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A NOTE ON THE VAN WIJNGAARDEN TRANSFORMATION

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A note on the van Wijngaarden transformation

H.A. Lauwerier

Introduction

In a paper on a transformation of formal series van Wijngaarden (2) suggested the following method for summing a slowly convergent or even divergent series Σa_k *). Introduce a sequence of non-vanishing multipliers λ_k and transform the series $\Sigma\lambda_k a_k$ by means of the Euler method. If the transformed series is denoted by Σb_k there exists a sequence of conjugate multipliers μ_k such that $\Sigma\mu_k b_k$ has the same (generalized) sum as Σa_k. In particular van Wijngaarden takes the special case $\lambda_k = \lambda^k/k!$ and finds that

$$\lambda_k = 2^{k+1}\lambda \int_0^\infty e^{-\lambda t}(1+t)^{-k-1}t^k dt.$$

In a number of interesting cases the new series turns out to be rapidly convergent which makes this process particularly well adapted to numerical computations.

In view of recent interest the problem is taken up again and considered from a different point of view.

The first section deals with the main properties of the Euler transformation of a formal series Σa_k. The analysis becomes very transparent by using the generating functions $\Sigma a_k x^k$ and $\Sigma a_k x^k/k!$.

The second section shows the importance of the Borel summation and the concept of the Borel sum

$$\int_0^\infty e^{-x}(\Sigma a_k x^k/k!)dx.$$

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*) In all summations denoted by Σ the index runs from 0 to infinity unless stated otherwise.

For convergent series the Borel sum equals the ordinary sum. For a wide class of divergent series the Borel sum exists and can be taken as the generalized sum. The importance of the Borel sum is that it is invariant for an Euler transformation.
In the third section it is shown that the use of generating functions permits a rather simple treatment of the van Wijngaarden transformation.
It is shown that also this transformation leaves the Borel sum invariant.
The properties of the multipliers μ_k of the special van Wijngaarden transformation on given above are the subject of a forthcoming paper by N.M. Temme who concentrates in particular to their numerical computation.

In many applications one wishes to compute a Laplace integral

$$\omega \int_0^\infty e^{-\omega x} F(x)dx.$$

by termwise integration of some series expansion of the integrand function F(x). Sometimes the resulting series is slowly convergent or even divergent. One might have the idea to subject this series to an Euler transformation or a van Wijngaarden transformation. However, the same result can be obtained in a much more direct and simpler way. It suffices to expand $F(x) = \Sigma a_k x^k$ as suggested by the generating function of Σa_k when subjected to an Euler transformation. We write

$$F(x) = 2^{k+1} \Sigma b_k \frac{x^k}{(1+x)^{k+1}} ,$$

where Σb_k is the Eulerized series of Σa_k.
Termwise integration of the latter expansion gives at once $\Sigma\mu_k a_k$.
This result may be formulated by saying that the special van Wijngaarden transformation is the Laplace transformation of the Euler transformation.

1. The Euler transformation

We consider a formal series Σa_k and introduce the forward shift operator S and the weighted mean operator M by means of

$$Sa_k = a_{k+1}$$

$$M = p + qS \quad \text{with } |p| < 1 \quad \text{and} \quad q = 1-p.$$

Then we have formally

$$\Sigma a_k = \Sigma S^k a_0 = \frac{a_0}{1-S} = \frac{qa_0}{1-M} = q\Sigma M^k a_0.$$

This suggests the so-called Euler transformation E(q)

$$b_k = qM^k a_0$$

or explicitly

$$(1.1) \qquad b_k = \sum_{j=0}^{k} \binom{k}{j} p^{k-j} q^{j+1} a_j.$$

In numerical practise one uses the Euler method preferably with $p = q = \frac{1}{2}$. According to the folklore of the numerical analyst the Euler transformation turns slowly convergent series into rapidly convergent series and transforms divergent series into less divergent or even convergent series.

In order to get a better insight into what is really going on we consider the generating functions

$$(1.2) \qquad a(z) = \Sigma a_k z^{k+1}, \quad b(z) = \Sigma b_k z^{k+1}.$$

We restrict our discussion preliminary to those series for which a(z) has a non-vanishing radius of convergence R_a. This enables us to handle divergent series such as 1-2+3-4+... but a series like 1!-2!+3!-4!+... falls outside this class.
It is easily seen by comparing equal powers of z that the relation (1.1)

is equivalent with

$$(1.3)\qquad b(z) = a\left(\frac{qz}{1-pz}\right).$$

The radius of convergence of a(w) where

$$(1.4)\qquad w = \frac{qz}{1-pz}\,, \qquad z = \frac{w}{q+pw}$$

is determined by the singularities $w = s$ of the holomorphic function a(w) as

$$(1.5)\qquad R_a = \inf |s|.$$

Then the radius of convergence of b(z) is given by

$$(1.6)\qquad R_b = \inf \left|\frac{s}{q+ps}\right|.$$

The Euler method is most effective if R_b/R_a is as large as possible.

<u>Example 1.1</u>

If a(w) is singular at $w = -1$ and $w = \infty$ then b(z) is singular at $z = (p-q)^{-1}$ and $z = p^{-1}$.
The ordinary method with $p = q = \frac{1}{2}$ gives $R_a = 1$ and $R_b = 2$. However, the method with $p = 1/3$ gives even $R_b = 3$. If this is applied to e.g. $a(w) = w(1+w)^{-\frac{1}{2}}$ we find indeed $b(z) = 2z(9-z^2)^{-\frac{1}{2}}$.

<u>Example 1.2</u>

The ordinary Euler transformation changes the divergent series 1-2+3-4+... into the convergent series $\frac{1}{2}-\frac{1}{4}+0+0+\ldots$ of which the first two terms only differ from zero. This rather surprising phenomenon is explained by the fact that $a(z) = z(1+z)^{-2}$ is singular only at $z = -1$ which gives $R_b = \infty$. Indeed $b(z) = \frac{1}{2}z(1-\frac{1}{2}z)$.

If Σa_k converges with sum A then we know that $R_a \geq 1$ and that $a(1) = A$. It follows from (1.3) that also $b(1) = A$. Hence it is tempting to conclude that Σb_k converges with the same sum A. However, it is not a priori obvious that Σb_k is convergent. But if this series converges Abel's theorem says that its sum must be A.

When the singularities of $a(z)$ are distributed in such a way that $R_b > 1$ there is no problem. However, when also $R_b = 1$ some further analysis is needed.
A summation method which sums every convergent series to its ordinary sum is called <u>regular</u>.
Hardy *) gives necessary and sufficient conditions for the regularity of a wide class of summation methods. The regularity of the Euler transformation then follows by checking the conditions. We shall give here a direct proof.

<u>Theorem 1.1</u>

The Euler method E(q) is regular.

<u>Proof</u>

Let $A_n = \sum_{k=0}^{n-1} a_k$, $B_n = \sum_{k=0}^{n-1} b_k$, $A_n \to A$. Since (1.4) implies

$$\frac{z}{1-z} = \frac{w}{q(1-w)}$$

the relation (1.3) may be replaced by

$$\frac{z}{1-z}\, b(z) = \frac{1}{q}\, \frac{w}{1-w}\, a(w).$$

Expanding both sides into a power series we have

$$\Sigma B_k z^{k+1} = q^{-1} \Sigma A_j w^{j+1} = q^{-1} \Sigma A_j \left(\frac{qz}{1-pz}\right)^{j+1}.$$

*) Hardy, Divergent Series, Oxford 1949.

Taking the coefficient of z^{k+1} we obtain

$$(1.7) \qquad B_k = \sum_{j=1}^{k} \binom{k}{j} p^{k-j} q^j A_j .$$

If $A_j = A$ for all j then $B_k = (1-p^k)A$. Thus also $B_k \to A$ since $|p| < 1$. Henceforward we may assume $A = 0$. The relation (1.7) will be written as

$$B_k = \sum_{j=1}^{m} + \sum_{j=m+1}^{k} = U + V.$$

Given an $\varepsilon > 0$ we can choose $m = m(\varepsilon)$ so that $|A_j| < \varepsilon$ for $j > m$. Then

$$|V| \leq \varepsilon \sum_{j=0}^{k} \binom{k}{j} p^{k-j} q^j = \varepsilon.$$

For fixed m we have $U \to 0$ as $k \to \infty$ since each term of U tends to zero. Hence for k sufficiently large $|U| < \varepsilon$ so that $|B_k| < 2\varepsilon$. This means $B_k \to 0$ which proves the theorem.

2. Borel summation

In an alternative way the Euler method may be discussed by considering the generating functions

$$(2.1) \qquad \alpha(z) = \Sigma a_k z^k/k!, \qquad \beta(z) = \Sigma b_k z^k/k!.$$

The relation (1.1) is easily seen to be equivalent with the functional equation

$$(2.2) \qquad e^{-z}\beta(z) = qe^{-qz}\alpha(qz).$$

It clearly suffices to consider the coefficient of z^k in the expansions of $q\alpha(qz) \exp pz$.

From the relation (2.2) we obtain the following interesting result.

Theorem 2.1

The Euler methods $E(q)$ form a commutative semigroup with

$$E(q_1)E(q_2) = E(q_1 q_2).$$

We shall now extend the discussion of the Euler method to those series Σa_k for which $\alpha(z)$ is holomorphic in a domain which contains the positive real axis. If

$$(2.3) \qquad A = \int_0^\infty e^{-x}\alpha(x)dx$$

exists the series is said to be Borel summable with A as its Borel sum. From (2.2) en (2.3) it follows at once

Theorem 2.2

The Euler transformation $E(q)$ with q real and positive does not change the Borel sum.

Further we have the following property which is proved in Hardy (l.c. 8.5) in a more general context.

Theorem 2.3

The Borel method (2.3) is regular.

Proof

We put for $k = 0,1,2,\ldots$

$$\phi_k(x) = \frac{1}{k!}\int_x^\infty e^{-t}t^k dt = e^{-x}\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^k}{k!}\right)$$

and

$$\psi_k(x) = e^{-x}x^k/k!.$$

If $\Sigma a_k = A$ and $\sum_{j=k}^{\infty} a_k = A_k$ then

$$\int_0^x e^{-t}\alpha(t)dt = \Sigma \frac{a_k}{k!} \int_0^x e^{-t}t^k dt = \Sigma a_k(1-\phi_k) =$$

$$= A - \Sigma a_k\phi_k = A - \Sigma(A_k - A_{k+1})\phi_k = A - \Sigma A_k\psi_k.$$

Hence there remains to prove that

$$\lim_{x\to\infty} e^{-x}\Sigma A_k x^k/k! = 0$$

but this is elementary matter.

In many applications a series which has to be summed is derived from some integral expression. We consider in particular the integral

$$(2.4) \qquad \int_0^\infty e^{-x}F(x)dx,$$

where $F(x)$ is holomorphic in $x = 0$, say $F(x) = \Sigma a_k x^k/k!$. One may have the idea of applying an Euler transformation upon the series Σa_k which is obtained by termwise integration of the power series expansion of $F(x)$. However, the preceding analysis shows that the final result could be obtained in a shorter way be writing the integral in the form

$$\int_0^\infty e^{-\mu x}\{e^{-(1-\mu)x}F(x)\}dx$$

and termwise integration of the power series expansion of $F(x)\exp-(1-\mu)x$. Comparing this expression with (2.2) it appears that implicitly an Euler transformation with $q = 1/\mu$ has been applied.

3. The van Wijngaarden method

In his paper (2) van Wijngaarden advocates the following method of summing Σa_k. We introduce non-vanishing multipliers λ_k and subject $\Sigma\lambda_k a_k$ to an Euler transformation. Let the resulting series be Σb_k then there exist conjugate multipliers μ_k such that $\Sigma\mu_k b_k$ has the same sum as Σa_k.
The formal analysis is very simple. According to (1.3) and (1.4) we have

$$(3.1) \qquad \Sigma b_k \left(\frac{t}{q+pt}\right)^{k+1} = \Sigma\lambda_k a_k t^{k+1}.$$

We suppose that there exists a moment generating function $\phi(t)$ such that

$$(3.2) \qquad \lambda_k^{-1} = \int_0^\infty \phi(t) t^k dt.$$

Then formal integration of (3.1) after multiplication by $\phi(t)/t$ gives

$$(3.3) \qquad \Sigma\mu_k b_k = \Sigma a_k$$

when

$$(3.4) \qquad \mu_k = \int_0^\infty \frac{t^k}{(q+pt)^{k+1}} \phi(t) dt.$$

Van Wijngaarden considers in particular the multiplier set $\lambda_k = s^k/k!$. Then $\phi(t) = s \exp - st$ so that

$$(3.5) \qquad \mu_k = \frac{s}{p^{k+1}} \int_0^\infty \frac{t^k}{(1+t)^{k+1}} \exp - \frac{qst}{p} dt.$$

This expansion shows that nothing is lost by restricting ourselves to the ordinary Euler method $p = q = \frac{1}{2}$.
If this method is applied to the power series

$$(3.6) \qquad \Sigma a_k \omega^{-k}$$

with ω either real or complex with Re $\omega > 0$ we take $\lambda_k = \omega^k/k!$ and compute

(3.7) $$\Sigma 2^{-k-1} s_k(\omega) b_k$$

where

(3.8) $$s_k(\omega) = \omega \int_0^\infty e^{-\omega t} \frac{t^k}{(1+t)^{k+1}}\, dt.$$

The functions $s_k(\omega)$ are studied from a computational point of view in a forthcoming publication (1) by N.M. Temme.

Theorem 3.1

The functions $s_k(\omega)$ have for $k \to \infty$, $\omega/k \ll 1$ the following asymptotic behaviour

(3.9) $$s_k(\omega) \sim \pi^{1/2} k^{-1/4} \omega^{3/4} \exp(\tfrac{1}{2}\omega - 2\sqrt{k\omega}).$$

Proof

Writing

$$s_k(w) = \int_0^\infty e^{-kf(t)} \omega (1+t)^{-1} dt$$

with $f(t) = \log \frac{1+t}{t} + \frac{\omega}{k} t$ we apply the saddle point method. The positive real axis is the line of steepest descent with a saddle point t_0 determined by $c^2 t(1+t) = 1$ where $c^2 = \omega/k$, $c > 0$. Explicitly

$$t_0 = -\frac{1}{2} + \left(\frac{1}{c^2} + \frac{1}{4}\right)^{\frac{1}{2}} = c^{-1} - \frac{1}{2} + O(c).$$

This gives

$$f(t_0) = 2c - \tfrac{1}{2}c^2 + O(c^3)$$

and

$$f''(t_0) = 2c^3 + O(c^4).$$

Using these expressions it follows that $s_k(w)$ is asymptotically equivalent to

$$s_k(\omega) \sim c\omega \exp - k(2c-\tfrac{1}{2}c^2) \int_{-\infty}^{\infty} e^{-kc^3u^2} du.$$

which can be written in the form stated above.

Theorem 3.2

The van Wijngaarden transformation with $\lambda_k = s^k/k!$ does not change the Borel sum.

Proof

The relation between the generating functions of Σa_k and $\Sigma \lambda_k a_k$ is given by

$$\alpha(x) = \int_0^{\infty} f(xt)\phi(t)dt$$

where

$$f(x) = \Sigma \frac{\lambda_k a_k x^k}{k!} .$$

According to (2.2) the generating function of Σb_k is given by

$$\beta(x) = qe^{px}f(qx).$$

Thus we have

$$\alpha(x) = \int_0^{\infty} e^{-ptx}\beta(xt)\phi(qt)dt.$$

On the other hand we have for the generating function

$$g(x) = \Sigma \frac{\mu_k b_k x^k}{k!}$$

the expression

$$g(x) = \int_0^\infty \frac{1}{q+pt} \Sigma \frac{b_k}{k!} (\frac{xt}{q+pt})^k \phi(t)dt$$

or

$$g(x) = \int_0^\infty \frac{1}{1+pt} \beta(\frac{xt}{1+pt}) \phi(qt)dt.$$

Substitution of the expressions derived above for $\alpha(x)$ and $g(x)$ into $\int_0^\infty e^{-x}\alpha(x)dx$ and $\int_0^\infty e^{-x}g(x)dx$ shows their equality. The formalities are certainly justified if Σa_k is such that

$$\Sigma \frac{a_k x^k}{k!k!} = O(x^m)$$

for $x \to \infty$ with some constant m. The treatment of the general problem is left to the reader as a research problem.

Example 3.1

The rapidly divergent series $\Sigma(-1)^k k!\omega^{-k}$ is treated by the multipliers $\lambda_k = \omega^k/k!$. This gives the formal series $\Sigma(-1)^k$. If the latter series is subjected to the ordinary Euler method $E(\frac{1}{2})$ we find

$$\tfrac{1}{2} + 0 + 0 + 0 + \ldots .$$

Accordingly the Borel sum of the given series equals simply $\frac{1}{2}\mu_0$ or

$$\omega \int_0^\infty \frac{e^{-\omega t}}{1+t} dt.$$

Indeed the Borel sum of the original series as derived from the generating series $\alpha(x) = \Sigma(-1)^k(x/\omega)^k = \omega(x+\omega)^{-1}$ agrees with the above given integral expression.

If one wishes to compute an integral of the type

(3.10) $$\int_0^\infty \phi(x)F(x)dx$$

where F(x) is holomorphic in x = 0, say $F(x) = \Sigma a_k x^k$, we may subject the series which is obtained by termwise integration to a van Wijngaarden transformation with multipliers (3.2). This means that to Σa_k an Euler transformation is applied. According to (1.3) results in the expansion of xF(x) in powers of $x(q+px)^{-1}$

(3.11) $$F(x) = \Sigma b_k \frac{x^k}{(q+px)^{k+1}} .$$

In view of (3.4) the effect of the van Wijngaarden transformation is merely the substitution of (3.11) into (3.10) followed by termwise integration. Thus

(3.12) $$\int_0^\infty \phi(x)F(x)dx = \Sigma \mu_k b_k .$$

Of particular interest is the case of a Laplace integral where (3.10) takes the form

(3.13) $$f(\omega) = \omega \int_0^\infty e^{-\omega x} F(x)dx.$$

The usual treatment of termwise Laplace transformation of the power series $F(x) = \Sigma a_k x^k$ gives the asymptotic expansion

(3.14) $$f(\omega) \sim \Sigma k! a_k \omega^{-k}.$$

If, however, F(x) is expanded as in (3.11) we obtain the expansion

(3.15) $$f(\omega) = \frac{1}{q} \Sigma b_k p^{-k} s_k(\omega q/p)$$

where Σb_k is the Euler transform of Σa_k and the s_k are given by (3.8). The same expansion would be obtained by applying the special van Wijngaarden transformation with $\lambda_k = \omega^k/k!$ upon the asymptotic expansion (3.14). This result may be phrased as follows

Theorem 3.3

The special van Wijngaarden transformation is the Laplace transformation of the Euler transformation.

The expansion (3.15) may be convergent even if (3.14) is divergent for all ω. If e.g. the generating function $\Sigma a_k x^k$ and $\Sigma b_k x^k$ both have a finite radius of convergence the asymptotic behaviour (3.9) shows that (3.15) converges for all ω. On the other hand it shares with (3.14) the asymptotic character for $\omega \to \infty$ since for fixed k

(3.16) $$s_k(\omega) \sim k!\omega^{-k}.$$

Example 3.2

We consider the Laplace integral

$$\sqrt{\pi\omega}\ e^{\omega} \operatorname{erfc} \omega = \omega \int_0^{\infty} e^{-\omega t}(1+t)^{-\frac{1}{2}}dt.$$

To the integrand function $(1+t)^{-\frac{1}{2}}$ we apply the Euler transformation E(2/3) since by this choice the radius of convergence of $\Sigma b_k x^k$ takes the optimal value of 3.
In fact $\Sigma b_k x^k = \frac{2}{3}(1-x^2/9)^{-\frac{1}{2}}$ so that finally

$$\sqrt{\pi\omega}\ e^{\omega} \operatorname{erfc} \omega = \Sigma \frac{(\frac{1}{2})k}{k!} s_{2k}(2\omega)$$

convergent for all ω.

Example 3.3

An interesting generalization is indicated in the following integral for the modified Bessel function

$$e^{\frac{1}{2}\omega}K_0(\tfrac{1}{2}\omega) = \int_0^\infty e^{-\omega t}t^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}dt.$$

Applying the same transformation as in the previous example

$$(1+t)^{-\frac{1}{2}} = \Sigma \frac{3^{k+1}b_k t^k}{(2+t)^{k+1}}$$

we obtain the expansion

$$e^{\frac{1}{2}\omega}K_0(\tfrac{1}{2}\omega) = 2^{-\frac{1}{2}}\Sigma b_k 3^{k+1} \int_0^\infty e^{-2\omega t} \frac{t^{k-\frac{1}{2}}}{(1+t)^{k+1}}\, dt.$$

Introducing the following variant of (3.8)

$$\sigma_k(\omega) = \omega \int_0^\infty e^{-\omega t} \frac{t^{k-\frac{1}{2}}}{(1+t)^{k+1}}\, dt$$

the final result may be written as

$$\omega\sqrt{2}\, e^{\frac{1}{2}\omega}K_0(\tfrac{1}{2}\omega) = \Sigma \frac{(\frac{1}{2})_k}{k!}\, \sigma_{2k}(2\omega).$$

The asymptotic behaviour of $\sigma_k(\omega)$ for $k \to \infty$ can be obtained in the same way as for $s_k(\omega)$. The obvious result is

$$\sigma_k(\omega) \sim \pi^{1/2}k^{-3/4}\omega^{5/4}\, \exp(\tfrac{1}{2}\omega-2\sqrt{k\omega}).$$

This shows that also in this case a convergent expansion is obtained.

Literature

1. N.M. Temme, Numerical evaluation of functions arising from transformations of formal series, (1972), Report TW 134/72, Mathematical Centre, Amsterdam.

2. A. van Wijngaarden, A transformation of formal series. Ind. Math., 15, 522-543 (1953).