \in X),$$

due to Peetre [20]. Thus we have

5.13. Theorem. The process $\{A_t,\ t>0,\ t=\log \frac{1}{r}\}$ is saturated with the order t. The saturation class is $H(X,[n(n+\alpha+\beta+1)]^{1/2})$. Moreover, the following statements are equivalent for $0<\theta<1$, $1\le q\le\infty$ or $\theta=1$, $q=\infty$:

i) $f \in (X,D(D_1))_{\theta,q;K}$,

ii) $f \in X_{\theta,q;A}$.

In theorem 6.6 we shall give a characterization of the spaces $(X,D(D_1))_{\theta,q;K}$, $0<\theta<1$, $1\le q\le\infty$ in terms of the intermediate spaces of X and D(A), which coincides with the Lipschitz spaces (definition 4.8) for $q=\infty$.

5.14. The generalized Abel-Poisson approximation process

The generalized Abel-Poisson kernel, a special case of some general investigations by Bochner [8], is defined by

$$(5.12)\qquad A_t^\sigma(\cos\theta) = \sum_{n=0}^{\infty} e^{-n^\sigma t}\ \omega_n^{(\alpha,\beta)}\ R_n^{(\alpha,\beta)}(\cos\theta)\quad (t>0,\ 0<\sigma\le 1).$$

In the case $\sigma = 1$, this kernel reduces to the Abel-Poisson kernel (with the substitution $r = e^{-t}$), which we have treated in section 5.12. The conditions for a summability kernel are satisfied. Conditions a and c are obvious, condition b, the positivity of (5.12), can be proved from the positivity in the case $\sigma = 1$ by a method due to Bochner [8], p. 43-47. The sequence $\{A_t^\sigma,\ t>0,\ 0<\sigma\le 1\}$ is an approximation process for the identity I by theorem 3.3. By B_{-n^σ} we denote the operator of factor sequence type with factors $\psi(n) = -n^\sigma$. If we put $A_t^\sigma f(\cos\theta) = (A_t^\sigma * f)(\cos\theta)$, then for each $f \in D(B_{-n^\sigma})$ the following integral

$$U_t(\cos\theta) = \int_0^t A_\tau^\sigma\ B_{-n^\sigma}\ f\ d\tau$$

exists in the Bochner sense. As in section 5.12 we conclude that for $f \in D(B_{-n^\sigma})$

$$(5.13) \qquad A_t^\sigma f - f = \int_0^t A_\tau^\sigma B_{-n^\sigma} f \, d\tau$$

and

$$\lim_{t\to 0^+} \frac{A_t^\sigma f - f}{t} = B_{-n^\sigma} f.$$

Using the same method as in section 5.12 we finally obtain

5.15. Theorem. The process $\{A_t^\sigma, t>0, 0<\sigma\leq 1\}$ is saturated with the order t. The saturation class is $H(X,[n(n+\alpha+\beta+1)]^{\sigma/2})$. Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \leq q \leq \infty$ or $\theta = 1$, $q = \infty$:

i) $\qquad f \in (X,D(D_\sigma))_{\theta,q;K}$,

ii) $\qquad f \in X_{\theta,q;A^\sigma}$.

A further characterization of the space $(X,D(D_\sigma))_{\theta,q;K}$ shall be given in theorem 6.6.

5.16. A Bernstein-type inequality for fractional derivatives

In the next sections we shall need a generalization of the inequality (4.21) to fractional powers of the operator A. These fractional derivatives D_σ have been introduced in Bavinck [5], section 5. We shall use the function

$$g_\sigma(\cos\theta) \sim \sum_{n=1}^{\infty} [n(n+\alpha+\beta+1)]^{-\sigma/2}\, \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta)$$

and some estimates derived in [5], section 5.

5.17. Lemma. There exists a constant $C(\sigma)$, such that for each $p_n \in P_n$

(5.14) $\quad ||D_\sigma p_n||_p \le C(\sigma) n^\sigma \, ||p_n||_p \qquad (0<\sigma\le 2,\ 1\le p\le\infty).$

Proof.

$$||D_\sigma p_n||_p = \{\int_0^\pi |\int_0^\pi T_\phi A p_n(\cos\theta) g_{2-\sigma}(\cos\theta)\rho^{(\alpha,\beta)}(\phi)d\phi|^p \rho^{(\alpha,\beta)}(\theta)d\theta\}^{1/p}$$

$$\le ||\int_0^{1/n}||_p + ||\int_{1/n}^\pi||_p = I_1 + I_2.$$

If we put $p' = p/(p-1)$ and notice that $\rho^{(\alpha,\beta)}(\phi) = O(\phi^{2\alpha+1})$, $\phi \to 0^+$, then

$$I_1^p = \int_0^\pi |\int_0^{1/n} T_\phi A p_n(\cos\theta) g_{2-\sigma}(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi|^p \rho^{(\alpha,\beta)}(\theta)d\theta$$

$$\le C\,(\int_0^{1/n}(\phi^{-1+\frac{\lambda(p-1)+1}{p}})^{p'}d\phi)^{p/p'}$$

$$\int_0^\pi(\int_0^{1/n}|T_\phi A p_n(\cos\theta)|^p|g_{2-\sigma}(\cos\phi)|^p\phi^{(2\alpha+2)p-\lambda(p-1)-1}d\phi)\rho^{(\alpha,\beta)}(\theta)d\theta$$

$$\le Cn^{-\lambda(p-1)}\int_0^{1/n}||T_\phi A p_n||_p^p\,|g_{2-\sigma}(\cos\phi)|^p\,\phi^{(2\alpha+2)p-\lambda(p-1)-1}d\phi.$$

Now using (1.6), (4.21) and [5], formula (5.2), we have

$$I_1^p \le Cn^{-\lambda(p-1)}\,n^{2p}\,||p_n||_p^p\int_0^{1/n}\phi^{(2-\sigma)p-\lambda(p-1)-1}d\phi$$

$$\le Cn^{\sigma p}\,||p_n||_p^p\,, \qquad \text{if we choose } 0<\lambda<(2-\sigma)p'.$$

On the other hand, using (1.6) and [5] formula (5.7), we have

$$I_2^p = \int_0^\pi |\int_{1/n}^\pi T_\phi p_n(\cos\theta)\ A g_{2-\sigma}(\cos\theta)\ \rho^{(\alpha,\beta)}(\phi)d\phi|^p \rho^{(\alpha,\beta)}(\theta)d\theta$$

$$\leq C \left(\int_{1/n}^{\pi} \phi^{-1-\frac{\mu(p-1)-1}{p}} d\phi\right)^{p/p'}$$

$$\int_0^{\pi}\left(\int_{1/n}^{\pi} |T_\phi p_n(\cos\theta)|^p |Ag_{2-\sigma}(\cos\phi)|^p \phi^{(2\alpha+2)p+\mu(p-1)-1} d\phi\right)\rho^{(\alpha,\beta)}(\theta)d\theta$$

$$\leq Cn^{\mu(p-1)} \int_{1/n}^{\pi} ||T_\phi p_n||_p^p \phi^{-\sigma p+\mu(p-1)-1} d\phi$$

$$\leq Cn^{\mu(p-1)} ||p_n||_p^p \int_{1/n}^{\infty} \phi^{-\sigma p+\mu(p-1)-1} d\phi$$

$$\leq Cn^{\sigma p} ||p_n||_p^p , \qquad \text{if we choose } 0 < \mu < \sigma p'.$$

Combination of the estimates proves the inequality (5.14).

5.18. The Cesàro summability process

The Nth Cesàro mean (C,λ) of the Fourier-Jacobi expansion of a function $f \in X$ is defined by the convolution of f with the polynomial kernel

$$(5.15) \qquad \sigma_N^\lambda(\cos\theta) = (a_N^\lambda)^{-1} \sum_{n=0}^{N} a_{N-n}^\lambda \, \omega_n^{(\alpha,\beta)} \, R_n^{(\alpha,\beta)}(\cos\theta),$$

$$(N \text{ integer } \geq 0, \ \lambda>0),$$

where

$$a_n^\lambda = \frac{\Gamma(n+\lambda+1)}{n!\,\Gamma(\lambda+1)} \qquad (\lambda>0).$$

The kernel σ_N^λ clearly satisfies the conditions a and c of definition 3.2. Szegö [25], theorem 9.41 has shown, that σ_N^λ is a quasi-positive kernel (condition b') if $\lambda > \alpha + \frac{1}{2}$ and that condition b' is not satisfied if $\lambda \leq \alpha + \frac{1}{2}$. In the rest of this paper we shall always assume $\lambda > \alpha + \frac{1}{2}$. It has been conjectured by Askey, that the kernel σ_N^λ is positive for $\lambda > \alpha + \beta + 2$, but this is still unproven.

We now derive a limit relation of the form (2.8) for the Cesàro summability process. We use the notation $\sigma_N^\lambda f(\cos\theta) = (\sigma_N^\lambda * f)(\cos\theta)$. We apply the identity (cf. Zygmund [27], p. 269)

$$\sigma_N^{\lambda+1} f(\cos\theta) - \sigma_{N-1}^{\lambda+1} f(\cos\theta) =$$

$$= \frac{\lambda+1}{N(N+\lambda+1)} (a_N^\lambda)^{-1} \sum_{n=0}^{N} a_{N-n}^\lambda n f^\wedge(n) \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta),$$

$$N = 0, 1, \ldots .$$

If we write $B = B_{-\lambda n}$ for the operator of the factor sequence type with the factors $\psi(n) = -\lambda n$ $(n=0,1,\ldots)$, then

$$\sigma_N^{\lambda+1} f - \sigma_{N-1}^{\lambda+1} f = -\frac{\lambda+1}{\lambda} \frac{1}{N(N+\lambda+1)} \sigma_N^\lambda Bf \quad (N=1,2,\ldots).$$

Thus,

$$(5.16) \qquad f - \sigma_N^{\lambda+1} f = \sum_{l=N+1}^{\infty} \sigma_l^{\lambda+1} f - \sigma_{l-1}^{\lambda+1} f$$

$$= -\frac{\lambda+1}{\lambda} \sum_{l=N+1}^{\infty} \frac{1}{l(l+\lambda+1)} \sigma_l^\lambda Bf.$$

The following identity (Zygmund [27], p. 269) enables us to go over from $\lambda+1$ to λ:

$$(5.17) \qquad \sigma_N^\lambda f = \sigma_N^{\lambda+1} f - \frac{1}{\lambda(N+\lambda+1)} \sigma_N^\lambda Bf \qquad (N=0,1,\ldots).$$

From (5.16) and (5.17) we deduce

$$f - \sigma_N^\lambda f = \frac{1}{\lambda(N+\lambda+1)} \sigma_N^\lambda Bf - \frac{\lambda+1}{\lambda} \sum_{l=N+1}^{\infty} \frac{1}{l(l+\lambda+1)} \sigma_l^\lambda Bf.$$

If we put $C_N = \sum_{l=N+1}^{\infty} \frac{1}{l(l+\lambda+1)}$, then for $f \in D(B)$

$$C_N^{-1}(\sigma_N^\lambda f-f) - Bf = \frac{\lambda+1}{\lambda} C_N^{-1} \sum_{l=N+1}^{\infty} (\sigma_l^\lambda Bf-Bf) \frac{1}{l(l+\lambda+1)}$$

$$- \frac{1}{\lambda} (\frac{1}{N+\lambda+1} C_N^{-1} \sigma_N^\lambda Bf-Bf).$$

Therefore, for $N \to \infty$,

$$||C_N^{-1}(\sigma_N^\lambda f-f)-Bf||_X \leq \frac{\lambda+1}{\lambda} \sup_{l\geq N+1} ||\sigma_l^\lambda Bf-Bf||_X + \frac{C_N^{-1}}{\lambda(N+\lambda+1)} ||\sigma_N^\lambda Bf-Bf||_X$$

$$+ \frac{1}{\lambda} |1- \frac{1}{C_N(N+\lambda+1)}| \; ||Bf||_X$$

$$= o(1), \qquad\qquad (f\in D(B))$$

by (3.2) and the fact that

$$1 - \frac{C_N^{-1}}{N+\lambda+1} \simeq 1 - \frac{1}{N+\lambda+1} (\int_{N+1}^{\infty} \frac{dl}{l(l+\lambda+1)}) = o(1) \quad (N\to\infty).$$

Since $\lim_{N\to\infty} C_N N = 1$, we have for $f \in D(B)$

$$\text{(5.18)} \qquad \lim_{N\to\infty} N(\sigma_N^\lambda f-f) = Bf \text{ in } X.$$

As σ_N^λ is a polynomial of degree N, it is obvious that $\sigma_N^\lambda f \in D(B)$ for all integers $N \geq 0$ and all $f \in X$. Also, by lemma 3.8, we know that there exists a measure $\mu \in M$ such that $Bf = D_1(\mu * f)$, where D_1 is the operator of the factor sequence type with factors $\psi(n) = [n(n+\alpha+\beta+1)]^{1/2}$. Application of (5.14) and the quasi-positivity of σ_N^λ leads to

$$\text{(5.19)} \qquad ||B\sigma_N f||_X \leq ||D_1\sigma_N^\lambda(\mu * f)||_X \leq CN||\sigma_N^\lambda||_1 \; ||\mu||_M \; ||f||_X = C_1N||f||_X.$$

Now all the conditions of the theorems 2.9 and 2.10 are satisfied. Hence we have

5.19. Theorem. The process $\{\sigma_N^\lambda, N\geq 0, \lambda>\alpha+\frac{1}{2}\}$ is saturated with the order N^{-1}. The saturation class is $H(X,[n(n+\alpha+\beta+1)]^{1/2})$. Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \leq q \leq \infty$ or $\theta = 1$, $q = \infty$:

i) $f \in (X,D(D_1))_{\theta,q;K}$,

ii) $f \in X_{\theta,q;\sigma^\lambda}$.

5.20. The de la Vallée-Poussin summability process

This summability method was introduced by de la Vallée-Pousin [26] for Fourier series and was generalized to ultraspherical series by Kogbetliantz [16]. The Nth de la Vallée-Poussin mean of the Fourier-Jacobi expansion of a function $f \in X$ is defined by the convolution of f with the kernel

$$V_N(\cos\theta) = \omega_0^{(\alpha,\beta+N)}(\cos\frac{\theta}{2})^{2N} = \frac{\Gamma(\alpha+\beta+N+2)}{\Gamma(\alpha+1)\Gamma(\beta+N+1)}(\cos\frac{\theta}{2})^{2N}$$

(N integer ≥ 0).

The kernel V_N clearly satisfies the conditions a, b and c' and therefore, by theorem 3.4, the sequence $\{V_N, N>0\}$ is an approximation process for the identity I. The Fourier-Jacobi coefficients of $V_N(\cos\theta)$ can be computed by means of Rodrigues' formula (Szegö [25], formula (4.3.1)). We obtain

$$(5.21)\qquad V_N(\cos\theta) = \sum_{n=0}^{N} \frac{\Gamma(N+1)\Gamma(N+\alpha+\beta+2)}{\Gamma(N-n+1)\Gamma(N+n+\alpha+\beta+2)}\,\omega_n^{(\alpha,\beta)}\,R_n^{(\alpha,\beta)}(\cos\theta).$$

A direct calculation, based in comparison of the Fourier-Jacobi coefficients,leads to the following identity, which generalizes an identity due to Butzer and Pawelke [12].

$$(5.22)\qquad N(N+\alpha+\beta+1)\,[V_N(\cos\theta) - V_{N-1}(\cos\theta)] = -\,AV_N(\cos\theta)\ (N=1,2,\dots).$$

A here denotes the operator defined in section 4.2. Hence, if we put $V_N f(\cos\theta) = (V_N * f)(\cos\theta)$, then

$$f - V_N f = - \sum_{l=N+1}^{\infty} \frac{AV_l f}{l(l+\alpha+\beta+1)} .$$

If we write $C_N = \sum_{l=N+1}^{\infty} \frac{1}{l(l+\alpha+\beta+1)}$, we obtain for $f \in D(A)$

$$||C_N^{-1}(V_N f - f) - Af||_X \leq \sup_{l \geq N+1} ||V_l Af - Af||_X = o(1) \quad (N\to\infty).$$

Since $\lim_{N\to\infty} NC_N = 1$, we have the following limit relation

(5.23) $$\lim_{N\to\infty} N(V_N f - f) = Af \text{ in } X.$$

As $V_N f$ is a polynomial of degree N, it is obvious that $V_N f \in D(A)$ for all integers $N \geq 0$. For the de la Vallée-Poussin kernel, we have an inequality of the Bernstein-type, which is much stronger than (4.21).

$$||AV_N f||_X = ||\int_0^\pi T_\phi f(\cos\theta) AV_N(\cos\phi)\rho^{(\alpha,\beta)}(\phi)d\phi||_X$$

$$= N(N+\alpha+\beta+1)||\int_0^\pi (V_{N-1}(\cos\phi) - V_N(\cos\phi))T_\phi f(\cos\theta)\rho^{(\alpha,\beta)}(\phi)d\phi||_X$$

$$\leq N(N+\alpha+\beta+1)||f||_X \int_0^\pi |V_{n-1}(\cos\phi) - V_N(\cos\phi)|\rho^{(\alpha,\beta)}(\phi)d\phi$$

$$= N(N+\alpha+\beta+1)||f||_X \frac{\Gamma(N+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(N+\beta+1)}$$

$$\int_0^\pi (\cos\frac{\phi}{2})^{2N-2} |(N+\beta) - (N+\alpha+\beta+1)\cos^2\frac{\phi}{2}| \rho^{(\alpha,\beta)}(\phi)d\phi$$

$$\frac{N\Gamma(N+\alpha+\beta+2)}{\Gamma(N+\beta)\Gamma(\alpha+1)} ||f||_X \{\int_0^\pi (\sin\frac{\phi}{2})^{2\alpha+3}(\cos\frac{\phi}{2})^{2N+2\beta-1}d\phi$$

$$+ \int_0^\pi \frac{(\alpha+1)}{(N+\beta)} (\sin\frac{\phi}{2})^{2\alpha+1}(\cos\frac{\phi}{2})^{2N+2\beta+1}d\phi\}$$

$$= 2(\alpha+1)N \ ||f||_X.$$

Now all the conditions of the theorems 2.9 and 2.10 are satisfied. We conclude

5.21. Theorem. The process $\{V_N, N\geq 0\}$ is saturated with the order N^{-1}. The saturation class is $H(X,n(n+\alpha+\beta+1))$. Moreover, the following statements are equivalent for $0 < \theta < 1$, $1 \leq q \leq \infty$ or $\theta = 1$, $q = \infty$:

i) $f \in (X,D(A))_{\theta,q;K}$,

ii) $f \in X_{\theta,q;V}$.

The approximation processes treated in this section have been studied extensively in the case of Fourier series. An excellent reference is the book of Butzer-Nessel [11]. For special harmonic expansions, the saturation behaviour of most of these approximation processes has been dealt with in Berens-Butzer-Pawelke [7].

6. Characterization of some classes of functions

This last part is devoted to the charaterization of some classes of functions which occur in the preceding sections. In 6.1 we give necessary and sufficient conditions for a function f on $[0,\pi]$ to belong to the domain of the operator A, the operator of the factor sequence type with factors $n(n+\alpha+\beta+1)$, $n = 0, 1, \ldots$. Next we characterize the domain of the operator D_1, the fractional differentiation operator of order 1, by means of the conjugate function $\tilde{f}$, which can be introduced in a way similar to the work of Muckenhoupt and Stein [19] on ultraspherical expansions. The method to obtain the characterizations for D(A) and $D(D_1)$ is taken from Berens-Butzer-Pawelke [7]. In the last section we show that the intermediate spaces $(X,D(D_\gamma))_{\theta,q,K}$, $0 < \gamma < 2$, $0 < \theta < 1$, $1 \le q \le \infty$ coincide with the intermediate spaces $(X,D(A))_{\theta\gamma/2,q;K}$, which in the case $q = \infty$ are Lipschitz spaces.

We first give a characterization of D(A).

6.1. Theorem. For f, g $\in$ X the relation

$$(6.1) \qquad n(n+\alpha+\beta+1)\, f^{\wedge}(n) = g^{\wedge}(n) \qquad (n=0,1,\ldots)$$

is valid, if and only if $f(\cos\theta)$ is locally absolutely continuous on $(0,\pi)$, the function $\rho^{(\alpha,\beta)}(\theta)\,\frac{df}{d\theta}$ is absolutely continuous on $[0,\pi]$ and vanishes in the points 0 and π, and

$$(6.2) \qquad -\frac{d}{d\theta}\,[\rho^{(\alpha,\beta)}(\theta)\,\frac{d}{d\theta}\,f(\cos\theta)] = \rho^{(\alpha,\beta)}(\theta)\,g(\cos\theta).$$

Proof. If we presuppose (6.1), then the differential operator $P(\frac{d}{d\theta})$, defined in (4.2), works on the Weierstrass approximation process (section 5.8) in the following way

$$P(\frac{d}{d\theta})\,W_t f(\cos\theta) = W_t g(\cos\theta).$$

If we integrate twice, we obtain

$$W_t f(\cos\theta) = W_t f(\cos\varepsilon) - \int_\varepsilon^\theta \{\rho^{(\alpha,\beta)}(\eta)\}^{-1}\, d\eta \int_0^\eta \rho^{(\alpha,\beta)}(\tau)\, W_t g(\cos\tau)d\tau.$$

From the norm convergence of $W_t f$ to f for $t \to 0^+$ we conclude, that there exists a subsequence t_i, such that $W_t f(\cos\theta)$ converges to $f(\cos\theta)$ almost everywhere, for $t_i \to 0^+$. Hence

$$f(\cos\theta) = f(\cos\varepsilon) - \int_\varepsilon^\theta \{\rho^{(\alpha,\beta)}(\eta)\}^{-1}\, d\eta \int_0^\eta g(\cos\tau)\rho^{(\alpha,\beta)}(\tau)d\tau,$$

where the right-hand side converges for every $\varepsilon > 0$ and for all $\theta \in (0,\pi)$. Thus f is absolutely continuous on $(0,\pi)$ and

$$\frac{d}{d\theta} f(\cos\theta) = \{\rho^{(\alpha,\beta)}(\theta)\}^{-1} \int_0^\theta g(\cos\tau)\rho^{(\alpha,\beta)}(\tau)d\tau.$$

It follows that the function $\rho^{(\alpha,\beta)}(\theta)\,\frac{d}{d\theta} f(\cos\theta)$ is absolutely continuous on $[0,\pi]$, vanishes at $\theta = 0$ and by the hypothesis $g^\wedge(0) = 0$ also at $\theta = \pi$. Moreover,

$$P(\tfrac{d}{d\theta})\, f(\cos\theta) = g(\cos\theta)$$

almost everywhere (the assertions are valid everywhere if $X = C$), which proves (6.2).

For the converse we presuppose (6.2). Then

$$g^\wedge(n) = -\int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\,\frac{d}{d\theta}\,\{\rho^{(\alpha,\beta)}(\theta)\,\frac{d}{d\theta}\, f(\cos\theta)\}\, d\theta.$$

Integration by parts twice yields

$$\begin{aligned} g^\wedge(n) &= -\int_0^\pi \frac{d}{d\theta}\,[\rho^{(\alpha,\beta)}(\theta)\,\frac{d}{d\theta}\, R_n^{(\alpha,\beta)}(\cos\theta)]\, f(\cos\theta)d\theta \\ &= \int_0^\pi n(n+\alpha+\beta+1)\, R_n^{(\alpha,\beta)}(\cos\theta)\, f(\cos\theta)\, \rho^{(\alpha,\beta)}(\theta)d\theta \\ &= n(n+\alpha+\beta+1)\, f^\wedge(n). \end{aligned}$$

We used the fact that $\rho^{(\alpha,\beta)}(\theta)\,\frac{df}{d\theta}$ vanishes at $\theta = 0$ and $\theta = \pi$.

6.2. The conjugate function

It has been shown by Askey [2], that some parts of the work of Muckenhoupt and Stein [19] on the conjugate expansions of ultraspherical expansions can be generalized to Fourier-Jacobi series. If we take $f \in X$ with the expansion

$$f(\cos\theta) \sim \sum_{n=0}^{\infty} f^{\wedge}(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta)$$

and the Abel-Poisson sum (section 5.12)

$$(6.3) \qquad A_r f(\cos\theta) = \sum_{n=0}^{\infty} r^n f^{\wedge}(n)\omega_n^{(\alpha,\beta)}R_n^{(\alpha,\beta)}(\cos\theta),$$

then the conjugate Abel-Poisson sum $\tilde{A}_r f(\cos\theta)$ can be defined by

$$(6.4) \qquad \tilde{A}_r f(\cos\theta) = \frac{1}{\alpha+1}\sum_{n=1}^{\infty} r^n n f^{\wedge}(n)\omega_n^{(\alpha,\beta)}R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

If we put $u = A_r f(\cos\theta)$ and $v = r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)\,\tilde{A}_r f(\cos\theta)$, then u and v satisfy the generalized Cauchy-Riemann equations

$$(6.5.a) \qquad r v_r = - r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta)u_\theta,$$

$$(6.5.b) \qquad v_\theta = r^{\alpha+\beta+1}\rho^{(\alpha,\beta)}(\theta) r u_r.$$

By the method developed by Askey [2], theorem 1, the generalization of M. Riesz' theorem can be proved:

<u>6.3. Theorem</u>. If $f \in L^p$, $1 < p < \infty$, then we have:

a) $$||\tilde{A}_r f||_p \le M_p ||f||_p.$$

b) There exists a function $\tilde{f} \in L^p$ such that

$$\lim_{r\to 1^-} ||\tilde{A}_r f - \tilde{f}||_p = 0$$

and

$$||\tilde{f}||_p \leq M_p ||f||_p.$$

c) $$\lim_{r\to 1^-} \tilde{A}_r f(\cos\theta) = \tilde{f}(\cos\theta), \quad \text{for almost all } \theta,\ 0 \leq \theta \leq \pi.$$

6.4. Theorem. For f, g $\in$ X the relation

$$\text{(6.6)} \qquad n f^{\wedge}(n) = g^{\wedge}(n) \qquad\qquad (n=0,1,\ldots)$$

is valid if and only of $\tilde{f} \in X$, the function $\rho^{(\alpha,\beta)}(\theta)\ \tilde{f}\ (\cos\theta)$ is absolutely continuous on $[0,\pi]$ and vanishes at 0 and π, and the relation

$$\text{(6.7)} \qquad \frac{d}{d\theta}\,[\rho^{(\alpha,\beta)}(\theta) f(\cos\theta)] = \rho^{(\alpha,\beta)}(\theta) g(\cos\theta)$$

holds.

Proof. We first assume that (6.6) is satisfied for f and g $\in$ X. As we have mentioned in section 6.2, the functions $u = A_r f(\cos\theta)$ and $v = r^{\alpha+\beta+1}\ \rho^{(\alpha,\beta)}(\theta)\ \tilde{A}_r f(\cos\theta)$ satisfy the equations (6.5). From (6.5.b) we deduce, using (6.3) and (6.6),

$$\text{(6.8)} \qquad \frac{d}{d\theta}\,\{r^{\alpha+\beta+1}\ \rho^{(\alpha,\beta)}(\theta)\ \tilde{A}_r f(\cos\theta)\} = r^{\alpha+\beta+1}\ \rho^{(\alpha,\beta)}(\theta)\ A_r g(\cos\theta).$$

It follows from (6.6) that f has a representation of the form

$$\text{(6.9)} \qquad f(\cos\theta) = f^{\wedge}(0) + (g*h)\cos\theta,$$

where

$$h(\cos\theta) = \sum_{n=1}^{\infty} n^{-1}\ \omega_n^{(\alpha,\beta)} R_n^{(\alpha,\beta)}(\cos\theta).$$

In [5], section 4 we have shown that h is a continuous function in each compact subinterval of $(0,\pi]$ and that

$$h(\cos\theta) = O(\theta^{-2\alpha-1}), \qquad \theta \to 0^+.$$

Thus, for $1 < p(\alpha) < 1 + 1/(2\alpha+1)$ the function h belongs to $L^{p(\alpha)}$. By lemma 1.2.iv we may conclude from (6.9) that f belongs to $L^{p(\alpha)}$. Hence, by theorem 6.3 we have $\tilde{f} \in L^{p(\alpha)}$ and also $\tilde{f} \in L^1$. Integration of (6.8) yields

$$(6.10) \qquad \rho^{(\alpha,\beta)}(\theta)\tilde{A}_r f(\cos\theta) = \int_0^\theta A_r g(\cos\tau)\rho^{(\alpha,\beta)}(\tau)d\tau.$$

For $r \to 1^-$, the left-hand side converges almost everywhere to $\rho^{(\alpha,\beta)}(\theta)\tilde{f}(\cos\theta)$ by theorem 6.3.c. Since

$$\left|\int_0^\theta [A_r g(\cos\tau)-g(\cos\tau)]\ \rho^{(\alpha,\beta)}(\tau)d\tau\right| \le ||A_r g-g||_1 = o(1) \qquad (r\to1^-),$$

the right-hand side of (6.10) converges almost everywhere to $\int_0^\theta g(\cos\tau)\rho^{(\alpha,\beta)}(\tau)d\tau$, which implies that the following relation holds almost everywhere

$$(6.11) \qquad \rho^{(\alpha,\beta)}(\theta)\ \tilde{f}(\cos\theta) = \int_0^\theta g(\cos\tau)\rho^{(\alpha,\beta)}(\tau)d\tau.$$

At $\theta = 0$ the right-hand side vanishes and also at $\theta = \pi$, since $g^\wedge(0) = 0$. Moreover, it follows that

$$\frac{d}{d\theta}\{\rho^{(\alpha,\beta)}(\theta)\tilde{f}(\cos\theta)\} = g(\cos\theta)\rho^{(\alpha,\beta)}(\theta),$$

which establishes the first part of the theorem.

For the proof of the converse we deduce from (6.7)

$$g^\wedge(n) = \int_0^\pi g(\cos\theta)\ R_n^{(\alpha,\beta)}(\cos\theta)\rho^{(\alpha,\beta)}(\theta)d\theta$$

$$= \int_0^\pi R_n^{(\alpha,\beta)}(\cos\theta)\ \frac{d}{d\theta}\ [\rho^{(\alpha,\beta)}(\theta)\tilde{f}(\cos\theta)]d\theta.$$

If we take into account that $\rho^{(\alpha,\beta)}(\theta)\tilde{f}(\cos\theta)$ vanishes at $\theta = 0$ and $\theta = \pi$ and that the relation

$$(6.12)\quad \frac{d}{d\theta} R_n^{(\alpha,\beta)}(\cos\theta) = -\frac{n(n+\alpha+\beta+1)}{\alpha+1} R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

is valid, we obtain after integration by parts

$$(6.13)\qquad g^{\wedge}(n) = \frac{n(n+\alpha+\beta+1)}{\alpha+1}\int_0^{\pi} \tilde{f}(\cos\theta)\, R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\rho^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(\theta)d\theta.$$

Since $\rho^{(\alpha,\beta)}(\theta)\tilde{f}(\cos\theta)$ is absolutely continuous, relation (6.11) holds. Thus we may conclude

$$|\tilde{f}(\cos\theta)| \leq ||g||_1 \{\rho^{(\alpha,\beta)}(\theta)\}^{-1},$$

which implies that $\tilde{f} \in L^{p(\alpha)}$ for $1 < p(\alpha) < 1 + \frac{1}{2\alpha+1}$. Hence,

$$\int_0^{\pi} [\tilde{f}(\cos\theta) - \tilde{A}_r f(\cos\theta)]\, R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\rho^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(\theta)d\theta$$

$$\leq C_n ||\tilde{A}_r f - \tilde{f}||_{p(\alpha)} = o(1) \qquad (r\to 1^-).$$

We now investigate

$$(6.14)\qquad \frac{n(n+\alpha+\beta+1)}{\alpha+1}\int_0^{\pi} \tilde{A}_r f(\cos\theta)\, R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\rho^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(\theta)d\theta.$$

If we substitute for $\tilde{A}_r f(\cos\theta)$ the expansion (6.4) and integrate term by term, noticing that the polynomials $R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)$ are orthogonal with respect to $\rho^{(\alpha+1,\beta+1)}(\theta)$, we obtain

$$g^{\wedge}(n) = \lim_{r\to 1^-} \frac{n(n+\alpha+\beta+1)}{\alpha+1}\int_0^{\pi} \tilde{A}_r f(\cos\theta) R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\rho^{(\alpha+\frac{1}{2},\beta+\frac{1}{2})}(\theta)d\theta$$

$$= \lim_{r\to 1^-} \frac{n(n+\alpha+\beta+1)}{(\alpha+1)^2}\sum_{k=1}^{\infty} k f^{\wedge}(k) r^k \omega_k^{(\alpha,\beta)}$$

$$\int_0^{\pi} R_{k-1}^{(\alpha+1,\beta+1)}(\cos\theta) R_{n-1}^{(\alpha+1,\beta+1)}(\cos\theta)\rho^{(\alpha+1,\beta+1)}(\theta)d\theta$$

$$= \lim_{r \to 1^-} \frac{n^2(n+\alpha+\beta+1)}{(\alpha+1)^2} r^n f^\wedge(n)\omega_n^{(\alpha,\beta)} [\omega_{n-1}^{(\alpha+1,\beta+1)}]^{-1}$$

$$= n\ f^\wedge(n),$$

which completes the proof of theorem 6.4.

As a consequence of corollary 3.9, theorem 6.4 gives a characterization of $D(D_1)$, the domain of the fractional differential operator of order 1.

6.5. Characterization of the spaces $(X,D(D_\gamma))_{\theta,q;K}$

In this section we want to show that the spaces $(X,D(D_\gamma))_{\theta,q;K}$ ($0<\gamma<2$, $0<\theta<1$, $1\leq q\leq\infty$), occurring in section 5, coincide with the spaces $(X,D(A))_{\theta\gamma/2,q;K}$. This is a direct consequence of the reiteration property (section 2.1, property d), if we are able to show that

$$(6.15) \qquad (X,D(A))_{\gamma/2,1;K} \subset D(D_\gamma) \subset (X,D(A))_{\gamma/2,\infty;K}\ .$$

We first prove the second inclusion. If $f \in D(D_\gamma)$ there esists a function $g \in X$, such that $f = I_\gamma g$, where I_γ denotes the fractional integration operator of order γ, introduced in Bavinck [5], section 5. We have shown there (theorem 5.2), that $f = I_\gamma g \in Lip(\gamma,X)$. Hence, by section 4.8, the second inclusion follows.

For the proof of the first inclusion we need some theorems on spaces of best approximation quoted in section 2.2. For the subspaces P_n are chosen the spaces of the polynomials in $\cos\theta$ of degree $\leq n$. As a consequence of the inequality (5.14) and definition 2.3, we know that $D(D_\gamma)$ is a space of the class $D_\gamma^J(X)$ and thus, by lemma 2.4, we have for the space of best approximation $X_{\gamma,1}^K$ the inclusion

$$(6.16) \qquad X_{\gamma,1}^K \subset D(D_\gamma).$$

Furthermore, the space $D(A)$ is a space of the class $D_2(X)$. The fact that $D(A)$ is a space of the class $D_2^K(X)$ follows from definition 2.3 and the formulas (4.7) and (4.17). The space $D(A)$ belongs to the class

$D_2^J(X)$ by the inequality (4.21). We are now in a position to apply theorem 2.5 to conclude that

$$(6.17) \qquad X^K_{\gamma,1} \cong (X,D(A))_{\gamma/2,1;K} .$$

Combination of (6.16) and (6.17) leads to the first inclusion of (6.15).

We have proved

6.6. Theorem. For $0 < \gamma < 2$ and $0 < \theta < 1$, $1 \leq q \leq \infty$ the following statements are equivalent for $f \in X$:

i) $\quad f \in (X,D(D_\gamma))_{\theta,q;K}$,

ii) $\quad f \in (X,D(A))_{\gamma\theta/2;q;K}$.

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