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ORTHOGONAL POLYNOMIALS IN TWO VARIABLES WHICH ARE EIGENFUNCTIONS OF TWO INDEPENDENT PARTIAL DIFFERENTIAL OPERATORS, II

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 33A30, 33A65, 33A75, 35Q99, 43A85, 43A90

Contents

Orthogonal polynomials in two variables which are eigenfunctions of two independent differential operators, II *)

by

T.H. Koornwinder

Abstract

Let the region $S = \{(x,y) \mid \mu(x+iy,x-iy) > 0\}$ be the interior of Steiner's hypocycloid, where $\mu(z,\bar{z}) = -z^2\bar{z}^2 + 4z^3 + 4\bar{z}^3 - 18z\bar{z} + 27$. For each real $\alpha > -5/6$ an orthogonal system of polynomial $p^{\alpha}_{m,n}(z,\bar{z})$, $m, n \geq 0$, can be defined on this region S such that $p^{\alpha}_{m,n}(z,\bar{z}) - z^m\bar{z}^n$ has degree less than $m + n$ and

$$\iint_S p^{\alpha}_{m,n}(z,\bar{z})\,\overline{q(z,\bar{z})}(\mu(z,\bar{z}))^{\alpha}dx\,dy = 0$$

for each polynomial q of degree less than m + n. If $z = e^{i(s+t/\sqrt{3})} + e^{i(-s+t/\sqrt{3})} + e^{-2it/\sqrt{3}}$ then, in terms of s and t, the functions $p^{-\frac{1}{2}}_{m,n}$ and $\mu^{\frac{1}{2}}\,p^{\frac{1}{2}}_{m-1,\,n-1}$ are the regular eigenfunctions of the operator $\partial^2/\partial s^2 + \partial^2/\partial t^2$ which remain invariant or change sign, respectively, under the reflections in the edges of a certain equilateral triangle. Two explicit partial differential operators D^{α}_1 and D^{α}_2 in z and $\bar{z}$ of orders two and three, respectively, are obtained such that the polynomials $p^{\alpha}_{m,n}$ are eigenfunctions of D^{α}_1 and D^{α}_2. The operators D^{α}_1 and D^{α}_2 commute and are algebraically independent, and they generate the algebra of all differential operators for which the polynomials $p^{\alpha}_{m,n}$ are eigenfunctions. If $\alpha = 0$, 3/2 or 7/2 then the operator D^{α}_1 expressed in terms of s and t is the radial part of the Laplace-Beltrami operator on certain compact Riemannian symmetric spaces of rank two.

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*) This paper is not for review; it is meant for publication in a journal.

1. Introduction

This paper deals with orthogonal polynomials in two variables on a region bounded by a closed three-cusped algebraic curve of fourth degree which is known as Steiner's hypocycloid. The weight function is some power of the fourth degree polynomial which vanishes on this curve. The main result in this paper is the construction of two algebraically independent partial differential operators of orders two and three, respectively, for which these orthogonal polynomials are eigenfunctions. Because of the existence of such operators these polynomials can be considered as a generalization of the classical orthogonal polynomials in one variable.

Another generalization of this type was studied in the author's previous paper [6], which dealt with orthogonal polynomials on a region bounded by two lines and a parabola touching these lines. The main difference between these two classes of polynomials is the method of orthogonalization. In [6] we orthogonalized the sequence $u^{n-k}v^k$, $n \geq k \geq 0$, arranged by the lexicographic ordering of the pairs (n,k). In the present paper, if $z = x+iy, \bar{z} = x-iy$ then the polynomial $p_{m,n}(z,\bar{z})$ is defined such that $p_{m,n}(z,\bar{z})-z^m\bar{z}^n$ has degree less than $m + n$ and $p_{m,n}$ is orthogonal to all polynomials of degree less than $m + n$. Thus the polynomials $p_{m,n}$, $m + n = N$, form a basis for the class of all orthogonal polynomials of degree N. For the special region and class of weight functions considered here it can be proved that this is an orthogonal basis.

A special case of the orthogonal polynomials studied in [6] can be obtained by considering the functions $\cos ns \cos kt + \cos ks \cos nt$, which are eigenfunctions of the Laplace operator satisfying certain symmetry relations. Expressed in the variables $u = \cos s + \cos t$, $v = \cos s \cos t$, these functions are orthogonal polynomials with respect to the weight function $(1-u+v)^{-\frac{1}{2}}(1+u+v)^{-\frac{1}{2}}(u^2-4v)^{-\frac{1}{2}}$. Similarly, the point of departure of the present paper are the regular eigenfunctions of $\partial^2/\partial s^2 + \partial^2/\partial t^2$ which are invariant under the reflections in the edges of some equilateral triangle. Let the interior of this triangle be denoted by R. After a suitable linear transformation $(s,t) \to (\sigma,\tau)$ these eigenfunctions can be expressed as sums of at most six distinct terms $e^{i(k\sigma + l\tau)}$, k, l integers, and they constitue a complete orthogonal system on the region R. This is discussed in §2.

The two non-constant eigenfunctions corresponding to the largest eigenvalue are the functions $z = e^{i\sigma} + e^{-i\tau} + e^{i(-\sigma+\tau)}$ and its complex conjugate $\bar{z}$. If $z = x + iy$ then the mapping $(s,t) \to (x,y)$ is bijective from R onto a region bounded by Steiner's hypocycloid. This last region will be denoted by S. In terms of z and $\bar{z}$, the eigenfunctions of $\partial^2/\partial s^2 + \partial^2/\partial t^2$ satisfying the symmetry relations mentioned above are polynomials for which the term of highest degree takes the form $z^m\bar{z}^n$. These polynomials are orthogonal on the region S with respect to the weight function $(\mu(z,\bar{z}))^{-\frac{1}{2}}$, where $\mu(z,\bar{z})$ is a fourth degree polynomial such that $\mu(z,\bar{z}) > 0$ on S and $\mu(z,\bar{z}) = 0$ on ∂S. These results are contained in §3. It is also shown there that orthogonal polynomials on S with respect to the weight function $(\mu(z,\bar{z}))^{\frac{1}{2}}$ are related to the eigenfunctions of $\partial^2/\partial s^2 + \partial^2/\partial t^2$ which change sign under the reflections in the edges of R.

Because of the previous considerations the region S, the weight function $(\mu(z,\bar{z}))^{\alpha}$ and the method of orthogonalization described earlier are quite natural for the definition of a class of orthogonal polynomials. For reasons of convergence let $\alpha > -5/6$. Then $p^{\alpha}_{m,n}(z,\bar{z})$ is defined as a polynomial such that $p^{\alpha}_{m,n}(z,\bar{z}) - z^m\bar{z}^n$ has degree less than m + n and

$$\iint_S p^{\alpha}_{m,n}(z,\bar{z})\, \overline{q\quad(z,\bar{z})}\, (\mu(z,\bar{z}))^{\alpha}\, dx\, dy = 0$$ for each polynomial q of degree less than m + n. Some simple properties of the polynomials $p^{\alpha}_{m,n}$ are given in §4. It is also pointed out in this section that the so-called disk polymonials provide a more elementary example of the method of orthogonalization used for the polynomials $p^{\alpha}_{m,n}$.

By the elementary interpretation of the functions $p^{\alpha}_{m,n}$ for $\alpha = \pm\frac{1}{2}$ one easily obtains in this case differential operators D^{α}_1 and D^{α}_2 of orders two and three, respectively, for which the functions $p^{\alpha}_{m,n}$ are eigenfunctions. For other values of α these operators can be generalized such that they are self-adjoint with respect to the weight function $(\mu(z,\bar{z}))^{\alpha}$. In §5 and §6 such operators D^{α}_1 and D^{α}_2, respectively, are constructed and it is proved that for all $\alpha > -5/6$ the functions $p^{\alpha}_{m,n}$ are eigenfunctions of D^{α}_1 and D^{α}_2. As a corollary it follows that

$$\iint_S p^{\alpha}_{m,n}\, \overline{p^{\alpha}_{k,l}}\, \mu^{\alpha}\, dx\, dy = 0 \text{ if } (m,n) \neq (k,l),\ m+n = k+l.$$ In [6, §5] a partial differential operator D_2 of fourth order was obtained as the product $D^+ D^-$ of two second order operators D^- and D^+. Although the corresponding operator D^{α}_2 considered here has lower order, it cannot be factorized. For this reason its construction is more complicated.

For certain values of α the operator D^{α}_1 expressed in terms of s and t is the radial part of the Laplace-Beltrami operator on certain compact Riemannian symmetric spaces of rank two. Hence it is reasonable to expect that for such α the functions $p^{\alpha}_{m,n}$ are spherical functions and the operator D^{α}_2 is the radial part of some invariant differential operator on the corresponding symmetric space. However, this will not be proved her.

This paper concludes in §7 with a discussion of the algebra of all partial differential operators for which the polynomials $p^{\alpha}_{m,n}$ are eigenfunctions. It is proved that each differential operator of this kind can be expressed in one and only one way as a polynomial in D^{α}_1 and D^{α}_2.

If all calculations in this paper would be done in a straightforward way then they would be quite long and tedious. In many cases it is indicated how a considerable gain in time and effort can be made by exploiting the symmetries in the formulas. It should be clear to the reader that due to the symmetry of the region this is a very charming class of orthogonal polynomials which can be studied in an elegant way.

2. Eigenfunctions of the Laplace operator which satisfy certain symmetry relations

Consider a regular tessellation of the Euclidean plane by equilateral triangles (cf. Fig.1).

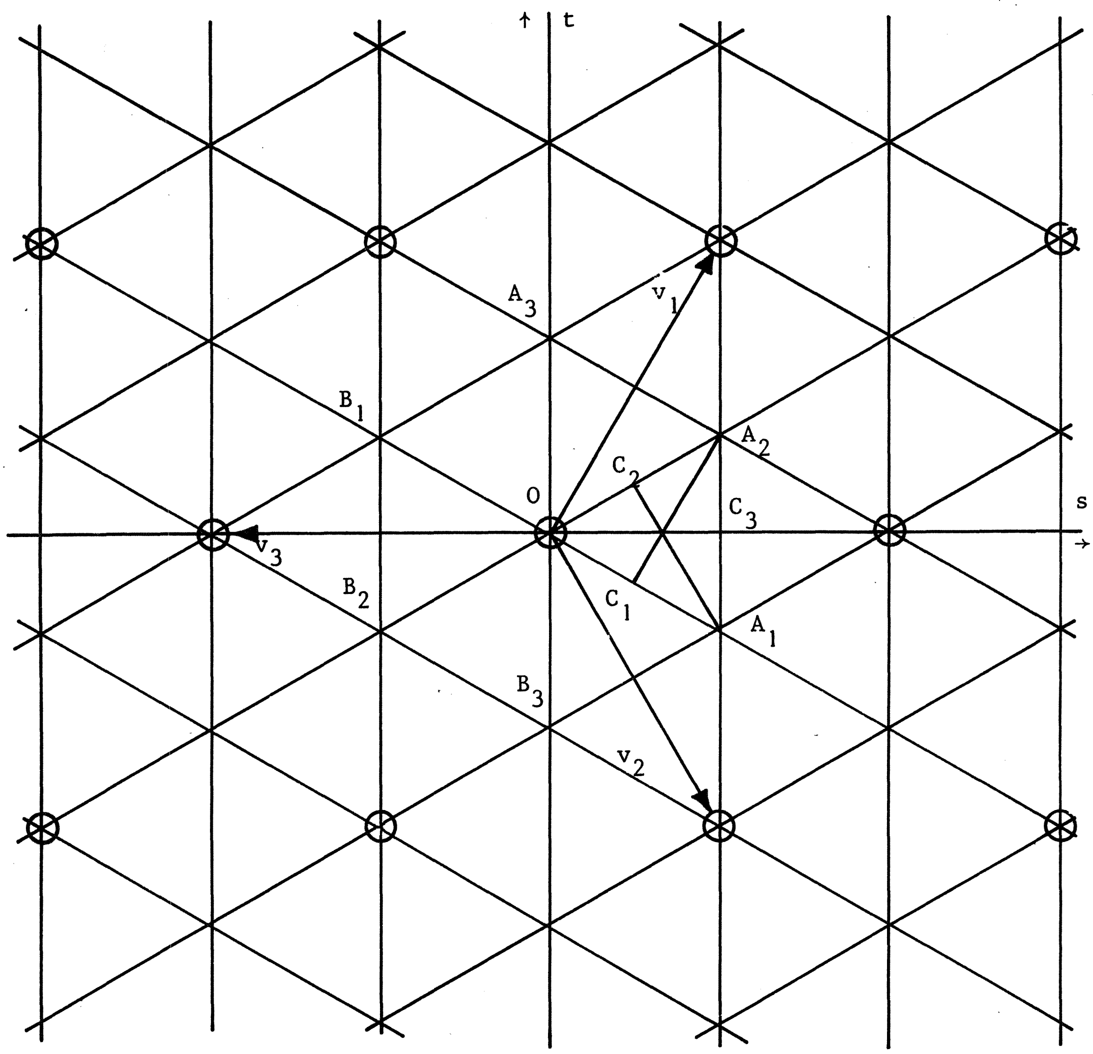

Fig. 1

Let G be the group of isometries which is generated by the reflections in in the edges of these triangles. Let R be a region bounded by one of these triangles, say, with vertices $O = (0,0)$, $A_1 = (\pi, -\pi/\sqrt{3})$, $A_2 = (\pi, \pi/\sqrt{3})$.

Let J_1, J_2 and J_3 denote the reflections in the edges OA_1, OA_2 and $A_1 A_2$, respectively, of R. Observe that the isometries $J_2 J_3 J_2 J_1$, $J_1 J_3 J_1 J_2$ and $J_1 J_2 J_1 J_3$ are the translations by the vectors $v_1 = (\pi, \pi\sqrt{3})$, $v_2 = (\pi, -\pi\sqrt{3})$ and $v_3 = (-2\pi, 0)$, respectively (cf.Fig. 1). It follows easily that the reflections J_1, J_2 and J_3 generate the group G. Alternatively, the translations by v_1 and v_2 and the reflections J_1 and J_2 generate G. Let H be the region bounded by the regular hexagon $A_1 A_2 A_3 B_1 B_2 B_3$ (cf. Fig. 1). The translations by v_1 and v_2 generate a translation group for which H is a fundamental region. The encircled points in Fig. 1 are the midpoints of the hexagons obtained by translation of H. The reflections J_1 and J_2 generate a transformation group of the region H for which R is a fundamental region. Hence R is a fundamental region for the group G.

In order to facilitate computations we transform the Euclidean coordinates s,t into new coordinates σ,τ defined by

$$\sigma = s + t/\sqrt{3} \quad , \quad \tau = s - t/\sqrt{3} \quad . \tag{2.1}$$

Note that in terms of these new coordinates $v_1 = (2\pi, 0)$ and $v_2 = (0, 2\pi)$. The reflections J_1, J_2, J_3 can be expressed by

$$\begin{cases} J_1(\sigma,\tau) = (-\sigma+\tau \,,\, \tau) \quad , \\ J_2(\sigma,\tau) = (\sigma, \sigma-\tau) \quad , \\ J_3(\sigma,\tau) = (2\pi-\tau \,,\, 2\pi-\sigma) . \end{cases} \tag{2.2}$$

Let the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijection. For each function f on $\mathbb{R}^2$ let the function Tf be defined by $(Tf)(\sigma,\tau) = f(T^{-1}(\sigma,\tau))$. If X is a partial differential operator in σ and τ of order r and if the mapping T is of class C^r, then let the operator dT(X) be defined such that $(dT(X))f = T(X(T^{-1}f))$ for each C^r - function f on $\mathbb{R}^2$.

In the following definition $\rho(T)$ denotes the Jacobian determinant of an isometry T, so $\rho(T) = \pm 1$.

DEFINITION 2.1. The function $f(\sigma,\tau)$ is called invariant (with respect to the group G) if $Tf = f$ for each $T \in G$. The function $f(\sigma,\tau)$ is called anti-invariant (with respect to G) if $Tf = \rho(T)f$ for each $T \in G$. The partial differential operator X in σ and τ is called invariant or anti-invariant (with respect to G) if $dT(X) = X$ or $dT(X) = \rho(T)X$, respectively, for each $T \in G$.

LEMMA 2.2. The function f is invariant if and only if f is 2π - periodic in σ and τ and $J_1 f = f = J_2 f$. The function f is anti-invariant if and only if f is 2π - periodic in σ and τ and $J_1 f = -f = J_2 f$.

This lemma is proved by using that the translations by v_1 and v_2 and the reflections J_1 and J_2 generate the group G.

LEMMA 2.3. Let the operators P^+ and P^- be defined by

$$(2.3)\qquad (P^{\pm}f)(\sigma,\tau) = \frac{1}{6}\,[f(\sigma,\tau) \pm f(\sigma,\sigma-\tau) + f(-\sigma+\tau,-\sigma) \pm f(-\tau,-\sigma) + f(-\tau,\sigma-\tau) \pm f(-\sigma+\tau,\tau)].$$

Then the operators P^+ and P^- are projections from the class of 2π -periodic functions in σ and τ onto the class of invariant, respectively anti-invariant functions.

Proof. By (2.2) and (2.3) we have

$$(2.4)\qquad P^{\pm}f = \frac{1}{6}\,[f \pm J_2 f + J_1 J_2 f \pm J_2 J_1 J_2 f + J_2 J_1 f \pm J_1 f].$$

The lemma follows from Lemma 2.2 by using that $J_1^2 = \text{id.} = J_2^2$ and $J_1 J_2 J_1 = J_2 J_1 J_2$. Q.e.d.

Let $\Delta = \partial^2/\partial s^2 + \partial^2/\partial t^2$ be the Laplace operator. Clearly this operator is invariant. In terms of σ and τ it is expressed by

$$(2.5)\qquad \Delta = \frac{4}{3}\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\sigma\partial\tau}\right).$$

THEOREM 2.4. Let the functions $e^+_{m,n}$, $m,n \geq 0$, and $e^-_{m,n}$, $m,n \geq 1$, be defined by

$$(2.6)\quad \begin{cases} e^{\pm}_{m,n}(\sigma,\tau) = e^{i(m\sigma+n\tau)} \pm e^{i((m+n)\sigma-n\tau)} + e^{i(-(m+n)\sigma+m\tau)} \\ \pm e^{i(-n\sigma-m\tau)} + e^{i(n\sigma-(m+n)\tau)} \pm e^{i(-m\sigma+(m+n)\tau)} \quad \text{if } m,n > 0, \\ e^{+}_{m,0}(\sigma,\tau) = e^{im\sigma} + e^{-im\tau} + e^{i(-m\sigma+m\tau)} \quad \text{if } m > 0\,, \\ e^{+}_{0,n}(\sigma,\tau) = e^{-in\sigma} + e^{in\tau} + e^{i(n\sigma-n\tau)} \quad \text{if } n > 0\,, \\ e^{+}_{0,0}(\sigma,\tau) = 1. \end{cases}$$

Then the functions $e^{+}_{m,n}$ and $e^{-}_{m,n}$ are invariant, respectively anti-invariant. Both systems $\{e^{+}_{m,n}\}$ and $\{e^{-}_{m,n}\}$ are complete orthogonal systems of eigenfunctions of Δ on the region R and

$$(2.7)\qquad \Delta e^{\pm}_{m,n} = -\frac{4}{3}(m^2+n^2+mn)\, e^{\pm}_{m,n}.$$

Proof. Let for arbitrary integers m,n $f_{m,n}(\sigma,\tau) = e^{i(m\sigma + n\tau)}$. It follows by (2.3) and (2.6) that $P^{\pm}f_{m,n} = 6\,e^{\pm}_{m,n}$ if $m,n > 0$, $P^{+}f_{m,0} = 3\,e^{+}_{m,0}$ if $m > 0$, $P^{+}f_{0,n} = 3\,e^{+}_{0,n}$ if $n > 0$, $P^{+}f_{0,0} = e^{+}_{0,0}$, $P^{-}f_{m,n} = 0$ if $m = 0$ or $n = 0$. Thus the invariance of the functions $e^{+}_{m,n}$ and the anti-invariance of the functions $e^{-}_{m,n}$ follows from Lemma 2.3. Formula (2.7) is obtained from (2.4) by using that $\Delta f_{m,n} = -(4/3)(m^2+n^2+mn)\, f_{m,n}$ and that $dJ_1(\Delta) = \Delta = dJ_2(\Delta)$. Next we have to prove the orthogonality and the completeness of the systems $\{e^{+}_{m,n}\}$ and $\{e^{-}_{m,n}\}$. Observe that each function g on R has an invariant extension g^{+} and an anti-invariant extension g^{-} to $\mathbb{R}^2$ and that these extensions are unique except on a set of measure zero. Let us denote by $\mathcal{H}$ the Hilbert space of 2π - periodic functions in σ and τ which are square integrable on the hexagonal region H. Then the mapping $g \to g^{+}$ identifies the Hilbert space $L^2(R)$ with the subspace $\mathcal{H}^{+}$ of $\mathcal{H}$ consisting of the invariant L^2 - functions on H. Similarly, the mapping $g \to g^{-}$ identifies $L^2(R)$ with the subspace $\mathcal{H}^{-}$ of $\mathcal{H}$ consisting of the anti-invariant L^2 - functions on H. Since for arbitrary $f,g \in \mathcal{H}$

$$\iint_H (P^{\pm}f)\bar{g} \; d\sigma \, d\tau = 6 \iint_R (P^{\pm}f) \; \overline{(P^{\pm}g)} \; d\sigma \, d\tau$$

$$= \iint_H f\overline{(P^{\pm}g)} \; d\sigma \, d\tau$$

it follows that the projections $P^+ : H \to H^+$ and $P^- : H \to H^-$ are self-adjoint. Let for m,n ≥ 0 $H_{m,n}$ be the subspace of H spanned by the functions $f_{m,n}$, $J_2 f_{m,n}$, $J_2 J_1 f_{m,n}$, $J_2 J_1 J_2 f_{m,n}$, $J_1 J_2 f_{m,n}$, $J_1 f_{m,n}$, i.e., by all functions $Tf_{m,n}$, $T \in G$. It follows by (2.2) that $H_{m,n}$ is spanned by the functions $f_{m,n}$, $f_{m+n,-n}$, $f_{-m-n,m}$, $f_{-n,-m}$, $f_{n,-m-n}$, $f_{-m,m+n}$. Hence each function $f_{k,l}$, k,l integers, is contained in one and only once class $H_{m,n}$, m,n ≥ 0. The functions $f_{k,l}$, k,l integers, form an orthogonal basis for H. So we have the orthogonal decompositions

$$H = \sum_{m,n=0}^{\infty}{}^{\oplus} H_{m,n} \,, \quad H^+ = \sum_{m,n=0}^{\infty}{}^{\oplus} P^+ H_{m,n} \,, \quad H^- = \sum_{m,n=0}^{\infty}{}^{\oplus} P^- H_{m,n} .$$

By (2.4) $P^+ H_{m,n}$ is spanned by $P^+ f_{m,n}$ = const. $e^+_{m,n}$ witn non-zero constant, $P^- H_{m,n}$ is spanned by $P^- f_{m,n} = 6\, e^-_{m,n}$ if m.n ≥ 1 and $P^- H_{m,n} = \{0\}$ if m = 0 or n = 0. This proves the orthogonality and the completeness of the systems $\{e^+_{m,n}\}$ and $\{e^-_{m,n}\}$. Q.e.d.

A function $g(\sigma,\tau)$ which is a finite linear combination of the functions $e^{i(k\sigma+l\tau)}$, k,l integers, will be called a trigonometric polynomial in σ and τ. The functions $e^+_{m,n}$ and $e^-_{m,n}$, defined by (2.6), are clearly trigonometric polynomials in σ and τ. Note that both $e^+_{m,n}$ and $e^-_{m,n}$ are expressed by a sum $\sum c_{k,l}\, e^{i(k\sigma+l\tau)}$, such that $c_{m,n} = 1$ and $c_{k,l} = 0$ if k,l ≥ 0 and (k,l) ≠ (m,n). This property is applied in the following useful lemma.

LEMMA 2.5 Let $g(\sigma,\tau) = \sum c_{m,n}\, e^{i(m\sigma+n\tau)}$ be a trigonometric polynomial. Then $g = \sum_{m \geq 0} \sum_{n \geq 0} c_{m,n}\, e^+_{m,n}$ if g is invariant and $g = \sum_{m \geq 1} \sum_{n \geq 1} c_{m,n}\, e_{m,n}$

if g is anti-invariant.

Proof. Let g be invariant. It follows from the proof of Theorem 2.4 that $g = P^+g = \sum c_{m,n} P^+f_{m,n} = \sum_{m \geq 0} \sum_{n \geq 0} b_{m,n} e^+_{m,n}$ for certain coefficients $b_{m,n}$. Hence $\sum c_{m,n} e^{i(m\sigma+n\tau)} = \sum_{m \geq 0} \sum_{n \geq 0} b_{m,n} e^+_{m,n}(\sigma,\tau)$.

This implies that $c_{m,n} = b_{m,n}$ if $m,n \geq 0$. If g is anti-invariant then a similar proof can be given. Q.e.d.

Let the lines A_2C_1, A_1C_2 and OC_3 bisect the angles of the triangle OA_1A_2 (cf. Fig. 1) and let I_1, I_2 and I_3 denote the reflections in the lines A_2C_1, A_1C_2 and OC_3, respectively. Then these reflections generate the group of isometries which map R onto itself. This group is isomorphic to the permutation group in three letters. It is also generated by the reflection I_3 and by the rotation $I_2I_1 = I_3I_2 = I_1I_3$. Observe that

$$(2.8)\qquad \begin{cases} I_1(\sigma,\tau) = (\sigma-\tau+2\pi/3, -\tau+4\pi/3), \\ I_2(\sigma,\tau) = (-\sigma+4\pi/3, -\sigma+\tau+2\pi/3), \\ I_3(\sigma,\tau) = (\tau,\sigma). \end{cases}$$

It follows by inspection from (2.6) that

$$(2.9)\qquad (I_3 e^{+}_{m,n})(\sigma,\tau) = e^{\pm}_{m,n}(\tau,\sigma) = \pm \overline{e^{\pm}_{m,n}(-\sigma,-\tau)}$$

$$= e^{\pm}_{n,m}(\sigma,\tau) = \pm e^{\pm}_{m,n}(\sigma,\tau),$$

$$(2.10)\qquad (I_1I_2 e^{\pm}_{m,n})(\sigma,\tau) = e^{\pm}_{m,n}(-\sigma+\tau+2\pi/3, -\sigma+4\pi/3)$$

$$= e^{i(m-n)2\pi/3} e^{\pm}_{m,n}(\sigma,\tau).$$

Let us introduce the first order differential operators

$$(2.11)\quad \begin{cases} X_1 = \frac{1}{i}\left(-\frac{3}{2}\frac{\partial}{\partial s} + \frac{1}{2}\sqrt{3}\frac{\partial}{\partial t}\right) = \frac{1}{i}\left(-\frac{\partial}{\partial \sigma} - 2\frac{\partial}{\partial \tau}\right), \\ X_2 = \frac{1}{i}\left(\frac{3}{2}\frac{\partial}{\partial s} + \frac{1}{2}\sqrt{3}\frac{\partial}{\partial t}\right) = \frac{1}{i}\left(2\frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau}\right), \\ X_3 = -\frac{1}{i}\sqrt{3}\frac{\partial}{\partial t} = \frac{1}{i}\left(-\frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau}\right). \end{cases}$$

Observe that

$$(2.12)\qquad X_1 + X_2 + X_3 = 0.$$

It follows from (2.2) and (2.8) that for each permutation (i,j,k) of (1,2,3) we have

$$(2.13)\quad \begin{cases} dJ_i(X_i) = X_i, \quad dJ_i(X_j) = X_k, \quad dJ_i(X_k) = X_j, \\ dI_i(X_i) = -X_i, \quad dI_i(X_j) = -X_k, dI_i(X_k) = -X_j. \end{cases}$$

THEOREM 2.6. Let Q be a symmetric polynomial in three variables. Then

$$(2.14)\qquad Q(X_1,X_2,X_3)e^{\pm}_{m,n} = Q(-m-2n,2m+n,-m+n)e^{\pm}_{m,n}.$$

Proof. By the symmetry property of Q and by (2.13) the operator $Q(X_1,X_2,X_3)$ is invariant. Hence the function $Q(X_1,X_2,X_3)e^{+}_{m,n}$ is invariant and the function $Q(X_1,X_2,X_3)e^{-}_{m,n}$ is anti-invariant. By (2.11) we have

$$Q(X_1,X_2,X_3)e^{i(k\sigma+l\tau)} = Q(-k-2l,2k+l,-k+l)e^{i(k\sigma+l\tau)}.$$

The theorem follows by using Lemma 2.5. Q.e.d.

Two particular cases of (2.14) are the differential equations

$$(2.15)\qquad (X_1^2+X_2^2+X_3^2)e^{\pm}_{m,n} = 6\,(m^2+n^2+mn)e^{\pm}_{m,n},$$

$$(2.16)\qquad (X_1X_2X_3)e^{\pm}_{m,n} = (m-n)(2m+n)(m+2n)e^{\pm}_{m,n}.$$

Note that $\Delta = -(2/9)(X_1^2+X_2^2+X_3^2)$. It follows from the lemma below that (2.15) and (2.16) generate all differential equations of type (2.14).

LEMMA 2.7 Let D be an invariant differential operator in σ and τ with constant coefficients. Then there exists a symmetric polynomial Q in three variables and a polynomial P in two variables such that $D = Q(X_1,X_2,X_3) = P(X_1^2+X_2^2+X_3^2,\ X_1X_2X_3)$. The polynomial P is uniquely determined by D.

Proof. Clearly there is a unique polynomial F in two variables such that $D = F(X_1,X_2)$. By (2.13) and by the invariance of D it follows that $D = (1/6) \sum F(X_i,X_j)$, where the summation runs over all permutations (i,j,k) of (1,2,3). Hence there exists a symmetric polynomial Q such that $D = Q(X_1,X_2,X_3)$. According to van der Waerden [9,§33] the symmetric polynomial $Q(X_1,X_2,X_3)$ can be expressed as a polynomial in the three elementary symmetric polynomials $X_1 + X_2 + X_3$, $X_1X_2 + X_2X_3 + X_3X_1$ and $X_1X_2X_3$. But $X_1 + X_2 + X_3 = 0$, hence $X_1^2 + X_2^2 + X_3^2 = -2(X_1X_2+X_2X_3+X_3X_1)$, so there exists a polynomial P in two variables such that $D = Q(X_1,X_2,X_3) = P(X_1^2+X_2^2+X_3^2,\ X_1X_2X_3)$. In order to prove the uniqueness of P, suppose that $P(X_1^2+X_2^2+X_3^2,\ X_1X_2X_3) = 0$ and that the polynomial P is non-zero. Substituting $X_3 = -X_1 - X_2$ we obtain that $P(2(X_1^2+X_2^2+X_1X_2),\ -X_1^2X_2-X_1X_2^2) = 0$. The polynomial $P(x,y)$ is a sum of terms $c_{k,l}(\frac{1}{2}x)^{\frac{1}{2}k-l}(-y)^l$, where $\frac{1}{2}k$ and l are integers and $\frac{1}{2}k \geq l \geq 0$. Among the pairs of integers (k,l) such that $c_{k,l} \neq 0$ there is a maximal element (m,n) with respect to lexicographic ordering. Then $c_{m,n}$ is the coefficient of $X_1^m X_2^n$ in the operator $P(2(X_1^2+X_2^2+X_1X_2),\ -X_1^2X_2 - X_1X_2^2)$ expressed as a polynomial in X_1 and X_2. Hence $c_{m,n} = 0$. This is a contradiction. Q.e.d.

3. A generalization of the Chebyshev polynomials

The classes of functions cos ns, n = 0,1,2,..., and sin ns, n = 1,2,..., are both complete orthogonal systems of eigenfunctions of the operator d^2/ds^2 on the interval $(0,\pi)$. These functions satisfy the symmetry relations $f(-s) = f(s) = f(2\pi-s)$ and $f(-s) = - f(s) = f(2\pi-s)$, respectively. Let the functions T_n and U_n be defined by the identities $T_n(\cos s) = \cos ns$ and $U_n(\cos s) = (\sin(n+1)s)\sin s$. Then $T_n(x)$ and $U_n(x)$ are both polynomials of degree n, the so-called Chebyshev polynomials of the first and of the second kind, respectively. They satisfy the orthogonality relations

$$\int_{-1}^{1} T_m(x)T_n(x)(1-x^2)^{-\frac{1}{2}}dx = 0, \ m \neq n, \text{ and}$$

$$\int_{-1}^{1} U_m(x)U_n(x)(1-x^2)^{\frac{1}{2}}dx = 0 \ , \ m \neq n.$$

The functions $e^+_{m,n}(\sigma,\tau)$ and $e^-_{m,n}(\sigma,\tau)$ can be considered as generalizations of the functions cos ns and sin ns, respectively. It will be proved in this section that the functions $e^+_{m,n}(\sigma,\tau)$ and $e^-_{m+1,\ n+1}(\sigma,\tau)/e^-_{1,1}(\sigma,\tau)$ can be expressed as polynomials in $e^+_{1,0}(\sigma,\tau)$ and $e^+_{0,1}(\sigma,\tau)$ and that both classes of polynomials obtained in this way are orthogonal systems on a region bounded by Steiner's hypocycloid. These orthogonal systems are a natural generalization of the Chebyshev polynomials.

Let us write

$$(3.1) \quad \begin{cases} z(\sigma,\tau) = e^+_{1,0}(\sigma,\tau) = e^{i\sigma} + e^{-i\tau} + e^{i(-\sigma+\tau)}, \\ \bar{z}(\sigma,\tau) = e^+_{0,1}(\sigma,\tau) = e^{-i\sigma} + e^{i\tau} + e^{i(\sigma-\tau)}. \end{cases}$$

Note that $\bar{z}(\sigma,\tau)$ is the complex conjugate of $z(\sigma,\tau)$.

LEMMA 3.1. Let Q be a polynomial in two variables such that $Q(z(\sigma,\tau), \bar{z}(\sigma,\tau)) = 0$ for all σ,τ. Then Q is the zero polynomial.

Proof. Suppose that Q is non-zero and has degree N. Then we can write

$Q(u,v) = \sum c_{k,l} u^k v^l$, where $c_{m,n} \neq 0$ for some pair (m,n), $m + n = N$. It follows from (3.1) that $c_{m,n}$ is the coefficient of $e^{i(m\sigma+n\tau)}$ in the trigonometric polynomial $Q(z(\sigma,\tau), \bar{z}(\sigma,\tau))$. Hence $c_{m,n} = 0$. This is a contradiction. Q.e.d.

Using (2.6) and Lemma 2.5 we derive the recurrence relations

$$(3.2)\qquad \begin{cases} e^+_{m+1,n} = z\, e^+_{m,n} - A_m\, e^+_{m-1,n+1} - A_n\, e^+_{m,n-1} & \text{if } m > 0 \text{ or } n > 1, \\ e^+_{m,n+1} = \bar{z}\, e^+_{m,n} - A_n\, e^+_{m+1,n-1} - A_m\, e^+_{m-1,n} & \text{if } m > 1 \text{ or } n > 0, \\ e^+_{1,1} = z\,\bar{z} - 3\,, \end{cases}$$

where $A_n = 1$ if $n \neq 1$, $A_1 = 2$, $e^+_{m,-1} = 0 = e^+_{-1,n}$.

From now on $\pi_n(z,\bar{z})$ will denote an arbitrary polynomial in z and $\bar{z}$ of degree $\leq n$.

THEOREM 3.2. For each pair (m,n) of nonnegative integers there is a unique polynomial in two variables, denoted by $p^{-\frac{1}{2}}_{m,n}$, such that

$$(3.3)\qquad p^{-\frac{1}{2}}_{m,n}(z(\sigma,\tau), \bar{z}(\sigma,\tau)) = e^+_{m,n}(\sigma,\tau).$$

Then $\qquad p^{-\frac{1}{2}}_{m,n}(z,\bar{z}) = z^m \bar{z}^n + \pi_{m+n-1}(z,\bar{z}).$

Proof. The uniqueness part follows from Lemma 3.1. The existence part and the last statement of the theorem follow from (3.2) by using complete induction with respect to $m + n$. Q.e.d.

By (2.9) there is the symmetry relation

$$(3.4)\qquad p^{-\frac{1}{2}}_{m,n}(z,\bar{z}) = p^{-\frac{1}{2}}_{n,m}(\bar{z},z).$$

It can be derived from (3.2) that, for instance,

$$(3.5)\quad \begin{cases} p^{-\frac12}_{0,0}(z,\bar z) = 1\ ,\ p^{-\frac12}_{1,0}(z,\bar z) = z\ , \\ p^{-\frac12}_{2,0}(z,\bar z) = z^2 - 2\bar z\ ,\ p^{-\frac12}_{1,1}(z,\bar z) = z\bar z - 3\ , \\ p^{-\frac12}_{3,0}(z,\bar z) = z^3 - 3z\bar z + 3\ ,\ p^{-\frac12}_{2,1}(z,\bar z) = z^2\bar z - 2\bar z^2 - z\ , \\ p^{-\frac12}_{4,0}(z,\bar z) = z^4 - 4z^2\bar z + 2\bar z^2 + 4z\ , \\ p^{-\frac12}_{3,1}(z,\bar z) = z^3\bar z - 3z\bar z^2 - z^2 + 5\bar z\ , \\ p^{-\frac12}_{2,2}(z,\bar z) = z^2\bar z^2 - 2z^3 - 2\bar z^3 + 4z\bar z - 3\ . \end{cases}$$

In a similar way as (3.2) one can derive the recurrence relations

$$(3.6)\quad \begin{cases} e^{-}_{m+1,n} = z\, e^{-}_{m,n} - e^{-}_{m-1,n+1} - e^{-}_{m,n-1}\ ,\ m,n \ge 1\ , \\ e^{-}_{m,n+1} = \bar z\, e^{-}_{m,n} - e^{-}_{m+1,n-1} - e^{-}_{m-1,n}\ ,\ m,n \ge 1\ , \end{cases}$$

where $e^{-}_{m,n} = 0$ if $m = 0$ or $n = 0$.

The following theorem can be proved in a similar way as Theorem 3.2.

THEOREM 3.3. For each pair (m,n) of nonnegative integers there is a unique polynomial in two variables, denoted by $p^{\frac12}_{m,n}$, such that

$$(3.7)\qquad p^{\frac12}_{m,n}(z(\sigma,\tau),\ \bar z(\sigma,\tau)) = \frac{e^{-}_{m+1,n+1}(\sigma,\tau)}{e^{-}_{1,1}(\sigma,\tau)}\ .$$

Then $$p^{\frac12}_{m,n}(z,\bar z) = z^m\bar z^n + \pi_{m+n-1}(z,\bar z).$$

Again we have by (2.9) a symmetry relation

$$(3.8)\qquad p^{\frac12}_{m,n}\ (z,\bar z) = p^{\frac12}_{n,m}\ (\bar z,z).$$

It can be derived from (3.6) that, for instance,

$$(3.9)\qquad \begin{cases} p^{\frac{1}{2}}_{0,0}(z,\bar{z}) = 1, \; p^{\frac{1}{2}}_{1,0}(z,\bar{z}) = z, \\ p^{\frac{1}{2}}_{2,0}(z,\bar{z}) = z^2 - \bar{z}, \; p^{\frac{1}{2}}_{1,1}(z,\bar{z}) = z\bar{z} - 1, \\ p^{\frac{1}{2}}_{3,0}(z,\bar{z}) = z^3 - 2z\bar{z} + 1, \; p^{\frac{1}{2}}_{2,1}(z,\bar{z}) = z^2\bar{z} - \bar{z}^2 - z. \end{cases}$$

LEMMA 3.4. Let the coordinate transformation $(s,t) \to (x,y)$ be defined by (2.1), (3.1) and

$$(3.10)\qquad x = \tfrac{1}{2}(z+\bar{z}), \; y = \tfrac{1}{2}i(-z+\bar{z}).$$

Then the Jacobian determinant of this transformation equals

$$(3.11)\qquad \frac{\partial(x,y)}{\partial(s,t)} = -(i/\sqrt{3})\, e^{-}_{1,1}(\sigma,\tau)$$

$$= (8/\sqrt{3}) \sin s \sin(\tfrac{1}{2}s+\tfrac{1}{2}\sqrt{3}t) \sin(\tfrac{1}{2}s-\tfrac{1}{2}\sqrt{3}t),$$

and it is non-zero on the region R.

Proof. We have

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{\partial(\sigma,\tau)}{\partial(s,t)} \frac{\partial(z,\bar{z})}{\partial(\sigma,\tau)} \frac{\partial(x,y)}{\partial(z,\bar{z})} = -\frac{i}{\sqrt{3}} \frac{\partial(z,\bar{z})}{\partial(\sigma,\tau)}.$$

The function $\partial(z,\bar{z})/\partial(\sigma,\tau)$ is anti-invariant and

$$\frac{\partial(z,\bar{z})}{\partial(\sigma,\tau)} = \frac{\partial z}{\partial\sigma}\frac{\partial\bar{z}}{\partial\tau} - \frac{\partial z}{\partial\tau}\frac{\partial\bar{z}}{\partial\sigma} = e^{-}_{1,1}(\sigma,\tau)$$

by (3.1) and Lemma 2.5. It follows from the explicit expression (2.6) of $e^{-}_{1,1}$ that

$$e^{-}_{1,1}(\sigma,\tau) = -8i \sin(\tfrac{1}{2}\sigma+\tfrac{1}{2}\tau) \sin(\sigma-\tfrac{1}{2}\tau) \sin(-\tfrac{1}{2}\sigma+\tau).$$

This proves (3.11). The zero lines of the function $e^{-}_{1,1}$ are just the edges of the triangles in the tessellation of Fig. 1. Q.e.d.

By the previous lemma the mapping $(s,t) \to (x,y)$ is a diffeomorphism from R onto a certain region S in the (x,y) - plane and the boundary ∂R of R is mapped onto the boundary ∂S of S. In terms of the coordinates σ,τ the edges of the triangle ∂R have the parameter representations

$$\begin{cases} OA_1 = \{(\theta,2\theta) \mid 0 \le \theta \le 2\pi/3\}, \\ A_1A_2 = \{(\theta,2\pi-\theta) \mid 2\pi/3 \le \theta \le 4\pi/3\}, \\ A_2O = \{(4\pi-2\theta,2\pi-\theta) \mid 4\pi/3 \le \theta \le 2\pi\}, \end{cases}$$

cf. Fig. 1. Hence, by (3.1), ∂S has a parameter representation

$$z = x + iy = 2\, e^{i\theta} + e^{-2i\theta},\ 0 \le \theta < 2\pi, \tag{3.12}$$

where the images of O, A_1 and A_2 correspond with the values $\theta = 0$, $2\pi/3$, $4\pi/3$, respectively. It follows easily from (3.12) that if a circle of radius 1 rolls on the inside of a fixed circle of radius 3 then ∂S is the orbit of a point on the smaller circle. The resulting curve (cf. Fig.2) has three cusps and it is known as Steiner's hypocycloid, see for instance Loria [7, §§ 73, 74] and Hilton [4, Chap. 17, §§ 2,5]. Then S is the region inside this curve.

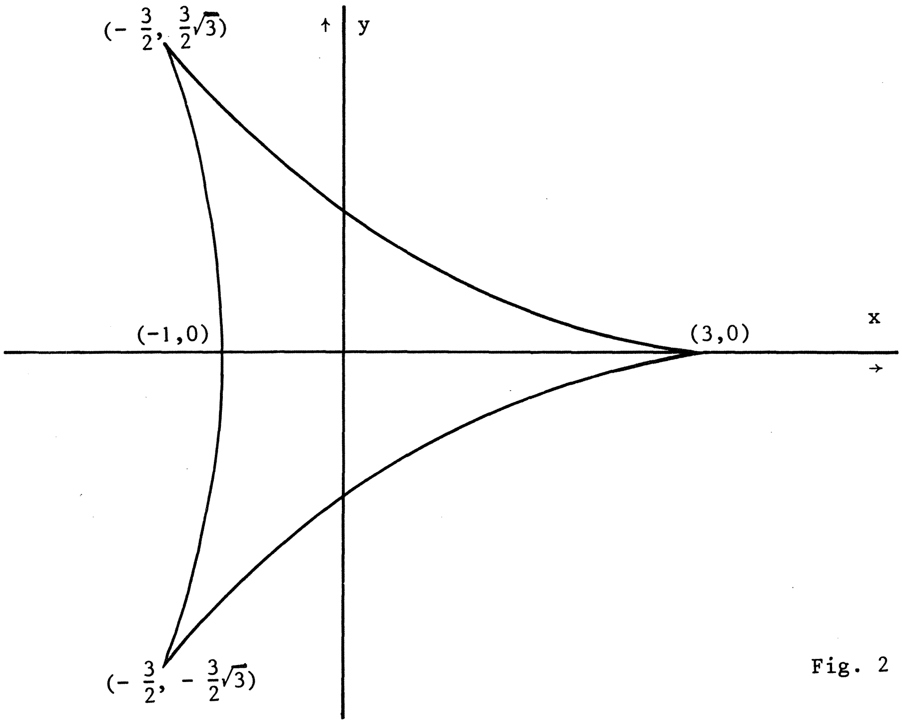

Fig. 2

By elimination of θ in (3.12) it can be shown that Steiner's hypocycloid is an algebraic curve of the fourth degree. This can also be proved in the following way. The function $(e^-_{1,1})^2$ is invariant. It follows from (2.6) and Lemma 2.5 that

$$(e^-_{1,1})^2 = e^+_{2,2} - 2e^+_{3,0} - 2e^+_{0,3} + 2e^+_{1,1} - 6e^+_{0,0}.$$

Hence, by (3,3), (3.5) and (3.4)

$$(3.13) \qquad (e^-_{1,1}(\sigma,\tau))^2 = z^2\bar{z}^2 - 4z^3 - 4\bar{z}^3 + 18z\bar{z} - 27.$$

By putting $e^-_{1,1}(\sigma,\tau) = 0$ the equation for Steiner's hypocycloid takes the form

$$(3.14) \qquad (x^2+y^2+9)^2 + 8(-x^3+3x^2y) - 108 = 0.$$

Instead of x,y we shall often use the coordinates $z = x + iy$, $\bar{z} = x - iy$ on the region S. Let

$$(3.15) \qquad w(\sigma,\tau) = \sin(\tfrac{1}{2}\sigma+\tfrac{1}{2}\tau)\ \sin(\sigma-\tfrac{1}{2}\tau)\ \sin(-\tfrac{1}{2}\sigma+\tau),$$

$$(3.16) \qquad \mu(z,\bar{z}) = -\ z^2\bar{z}^2 + 4z^3 + 4\bar{z}^3 - 18z\bar{z} + 27.$$

Then w is positive on R, μ is positive on S, and by (3.11) and (3.13) we have

$$(3.17) \qquad \mu(z,\bar{z}) = -\ (e^-_{1,1}(\sigma,\tau))^2 = 64(w(\sigma,\tau))^2.$$

THEOREM 3.5. The polynomials $p^{-\frac{1}{2}}_{m,n}$ are orthogonal on S with respect to the weight function $\mu^{-\frac{1}{2}}$. The polynomials $p^{\frac{1}{2}}_{m,n}$ are orthogonal on S with respect to the weight function $\mu^{\frac{1}{2}}$.

Proof. By (3.11) and (3.17) $(\mu(z,\bar{z}))^{-\frac{1}{2}}dx\,dy = \text{const.}\ d\sigma\,d\tau$. It follows that

$$\iint_R e^+_{m,n}(\sigma,\tau)\ \overline{e^+_{k,l}(\sigma,\tau)}\ d\sigma\,d\tau$$

$$= \text{const.} \iint_S p^{-\frac{1}{2}}_{m,n}(z,\bar{z})\ \overline{p^{-\frac{1}{2}}_{k,l}(z,\bar{z})}(\mu(z,\bar{z}))^{-\frac{1}{2}}\ dx\,dy$$

and

$$\iint_R e_{m+1,n+1}(\sigma,\tau)\ \overline{e^-_{k+1,l+1}(\sigma,\tau)}\ d\sigma\,d\tau$$

$$= \text{const.} \iint_R p^{\frac{1}{2}}_{m,n}(z,\bar{z})\overline{p^{\frac{1}{2}}_{k,l}(z,\bar{z})}(e^-_{1,1}(\sigma,\tau))^2\ d\sigma\,d\tau$$

$$= \text{const.} \iint_S p^{\frac{1}{2}}_{m,n}(z,\bar{z})\overline{p^{\frac{1}{2}}_{k,l}(z,\bar{z})}\ (\mu(z,\bar{z}))^{\frac{1}{2}}\ d\sigma\,d\tau.$$

The theorem is then proved by using Theorem 2.4. Q.e.d.

4. Orthogonal polynomials on the interior of Steiner's hypocycloid with respect to a more general weight function

If the weight functions $(1-x^2)^{-\frac{1}{2}}$ and $(1-x^2)^{\frac{1}{2}}$ for Chebyshev polynomials are generalized to $(1-x)^{\alpha}(1+x)^{\beta}$, $-1 < x < 1$, then the corresponding orthogonal polynomials are the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, for which many properties of the Chebyshev polynomials can be generalized. In this section the polynomials $p_{m,n}^{-\frac{1}{2}}(z,\bar{z})$ and $p_{m,n}^{\frac{1}{2}}(z,\bar{z})$, introduced in §3, will be generalized in a similar way by considering the weight function $(\mu(z,\bar{z}))^{\alpha}$ on the region S.

LEMMA 4.1. Let α be a real number. Then

$$\iint_S (\mu(z,\bar{z}))^{\alpha} dx\, dy < \infty \text{ if and only if } \alpha > -5/6.$$

Proof. It is an equivalent problem to find all real values of α for which

$$\iint_R (w(\sigma,\tau))^{2\alpha+1} d\sigma\, d\tau < \infty$$

where $w(\sigma,\tau)$ is given by (3.15). This integral clearly converges if $\alpha \geq -\frac{1}{2}$. If small neighbourhoods of the three vertices of R are excluded from the region of integration then the integral converges if and only if $\alpha > -1$, due to the singularities on the edges of R. Next consider a small neighbourhood V of the vertex 0 of R. Introducing polar coordinates r,ϕ such that $\sigma = r\cos\phi$, $\tau = r\sin\phi$, we have $(w(\sigma,\tau))^{2\alpha+1} = O(r^{6\alpha+3})$ if $r \downarrow 0$. Hence, $\iint_V (w(\sigma,\tau))^{2\alpha+1} d\sigma\, d\tau < \infty$ if and only if $\int_0^{\delta} r^{6\alpha+4}\, dr < \infty, \delta > 0$, i.e., if and only if $\alpha > -5/6$. For the two other vertices of R the same inequality can be obtained. Q.e.d.

DEFINITION 4.2. For real $\alpha > -5/6$ and for integers $m,n \geq 0$ the function $p_{m,n}^{\alpha}(z,\bar{z})$ is a polynomial such that

(i) $$p_{m,n}^{\alpha}(z,\bar{z}) = z^m \bar{z}^n + \pi_{m+n-1}(z,\bar{z}),$$

(ii) $$\iint_S p_{m,n}^{\alpha}(z,\bar{z})\, \overline{q(z,\bar{z})}\, (\mu(z,z))^{\alpha}\, dx\, dy = 0$$

if q is a polynomial of degree less than m+n.

The conditions (i) and (ii) of Definition 4.2 uniquely determine the polynomial $p^{\alpha}_{m,n}$. The polynomials $p^{-\frac{1}{2}}_{m,n}$ and $p^{\frac{1}{2}}_{m,n}$ defined by (3.3) and (3.7) satisfy these conditions for $\alpha = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$, respectively.

It is clear from Definition 4.2 that

$$\iint_S p^{\alpha}_{m,n}(z,\bar{z})\overline{p^{\alpha}_{k,l}(z,\bar{z})}(\mu(z,\bar{z}))^{\alpha}\,dx\,dy = 0 \tag{4.1}$$

if $m+n \neq k+l$. The proof that (4.1) also holds if $m+n = k+l$, $(m,n) \neq (k,l)$, will be postponed to §5 and §6.

In the author's previous paper [6, (3.14)] certain orthogonal polynomials in two variables u and v were defined by orthogonalization of the sequence $1,u,v,u^2,uv,v^2,v^3,u^2v,\ldots$. The method of orthogonalization in Definition 4.2 is quite different. The so-called disk polynomials are a more elementary example of this method of orthogonalization. Let for $\alpha > -1$ and for $m,n \geq 0$ the polynomial $q^{\alpha}_{m,n}(z,\bar{z})$ be defined by

$$q^{\alpha}_{m,n}(z,\bar{z}) = \begin{cases} \dfrac{n!}{(m+\alpha+1)_n} P_n^{(\alpha,m-n)}(2z\bar{z}-1)z^{m-n} & \text{if } m \geq n, \\[2ex] \dfrac{m!}{(n+\alpha+1)_m} P_m^{(\alpha,n-m)}(2z\bar{z}-1)\bar{z}^{n-m} & \text{if } m < n, \end{cases}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes a Jacobi polynomial. Then it can easily be proved that

(i) $$q^{\alpha}_{m,n}(z,\bar{z}) = z^m\bar{z}^n + \pi_{m+n-1}(z,\bar{z}),$$

(ii) $$\iint_{x^2+y^2<1} q^{\alpha}_{m,n}(x+iy,x-iy)\,\overline{q^{\alpha}_{k,l}(x+iy,x-iy)}\,.(1-x^2-y^2)^{\alpha}dx\,dy = 0 \qquad \text{if } (m,n) \neq (k,l).$$

These polynomials were discussed by Zernike und Brinkman [10], Šapiro [8], Koornwinder [5, pp. 18, 19], Boyd [2].

As a consequence of the obvious symmetries of the region S and the weight function $(\mu(z,\bar{z}))^\alpha$ there are the symmetry relations

$$(4.2) \qquad p^\alpha_{m,n}(z,\bar{z}) = p^\alpha_{n,m}(\bar{z},z) = \overline{p^\alpha_{n,m}(z,\bar{z})}$$

$$= e^{(n-m)2\pi i/3}\, p^\alpha_{m,n}(e^{2\pi i/3}z, e^{-2\pi i/3}\bar{z})$$

This is proved by verifying that the last three members of these equalities all satisfy conditions (i) and (ii) of Definition 4.2.

COROLLARY 4.3. Let $p^\alpha_{m,n}(z,\bar{z}) = \sum c_{k,l} z^k \bar{z}^l$. Then $c_{k,l}$ is real, and $c_{k,l} = 0$ if $k - l \neq m - n \pmod 3$.

5. The polynomials $p^{\alpha}_{m,n}$ as eigenfunctions of a second order differential operator D^{α}_{1}

In this section an explicit second order partial differential operator D^{α}_{1} is introduced for which the polynomials $p^{\alpha}_{m,n}$ are eigenfunctions. First the case that $\alpha = \pm \frac{1}{2}$ is obtained by transforming (2.15) in terms of the coordinates $z,\bar{z}$. Next the operator D^{α}_{1} is constructed as a generalization of $D_1^{-\frac{1}{2}}$ and $D_1^{\frac{1}{2}}$ such that it is a self-adjoint operator with respect to the appropriate weight function. Finally it is proved that $D^{\alpha}_{1}p^{\alpha}_{m,n}$ = const. $p^{\alpha}_{m,n}$. This is done in a similar way as in [6, §4].

Given a partial differential operator D in terms of σ and τ it is a tedious job to express D in terms of z and $\bar{z}$ by a straightforward transformation of variables. The recurrence relation (5.1) below can be very helpful in doing such calculations.

LEMMA 5.1. For fixed $\alpha > -5/6$ let $D = \sum c_{m,n}(z,\bar{z})(\partial/\partial z)^m(\partial/\partial\bar{z})^n$ be a differential operator of order N such that $Dp^{\alpha}_{m,n} = \lambda_{m,n}\, p^{\alpha}_{m,n}$, $m,n \geq 0$. Then the coefficients $c_{m,n}$ are uniquely determined by the eigenvalues $\lambda_{k,l}$, $k + l \leq N$, and $c_{m,n}(z,\bar{z}) = \text{const.}\, z^m\bar{z}^n + \pi_{m+n-1}(z,\bar{z})$.

Proof. Substituting the explicit expression for D in the differential equation $Dp^{\alpha}_{m,n} = \lambda_{m,n}\, p^{\alpha}_{m,n}$ and using that $p^{\alpha}_{m,n}(z,\bar{z}) = z^m\bar{z}^n + \pi_{m+n-1}(z,\bar{z})$ we obtain the recurrence relation

$$m!n!c_{m,n} = \lambda_{m,n}p^{\alpha}_{m,n} - \sum_{k+l\,<\,m+n} c_{k,l}\,\frac{\partial^{k+l}p^{\alpha}_{m,n}}{\partial z^k\,\partial\bar{z}^l}\,. \tag{5.1}$$

The lemma follows by complete induction with respect to m + n. Q.e.d.

If D is a differential operator and f is a function then let $D \circ f$ be defined as an operator such that $(D\circ f)(g) = D(fg)$ for each function g, whenever the differentiations can be performed.

Let us define

$$(5.2) \qquad D_1^{-\frac{1}{2}} = \frac{1}{6}(X_1^2 + X_2^2 + X_3^2),$$

$$(5.3) \qquad D_1^{\frac{1}{2}} = \frac{1}{6}(e_{1,1}^-)^{-1}(X_1^2 + X_2^2 + X_3^2) \circ e_{1,1}^- - 3.$$

Since $(X_1^2 + X_2^2 + X_3^2)\, e_{1,1}^- = 18\, e_{1,1}^-$ by (2.15) and using that $w = \text{const.}\ e_{1,1}^-$ we can rewrite formula (5.3) as

$$(5.4) \qquad D_1^{\frac{1}{2}} = \frac{1}{6}(X_1^2 + X_2^2 + X_3^2) + \frac{1}{3}w^{-1}[(X_1 w)X_1 + (X_2 w)X_2 + (X_3 w)X_3]$$

$$= \frac{1}{6}w^{-2}[(X_1 \circ w^2)X_1 + (X_2 \circ w^2)X_2 + (X_3 \circ w^2)X_3].$$

It follows from (2.15), (3.3), (5.2), (3.7) and (5.3) that

$$(5.5) \qquad D_1^{-\frac{1}{2}}\, p_{m,n}^{-\frac{1}{2}} = (m^2+n^2+mn)\, p_{m,n}^{-\frac{1}{2}},$$

$$(5.6) \qquad D_1^{\frac{1}{2}}\, p_{m,n}^{\frac{1}{2}} = (m^2+n^2+mn+3m+3n)\, p_{m,n}^{\frac{1}{2}}.$$

The operators $D_1^{-\frac{1}{2}}$ and $D_1^{\frac{1}{2}}$ can be expressed in terms of z and $\bar{z}$ by using (5.5), (5.6) and (5.1). It already follows from (5.2) and (5.4) that $D_1^{-\frac{1}{2}}$ and $D_1^{\frac{1}{2}}$ have the same second order part. Furthermore, since $dI_3(D_1^{-\frac{1}{2}}) = D_1^{-\frac{1}{2}}$ and $dI_3(D_1^{\frac{1}{2}}) = D_1^{\frac{1}{2}}$ by (2.13), (2.9), (5.2) and (5.3), both operators $D_1^{-\frac{1}{2}}$ and $D_1^{\frac{1}{2}}$ remain invariant if z and $\bar{z}$ are interchanged. We obtain that

$$(5.7) \qquad D_1^{-\frac{1}{2}} = (z^2-3\bar{z})\frac{\partial^2}{\partial z^2} + (z\bar{z}-9)\frac{\partial^2}{\partial z \partial \bar{z}} + (\bar{z}^2-3z)\frac{\partial^2}{\partial \bar{z}^2}$$

$$+ z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial z},$$

$$(5.8) \qquad D_1^{\frac{1}{2}} = (z^2-3\bar{z})\frac{\partial^2}{\partial z^2} + (z\bar{z}-9)\frac{\partial^2}{\partial z \partial \bar{z}} + (\bar{z}^2-3z)\frac{\partial^2}{\partial \bar{z}^2}$$

$$+ 4z\frac{\partial}{\partial z} + 4\bar{z}\frac{\partial}{\partial \bar{z}}.$$

It follows from (5.7), (5.8), (5.2) and (5.4) that

$$(5.9)\qquad w^{-1}[(X_1w)X_1 + (X_2w)X_2 + (X_3w)X_3] = 9\,(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}).$$

Next we define as a generalization of (5.2) and (5.4) the operator

$$(5.10)\qquad D_1^\alpha = \frac{1}{6}(X_1^2+X_2^2+X_3^2)+\frac{1}{6}(2\alpha+1)w^{-1}[(X_1w)X_1 +(X_2w)X_2+(X_3w)X_3]$$

$$= \frac{1}{6}w^{-2\alpha-1}[(X_1\circ w^{2\alpha+1})X_1 + (X_2\circ w^{2\alpha+1})X_2 + (X_3\circ w^{2\alpha+1})X_3].$$

This choice of D_1^α is motivated by the following lemma.

LEMMA 5.2. Let $\alpha > -5/6$. Let f and g be invariant trigonometric polynomials in σ and τ. Then the same holds for $D_1^\alpha f$ and $D_1^\alpha g$ and

$$(5.11)\qquad \iint_R (D_1^\alpha f)\ \bar{g}\ w^{2\alpha+1} d\sigma\, d\tau = \iint_R f\,\overline{(D_1^\alpha g)}\, w^{2\alpha+1} d\sigma\, d\tau$$

Proof. The invariance of $D_1^\alpha f$ follows by (2.13) and the fact that $w = \text{const.}\ e^-_{1,1}$. The functions $(X_1^2+X_2^2+X_3^2)f$ and $(X_1w)(X_1f)+(X_2w)(X_2f) + (X_3w)(X_3f)$ are clearly trigonometric polynomials. This last function, which we denote by F, is anti-invariant. Hence, by Lemma 2.5 and Theorem 3.3 the function $w^{-1}F$ is a trigonometric polynomial in σ and τ, and thus the same is true for $D_1^\alpha f$. Next we have to prove (5.11). By (2.11) and (5.10) we have $\overline{D_1^\alpha} = D_1^\alpha$ so we may prove as well that

$$(5.12)\qquad \iint_R D_1^\alpha f)\ g\ w^{2\alpha+1} d\sigma d\tau = \iint_R f(D_1^\alpha g)\ w^{2\alpha+1}\ d\sigma\ d\tau$$

where f and g are trigonometric polynomials in σ and τ. Both sides of (5.12) are well-defined and analytic in α if $\operatorname{Re}\alpha > -5/6$. If $\alpha > -\frac{1}{2}$ then $w^{2\alpha+1}$ vanishes on ∂R and it follows by integration by parts and by application of Gauss's theorem that

$$\iint_R (D_1^\alpha f)\ g\ w^{2\alpha+1} d\sigma\ d\tau = \frac{1}{6} \iint_R [(X_1 \circ w^{2\alpha+1}) X_1 f$$

$$+ (X_2 \circ w^{2\alpha+1}) X_2 f + (X_3 \circ w^{2\alpha+1}) X_3 f]\ g\ d\sigma\ d\tau$$

$$= -\frac{1}{6} \iint_R [(X_1 f)(X_1 g) + (X_2 f)(X_2 g) + (X_3 f)(X_3 g)] w^{2\alpha+1} d\sigma\ d\tau.$$

By reversing the roles of f and g it is seen that this expression equals $\iint_R f(D_1{}^\alpha g)\ w^{2\alpha+1} d\sigma\ d\tau$. So (5.12) is proved for $\alpha > -\frac{1}{2}$. The general case follows by analytic continuation. Q.e.d.

For particular values of α the operator D_1^α has an interpretation on certain symmetric spaces. First note that by (2.11) the operator D_1^α can be expressed in terms of s and t as

$$(5.13) \qquad -\frac{4}{3} D_1^\alpha = w^{-2\alpha-1} [(\frac{\partial}{\partial s} \circ w^{2\alpha+1}) \frac{\partial}{\partial s} + (\frac{\partial}{\partial t} \circ w^{2\alpha+1}) \frac{\partial}{\partial t}],$$

where $w(\sigma,\tau) = \sin s\ \sin(\frac{1}{2}s+\frac{1}{2}\sqrt{3}t)\ \ \sin(\frac{1}{2}s-\frac{1}{2}\sqrt{3}t)$.

Consider a compact Riemannian symmetric space of rank two for which the restricted root vectors have Dynkin diagram 0 – 0 and multiplicity $2\alpha + 1$. Then it follows from Harish-Chandra [3,p.270, Cor.1] that the operator $-(4/3)\ D_1^\alpha$ given by (5.13) denotes the radial part of the Laplace-Beltrami operator on such a symmetric space. By Araki [1, pp. 32,33] the only possibilities are the spaces SU(3)/SO(3) (α=0), SU(6)/Sp(3) (α=3/2) and the exceptional space E IV (α=7/2).

Using (5.2), (5.7), (5.9) and (5.10) we find that D_1^α can be expressed in terms of z and $\bar{z}$ as

(5.14) $$D_1^\alpha = (z^2-3\bar{z})\frac{\partial^2}{\partial z^2} + (z\bar{z}-9)\frac{\partial^2}{\partial z\partial\bar{z}} + (\bar{z}^2-3z)\frac{\partial^2}{\partial\bar{z}^2}$$

$$+ 3(\alpha+\tfrac{5}{6})\ (z\frac{\partial}{\partial z}+\bar{z}\frac{\partial}{\partial\bar{z}}).$$

It follows from (5.14) that

(5.15) $$D_1^\alpha(z^m\bar{z}^n) = (m^2+mn+n^2+3(\alpha+\tfrac{1}{2})(m+n))z^m\bar{z}^n + \pi_{m+n-1}(z,\bar{z}).$$

Since $w^{2\alpha+1}d\sigma d\tau$ = const. $\mu^\alpha dx\,dy$ we conclude from Lemma 5.2 that

(5.16) $$\iint_S (D_1^\alpha p)\bar{q}\ \mu^\alpha\ dx\ dy = \iint_S p\overline{(D_1^\alpha q)}\ \mu^\alpha\ dx\ dy$$

for arbitrary polynomials $p(z,\bar{z})$ and $q(z,\bar{z})$.

THEOREM 5.3. Let $\alpha > -5/6$. Then

(5.17) $$D_1^\alpha\ p_{m,n}^\alpha = (m^2+mn+n^2+3(\alpha+\tfrac{1}{2})(m+n))p_{m,n}^\alpha.$$

Proof. It follows from (5.15) that

$$D_1^\alpha\ p_{m,n}^\alpha(z,\bar{z}) = \text{const. } z^m\bar{z}^n + \pi_{m+n-1}(z,\bar{z}).$$

Let $q(z,\bar{z})$ be a polynomial of degree less than m+n. Then the same holds for $D_1^\alpha q$ and by (5.16) and Definition 4.2 we have

$$\iint_S (D_1^\alpha p_{m,n}^\alpha)\ \bar{q}\ \mu^\alpha dx\ dy = \iint_S p_{m,n}^\alpha\overline{(D_1^\alpha q)}\ \mu^\alpha dx\ dy = 0.$$

Hence, by Definition 4.2, $D_1^\alpha p_{m,n}^\alpha$ = const. $p_{m,n}^\alpha$, where the constant is given by (5.15). Q.e.d.

COROLLARY 5.4. Formula (4.1) holds if $m + n = k + 1$ and $(m,n) \neq (k,l) \neq (n,m)$.

Proof. Let $\lambda^{\alpha}_{m,n}$ denote the eigenvalue in (5.17). Then $\lambda^{\alpha}_{m,n} = \frac{3}{4}(m+n)^2 + 3(\alpha+\frac{1}{2})(m+n) + \frac{1}{4}(m-n)^2$, so $\lambda^{\alpha}_{m,n} \neq \lambda^{\alpha}_{k,l}$ under the given conditions. The result follows by (5.16) and (5.17). Q.e.d.

6. The polynomials $p^{\alpha}_{m,n}$ as eigenfunctions of a third order differential operator D^{α}_2

It is clear from (2.16) that there exist third order operators $D_2^{-\frac{1}{2}}$ and $D_2^{\frac{1}{2}}$ for which the polynomials $p^{-\frac{1}{2}}_{m,n}$ and $p^{\frac{1}{2}}_{m,n}$, respectively, are eigenfunctions. In this section a third order operator D^{α}_2 will be constructed as a generalization of $D_2^{-\frac{1}{2}}$ and $D_2^{\frac{1}{2}}$, such that D^{α}_2 is self-adjoint on R with respect to the weight function $w^{2\alpha+1}$. Then it will be proved that the polynomials $p^{\alpha}_{m,n}$ are eigenfunctions of D^{α}_2.

Let us define

(6.1) $$D_2^{-\frac{1}{2}} = X_1X_2X_3,$$

(6.2) $$D_2^{\frac{1}{2}} = (e^{-}_{1,1})^{-1}\ X_1X_2X_3\circ e^{-}_{1,1}.$$

Since $X_1X_2X_3e^{-}_{1,1} = 0$ by (2.16) and using that $w = \text{const.}\ e^{-}_{1,1}$ we can rewrite formula (6.2) as

(6.3) $$D_2^{\frac{1}{2}} = X_1X_2X_3 + w^{-1}[(X_1w)X_2X_3 + (X_2w)X_3X_1 + (X_3w)X_1X_2] + w^{-1}[(X_1X_2w)X_3 + (X_2X_3w)X_1 + (X_3X_1w)X_2].$$

It follows from (2.16), (2.3), (6.1), (3.7) and (6.2) that

(6.4) $$D_2^{-\frac{1}{2}}\ p^{-\frac{1}{2}}_{m,n} = (m-n)(2m+n)(m+2n)\ p^{-\frac{1}{2}}_{m,n},$$

(6.5) $$D_2^{\frac{1}{2}}\ p^{\frac{1}{2}}_{m,n} = (m-n)(2m+n+3)(m+2n+3)\ p^{\frac{1}{2}}_{m,n}.$$

By comparing (6.1) and (6.3) it follows that $D_2^{-\frac{1}{2}}$ and $D_2^{\frac{1}{2}}$ have the same third order part. By (2.13), (2.9), (6.1) and (6.2) we have $dI_3\,(D_2^\alpha) = -\,D_2^\alpha$, $\alpha = \pm\frac{1}{2}$. In terms of z and $\bar z$ this means that the operator D_2^α, $\alpha = \pm\frac{1}{2}$, is transformed into $-D_2^\alpha$ if z and $\bar z$ are interchanged. Using these properties and formulas (6.4), (6.5), (5.1) we can express $D_2^{-\frac{1}{2}}$ and $D_2^{\frac{1}{2}}$ in terms of z and $\bar z$ as

$$(6.6)\qquad D_2^{-\frac{1}{2}} = (2z^3-9z\bar z+27)\frac{\partial^3}{\partial z^3} + (3z^2\bar z-18\bar z^2+27z)\frac{\partial^3}{\partial z^2\partial\bar z} + (-3z\bar z^2+18z^2-27\bar z)\frac{\partial^3}{\partial z\partial\bar z^2} + (-2\bar z^3+9z\bar z-27)\frac{\partial^3}{\partial\bar z^3} + (6z^2-18\bar z)\frac{\partial^2}{\partial z^2} + (-6\bar z^2+18z)\frac{\partial^2}{\partial\bar z^2} + 2z\frac{\partial}{\partial z} - 2\bar z\frac{\partial}{\partial\bar z},$$

$$(6.7)\qquad D_2^{\frac{1}{2}} = (2z^3-9z\bar z+27)\frac{\partial^3}{\partial z^3} + (3z^2\bar z-18\bar z^2+27z)\frac{\partial^3}{\partial z^2\partial\bar z} + (-3z\bar z^2+18z^2-27\bar z)\frac{\partial^3}{\partial z\partial\bar z^2} + (-2\bar z^3+9z\bar z-27)\frac{\partial^3}{\partial\bar z^3} + (15z^2-45\bar z)\frac{\partial}{\partial z^2} + (-15\bar z^2+45z)\frac{\partial^2}{\partial\bar z^2} + 20z\frac{\partial}{\partial z} - 20\bar z\frac{\partial}{\partial\bar z}.$$

A calculation shows that

$$(6.8)\qquad w^{-1}[(X_1X_2w)X_3 + (X_2X_3w)X_1 + (X_3X_1w)X_2] = 9z\frac{\partial}{\partial z} - 9\bar z\frac{\partial}{\partial\bar z}.$$

This can be done in the following way. It is clear that the left hand side of (6.8) takes the form $c(z,\bar z)(\partial/\partial z) - c(\bar z,z)(\partial/\partial\bar z)$ for some function $c(z,\bar z)$. By using Lemma 2.5 it follows that $(X_1X_2e^-_{1,1})(X_3e^+_{1,0}) + (X_2X_3e^-_{1,1})\,(X_1e^+_{1,0}) + (X_3X_1e^-_{1,1})(X_2e^+_{1,0}) = 9\,e^-_{2,1}$. Hence $c(z,\bar z) = 9\,e^-_{2,1}\,/\,e^-_{1,1} = 9\,p^{\frac{1}{2}}_{1,0}\,(z,\bar z) = 9\,z$.

By comparing (6.1), (6.3), (6.6), (6.7) and (6.8) we obtain that

(6.9) $$w^{-1}[(X_1w)\,X_2X_3 + (X_2w)\,X_3X_1 + (X_3w)\,X_1X_2]$$
$$= 9\,(z^2-3\bar{z})\,\frac{\partial^2}{\partial z^2} + 9(-\bar{z}^2+3z)\,\frac{\partial^2}{\partial \bar{z}^2} + 9z\,\frac{\partial}{\partial z} - 9\bar{z}\,\frac{\partial}{\partial z}\,.$$

Consider the differential operator $D = \sum c_{m,n}(\sigma,\tau)(\partial/\partial\sigma)^m(\partial/\partial\tau)^n$, where $c_{m,n}$ is a C^{m+n}-function on R. Then the formal adjoint D^* of D is defined as the operator

$$D^* = \sum_{m,n} (-1)^{m+n}\,\frac{\partial^{m+n}}{\partial\sigma^m\partial\tau^n} \circ \overline{c_{m,n}(\sigma,\tau)}.$$

By using that $\overline{X}_k = -X_k$ and $X_k^* = X_k$ (k=1,2,3) it follows from (6.1) and (6.2) that $(D_2^{-\frac{1}{2}})^* = D_2^{-\frac{1}{2}}$ and $(D_2^{\frac{1}{2}})^* = w^2D_2^{\frac{1}{2}} \circ w^{-2}$. Generalizing $D_2^{-\frac{1}{2}}$ and $D_2^{\frac{1}{2}}$ we try to find an operator D_2^α which takes the form

(6.10) $$D_2^\alpha = X_1X_2X_3 + A_\alpha w^{-1}[(X_1w)X_2X_3 + (X_2w)X_3X_1 + (X_3w)X_1X_2]$$
$$+ B_\alpha w^{-1}[(X_1X_2w)X_3 + (X_2X_3w)X_1 + (X_3X_1w)X_2]$$

for certain constant coefficients A_α and B_α and which satisfies

(6.11) $$(D_2^\alpha)^* = w^{2\alpha+1}D_2^\alpha \circ w^{-2\alpha-1}.$$

LEMMA 6.1. Formula (6.11) holds if $A_\alpha = \alpha + \frac{1}{2}$, $B_\alpha = (\alpha+\frac{1}{2})^2$. If $\alpha \neq 0$ then A_α and B_α are uniquely determined by (6.10) and (6.11).

Proof. It follows from (6.10) that

$$w^{-2\alpha-1}(D_2^{\alpha})^* \circ w^{2\alpha+1} = w^{-2\alpha-1} X_1X_2X_3 \circ w^{2\alpha+1}$$

$$-A_{\alpha}w^{-2\alpha-1}[X_2X_3\circ(X_1w) + X_3X_1\circ(X_2w) + X_1X_2\circ(X_3w)]\circ w^{2\alpha}$$

$$+ B_{\alpha}w^{-2\alpha-1}[X_3\circ(X_1X_2w) + X_1\circ(X_2X_3w) + X_2\circ(X_3X_1w)]\circ w^{2\alpha}.$$

By a straightforward calculation this becomes

$$(6.12) \qquad w^{-2\alpha-1}(D_2^{\alpha})^*\circ w^{2\alpha+1} = X_1X_2X_3$$

$$+ (2\alpha+1-A_{\alpha})w^{-1}[(X_1w)X_2X_3 + (X_2w)X_3X_1 + (X_3w)X_1X_2]$$

$$+ (2\alpha+1-2A_{\alpha}+B_{\alpha})w^{-1}[(X_1X_2w)X_3 + (X_2X_3w)X_1 + (X_3X_1w)X_2]$$

$$+ (2\alpha(2\alpha+1)-4\alpha A_{\alpha})w^{-2}[(X_1w)(X_2w)X_3 +(X_2w)(X_3w)X_1+(X_3w)(X_1w)X_2]$$

$$+ (2\alpha(2\alpha+1)^2-6\alpha(2\alpha+1)A_{\alpha}+ 4\alpha B_{\alpha})w^{-3}(X_1w)(X_2w)(X_3w) .$$

Here it is used that $X_1X_2X_3w = 0$ and that

$$w^{-2}[(X_1w)(X_2X_3w) + (X_2w)(X_3X_1w) + (X_3w) (X_1X_2w)]$$

$$= 2w^{-3}(X_1w) (X_2w) (X_3w) .$$

This last identity holds since, using Lemma 2.5, we have
$(X_1e^-_{1,1}) (X_2X_3 e^-_{1,1}) + (X_2e^-_{1,1})(X_3X_1e^-_{1,1}) + (X_3e^-_{1,1})(X_1X_2e^-_{1,1})$
$= -54 (e^+_{3,0}-e^+_{0,3}) = - 54(p^{-\frac{1}{2}}_{3,0}-p^{-\frac{1}{2}}_{0,3}) = - 54(z^3-\bar{z}^3)$ and

$(e^-_{1,1})^{-1}(X_1e^-_{1,1})(X_2e^-_{1,1})(X_3e^-_{1,1})$

$= -27(\bar{e}_{1,1})^{-1}(\bar{e}_{4,1}-\bar{e}_{1,4}) = -27(p^{\frac{1}{2}}_{3,0}-p^{\frac{1}{2}}_{0,3}) = -27(z^3-\bar{z}^3)$.

The differential operators given by (6.10) and (6.12) must be equal. Comparing the second order parts we find that $A_\alpha = \alpha + \frac{1}{2}$, and next comparing the parts of zero order we obtain that $B_\alpha = (\alpha+\frac{1}{2})^2$ if $\alpha \neq 0$ and B_α is arbitrary if $\alpha = 0$. Q.e.d.

Because of the previous lemma we define D^α_2 as the operator

$$(6.13)\qquad D^\alpha_2 = X_1X_2X_3 + (\alpha+\tfrac{1}{2})w^{-1}[(X_1w)X_2X_3+(X_2w)X_3X_1+(X_3w)X_1X_2]$$
$$+ (\alpha+\tfrac{1}{2})^2w^{-1}[(X_1X_2w)X_3+(X_2X_3w)X_1+(X_3X_1w)X_2].$$

Formulas (6.1) and (6.3) are special cases of (6.13).

LEMMA 6.2. Let $\alpha > -5/6$. Let f and g be invariant trigonometric polynomials in σ and τ. Then the same holds for $D^\alpha_2 f$ and $D^\alpha_2 g$, and

$$(6.14)\qquad \iint_R (D^\alpha_2 f)\, \bar{g}\, w^{2\alpha+1} d\sigma\, d\tau = \iint_R f\,\overline{(D^\alpha_2 g)}\, w^{2\alpha+1} d\sigma\, d\tau$$

Proof. Similar to the proof of Lemma 5.2 it can be shown that $D^\alpha_2 f$ and $D^\alpha_2 g$ are invariant trigonometric polynomials in σ and τ. Because of (6.11) we try to prove (6.14) by repeated integration by parts and by application of Gauss's theorem. Thus we obtain a string of equalities connecting both sides of (6.14) with each other. However, this only proves (6.14) if all surface integrals occurring in these equalities converge and if all line integrals occurring in the equalities vanish. It follows from (6.13) that these integrals take the forms $\iint_R h\, w^{2\alpha-2} d\sigma\, d\tau$ and $\iint_{\partial R} h\, w^{2\alpha-1}(c_1 d\sigma + c_2 d\tau)$, where h is some trigonometric polynomial in σ and τ, and c_1 and c_2 are constants. Thus the result is clear if $\alpha > 2/3$. The case that $-5/6 < \alpha \leq 2/3$ can be proved as follows. By (2.11) and (6.13) we have $D^\alpha_2 = -\overline{D^\alpha_2}$, so (6.14) is equivalent to

$$(6.15)\qquad \iint_R (D_2^\alpha f) g w^{2\alpha+1} d\sigma\, d\tau = - \iint_R f(D_2^\alpha g) w^{2\alpha+1} d\sigma\, d\tau,$$

f,g trigonometric polynomials. Both sides of (6.15) are well-defined and analytic in α if Re $\alpha > -5/6$. Since (6.15) is already proved for $\alpha > 2/3$, the general case of (6.15) follows by analytic continuation. Q.e.d.

Using (6.1), (6.6), (6.8), (6.9) and (6.13) we find that D_2^α can be expressed in terms of z and $\bar{z}$ as

$$(6.16)\qquad D_2^\alpha = (2z^3-9z\bar{z}+27)\frac{\partial^3}{\partial z^3} + (3z^2\bar{z}-18\bar{z}^2+27z)\frac{\partial^3}{\partial z^2\partial\bar{z}}$$

$$+ (-3z\bar{z}^2+18z^2-27\bar{z})\frac{\partial^3}{\partial z\partial\bar{z}^2} + (-2\bar{z}^3+9z\bar{z}-27)\frac{\partial^3}{\partial\bar{z}^3}$$

$$+9(\alpha+\tfrac{7}{6})[(z^2-3\bar{z})\frac{\partial^2}{\partial z^2} + (-\bar{z}^2+3z)\frac{\partial^2}{\partial\bar{z}^2}]$$

$$+9(\alpha+\tfrac{7}{6})(\alpha+\tfrac{5}{6})(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial\bar{z}}) .$$

It follows from (6.16) that

$$(6.17)\qquad D_2^\alpha(z^m\bar{z}^n) = (m-n)(2m+n+3\alpha+3/2)(m+2n+3\alpha+3/2)\, z^m\,\bar{z}^n$$

$$+ \pi_{m+n-1}(z,\bar{z}).$$

Since $w^{2\alpha+1}d\sigma\, d\tau = \text{const.}\ \mu^\alpha dx\, dy$ we conclude from Lemma 6.2 that

$$(6.18)\qquad \iint_S (D_2^\alpha p)\,\bar{q}\,\mu^\alpha\, dx\, dy = \iint_S p\,\overline{(D_2^\alpha q)}\,\mu^\alpha\, dx\, dy$$

for arbitrary polynomials $p(z,\bar{z})$ and $q(z,\bar{z})$. By using (6.17), (6.18) and

Definition 4.2 the following theorem is proved in the same way as Theorem 5.3.

THEOREM 6.3. Let $\alpha > -5/6$. Then

$$D_2^{\alpha} p_{m,n}^{\alpha} = (m-n)(2m+n+3\alpha+3/2)(m+2n+3\alpha+3/2)\, p_{m,n}^{\alpha}. \tag{6.19}$$

COROLLARY 6.4. Formula (4.1) holds if $(k,l) = (n,m)$ and $m \neq n$.

Proof. Let $\lambda_{m,n}^{\alpha}$ denote the eigenvalue in (6.19). Then $\lambda_{m,n}^{\alpha} = -\lambda_{n,m}^{\alpha}$, and $\lambda_{m,n}^{\alpha} \neq 0$ since $m \neq n$ and $\alpha > -5/6$. Hence $\lambda_{m,n}^{\alpha} \neq \lambda_{n,m}^{\alpha}$. The result follows by (6.18) and (6.19). Q.e.d.

Thus it is finally proved that the orthogonality relations (4.1) hold whenever $(m,n) \neq (k,l)$.

7. The algebra of differential operators generated by D_1^α and D_2^α

In this last section we consider the class, denoted by A^α, of all partial differential operators in z and $\bar{z}$ which admit the polynomials $p_{m,n}^\alpha$ as eigenfunctions. The methods and results are similar to [6,§6]. From now on it is supposed that the parameter α is a fixed real number larger than $-5/6$. We shall write $p_{m,n}$, D_1, D_2, A instead of $p_{m,n}^\alpha$, D_1^α, D_2^α, A^α, respectively.

Clearly, the class A is an algebra of operators and, by Lemma 5.1, this algebra is commutative. The operators D_1 and D_2 are elements of A and they generate a subalgebra A_0 of A which consists of all polynomials in D_1 and D_2. It will be proved that $A_0 = A$.

LEMMA 7.1. Let $D = \sum c_{m,n}(z,\bar{z})(\partial/\partial z)^m(\partial/\partial\bar{z})^n$ be a differential operator of order N, where the coefficients $c_{m,n}$ are polynomials in z and $\bar{z}$. In terms of σ and τ let $D = \sum b_{k,l}(\sigma,\tau)(\partial/\partial\sigma)^k(\partial/\partial\tau)^l$, $(\sigma,\tau) \in R$. Then the functions $b_{k,l}$ have unique extensions to one-valued analytic functions, regular for all complex values of σ and τ except possibly on the lines $w(\sigma,\tau) = 0$. Furthermore, if the operator D is extended in this way to the region $\{(\sigma,\tau)\,|\,w(\sigma,\tau) \neq 0\}$ then D is invariant with respect to G.

Proof. A calculation shows that

$$\frac{\partial}{\partial z} = \left(\frac{\partial(z,\bar{z})}{\partial(\sigma,\tau)}\right)^{-1}\left(\frac{\partial\bar{z}}{\partial\tau}\frac{\partial}{\partial\sigma} - \frac{\partial\bar{z}}{\partial\sigma}\frac{\partial}{\partial\tau}\right)$$

$$= \frac{1}{24}\, i(w(\sigma,\tau))^{-1}[e^{i\tau}(X_2-X_3) + e^{-i\sigma}(X_3-X_1) + e^{i(\sigma-\tau)}(X_1-X_2)].$$

Hence the lemma is true for the operator $\partial/\partial z$, and similarly for the operator $\partial/\partial\bar{z}$. The lemma clearly holds for the zero operators z and $\bar{z}$. Then the general result follows immediately. Q.e.d.

In the remainder of this section δ_n will denote an arbitrary partial differential operator in σ and τ of order $\leq n$. It follows from (5.10), (6.13) and (2.12) that

$$(7.1) \qquad D_1 = \frac{1}{3}(X_1^2+X_1X_2+X_2^2) + \delta_1 ,$$

$$(7.2) \qquad D_2 = -(X_1^2X_2+X_1X_2^2) + \delta_2 .$$

LEMMA 7.2. Let $D \in A$ and write

$$(7.3) \qquad D = \sum_{k=0}^{N} c_k(\sigma,\tau)X_1^kX_2^{N-k} + \delta_{N-1} .$$

Then the coefficients c_k are constants.

Proof. Since $D \in A$, D commutes with D_1 and D_2. Put $c_k = 0$ if $k < 0$ or $k > N$. By (7.1) and (7.3) the vanishing of the terms of order $N + 1$ in the operator $DD_1 - D_1D$ implies that $(2X_1+X_2)c_{k-1} + (X_1+2X_2)c_k = 0$. Similarly, by (7.2) and (7.3) the vanishing of the terms of order $N + 2$ in the operator $DD_2 - D_2D$ implies that $X_1c_k + 2(X_1+X_2)c_{k-1} + X_2c_{k-2} = 0$. Adding and subtracting these two recurrence relations we obtain that $X_1(c_k+c_{k-1}-2c_{k-2}) = 0$, $X_2(2c_k-c_{k-1}-c_{k-2}) = 0$. Since $c_{-1} = c_{-2} = 0$, it follows by complete induction with respect to k that $X_1c_k = 0$ and $X_2c_k = 0$. Hence c_k is constant. Q.e.d.

THEOREM 7.3. Let $D \in A$. Then D can be expressed in one and only one way as a polynomial in the operators D_1 and D_2.

Proof. Suppose that there exist elements of A which can not be expressed as polynomials in D_1 and D_2 and let D be such an operator of minimal order N. Then by Lemma 7.2 $D = Q(X_1,X_2) + \delta_{N-1}$, where Q is a homogeneous polynomial of degree N. By Lemma 5.1 and Lemma 7.1 D is invariant with respect to G. Hence the operator $Q(X_1,X_2)$ is invariant, so it follows from Lemma 2.7 and formulas (7.1), (7.2) that $D = P(D_1,D_2) + \delta_{N-1}$ for some polynomial P. Hence $D - P(D_1,D_2) \in A$. Then, by hypothesis, $D - P(D_1,D_2)$ is a polynomial in D_1 and D_2, so D is a polynomial in D_1 and D_2. This is a contradiction.

Next we turn to the uniqueness part of the theorem. Suppose that there exists a non-zero polynomial P in two variables such that $P(D_1,D_2)$ is the zero operator. The polynomial $P(x,y)$ can be expressed as a sum of terms $c_{k,l}\,(3x)^{k/2-l}(-y)^{l}$, where $k/2$ and l are integers and $k/2 \geq l \geq 0$. Among the pairs of integers (k,l) such that $c_{k,l} \neq 0$ there is a maximal element (m,n) with respect to lexicographic ordering. Then by (7.1) and (7.2) $c_{m,n}$ is the coefficient of $X_1^m X_2^n$ in the operator $P(D_1,D_2)$. Hence $c_{m,n} = 0$. This is a contradiction. Q.e.d.

It follows from Theorem 7.3 that the operators D_1 and D_2 are algebraically independent, i.e., if P is a polynomial in two variables and if $P(D_1,D_2)$ is the zero operator then P is the zero polynomial.

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