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AFDELING TOEGEPASTE WISKUNDE TW 138/75 NOVEMBER

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THE BIRTH OF A BOUNDARY LAYER IN AN ELLIPTIC SINGULAR PERTURBATION PROBLEM

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 35B25, 35J25

The birth of a boundary layer in an elliptic singular perturbation problem

by

J. Grasman

ABSTRACT

In elliptic singular perturbation problems two types of boundary layers may arise. In a particular example it is shown that there exists a smooth transition from one type of boundary layer to the other.

KEY WORDS & PHRASES: *elliptic singular perturbation problem, parabolic boundary layer.*

1. INTRODUCTION

We investigate the Dirichlet problem for the second order, linear, elliptic differential equation

$$(1) \qquad L_\varepsilon \phi \equiv (\varepsilon L_2 + L_1)\phi = 0, \qquad 0 < \varepsilon \ll 1,$$

in a bounded convex domain G of $\mathbb{R}_2$. When L_1 is a first order differential operator with constant coefficients the solution of the problem exhibits a well-known boundary layer structure. There exists a large number of papers dealing with this subject, we mention Visik and Lyusternik (1962) and Eckhaus and De Jager (1966). In these studies it is remarked that some difficulties arise, if one tries to approximate ϕ in a neighborhood of a point where the subcharacteristic of L_1 is tangent to the boundary of G. In Grasman (1971) it is shown that the seemingly singular behaviour of the asymptotic approximation is due to the presence of corner layers, which are visualized by applying the coordinate stretching method to both coordinates. This technique has been suggested by Eckhaus (1968).

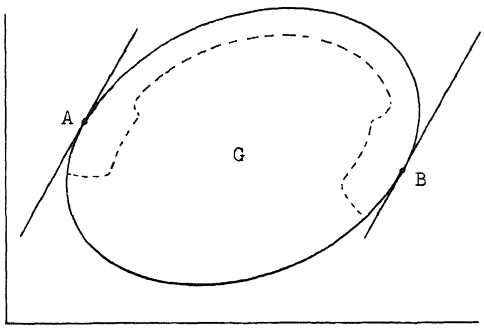

Fig. 1.

Let us first consider the case where we have $(2n-1)^{th}$ order of tangency (n=1,2,...) of the subcharacteristics in the points A en B, which divide the boundary into two segments. It appears that then we have a boundary layer of thickness $0(\varepsilon)$ along one of these segments depending on the coefficients of L_1. This can be proved with the maximum principle for differential equations. Near A and B there are layers of thickness $0(\varepsilon^{\nu})$ and length $0(\varepsilon^{\mu})$, $\nu = 2n/(4n-1)$ and $\mu = 1/(4n-1)$. For $n \to \infty$ such a layer tends to a so-called parabolic boundary layer. In this paper we will attain this limit situation in a different way.

The differential equation which will be studied in particular is of the form

$$(2) \qquad \varepsilon\Delta\phi + a\frac{\partial\phi}{\partial y} - b\frac{\partial\phi}{\partial y} - c\phi = 0 \qquad (a\geq 0,\ b>0).$$

For the domain $G = \{x,y;\ x \geq 0,\ y \geq 0\}$ the boundary values are

$$(3a) \qquad \phi(x,0) = 0 \qquad \text{for } x \geq 0,$$

and

$$(3b) \qquad \phi(0,y) = f(y) \qquad \text{for } y > 0.$$

Since the domain is not bounded, we only consider solutions that satisfy a boundedness condition at infinity. For the solution in rectangular domain we refer to Grasman (1974).

For $\varepsilon \to 0$ equation (2) degenerates to a first order differential equation. It can be demonstrated that the solution of this reduced equation only satisfies condition (3a). This trivial solution will hold in the greater part of the domain G, only a neighborhood of the line $x = 0$ needs to be excluded. Near this line ϕ has a boundary layer structure. Special attention will be given to the corner layer near the point (0,0). This is done by stretching both coordinates as follows

$$(4) \qquad x = \xi\varepsilon^{\alpha}, \qquad y = \eta\varepsilon^{\beta}.$$

Near $x = 0$ we may expect an ordinary boundary layer of thickness $0(\varepsilon)$ for $a \geq \delta > 0$ (δ arbitrary small but independent of ε) and a parabolic boundary layer of thickness $O(\sqrt{\varepsilon})$, for $a = 0$. This would suggest that a parabolic boundary layer is an unusual phenomenon in physical problems.

It is the aim of this study to demonstrate that there is a smooth transition from one type of boundary layer to the other for $a \to 0$. In such a transition interval the boundary layer will have properties of both the parabolic and the ordinary boundary layer.

2. THE ORDINARY BOUNDARY LAYER

Subsituting (4) into (2) and letting $\varepsilon \to 0$ we obtain various limiting equations depending on the values of α, β and a. Taking into account the matching principle and the boundary conditions we come to a certain number of significant asymptotic approximations satisfying some limiting equation.

For $a \geq \delta > 0$ we distinguish the following significant cases

(a) *The ordinary boundary layer*

For $\alpha = 1$ and $\beta = 0$ we have the limiting equation

$$(5) \qquad \frac{\partial^2 U}{\partial \xi^2} + a \frac{\partial U}{\partial y} = 0.$$

The significant approximation satisfying boundary condition (3b) has the form

$$(6) \qquad U(\xi,y) = f(y)e^{-a\xi}.$$

(b) *The corner layer*

For $\alpha = \beta = 1$ the limiting equation is

$$(7) \qquad \frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 W}{\partial \eta^2} + a \frac{\partial W}{\partial \xi} - b \frac{\partial W}{\partial \eta} = 0$$

The boundary values for $W(\xi,\eta)$ are

$$(8) \qquad W(\xi,0) = 0, \qquad W(0,\eta) = f(0).$$

Moreover the function $W(\xi,\eta)$ is required to satisfy the matching condition

(9) $$W(\xi,\eta) = f(0)e^{-a\xi} \qquad \text{for } \eta \to \infty.$$

We obtain the following expression for $W(\xi,\eta)$:

(10) $$W(\xi,\eta) = \frac{4f(0)}{\pi i} e^{-\frac{a\xi}{2}} \int_{-\infty}^{\infty} e^{-\frac{a\xi}{2}\sqrt{a^2+b^2+4\lambda^2} + \frac{\eta}{2}(b+2i\lambda)} \frac{\lambda}{4\lambda^2+b^2} d\lambda.$$

3. THE TRANSITION BOUNDARY LAYER

For $a = a_\gamma \varepsilon^\gamma$, $0 < \gamma < 1/2$ we observe the following modifications. The corner layer solution follows from (10) by taking $a = 0$. The ordinary boundary layer increases and will have a thickness of order $0(\varepsilon^{1-\gamma})$. Moreover a new layer arises:

(c) *The intermediate layer*

The function $X(\xi,\eta)$ has to satisfy the limiting equation

(11) $$\frac{\partial^2 X}{\partial \xi^2} + a_\gamma \frac{\partial X}{\partial \xi} - b\frac{\partial X}{\partial \eta} = 0 \qquad (\alpha=1-\gamma,\ \beta=1-2\gamma).$$

The matching conditions are

(12) $$\lim_{\varepsilon\to 0} W(\varepsilon^{\alpha-1}\xi, \varepsilon^{2(\alpha-1)}\eta) = \lim_{\varepsilon\to 0} X(\varepsilon^{\alpha-1+\gamma}\xi, \varepsilon^{2(\alpha-1+\gamma)}\eta) \text{ for } 1-\gamma < \alpha < 1$$

and

(13) $$\lim_{\eta\to\infty} X(\xi,\eta) = f(0)e^{-a_\gamma \xi}.$$

It appears that the solution of this problem already has some properties of the well-known parabolic boundary layer solution for characteristic boundaries:

(14) $$X(\xi,\eta) = \frac{f(0)}{2}\{e^{-a_\gamma \xi} \operatorname{erfc}(\frac{\xi}{2}\sqrt{\frac{b}{\eta}} - \frac{a_\gamma}{2}\sqrt{\frac{\eta}{b}}) + \operatorname{erfc}(\frac{\xi}{2}\sqrt{\frac{b}{\eta}} + \frac{a_\gamma}{2}\sqrt{\frac{\eta}{b}})\}.$$

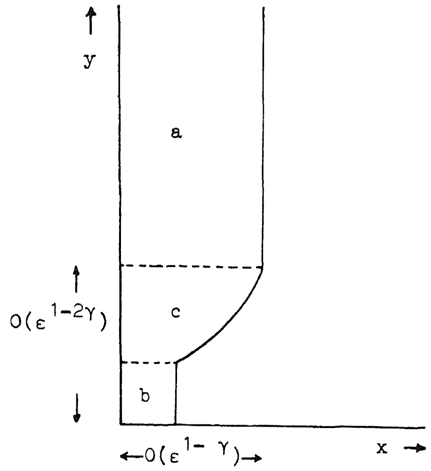

Fig. 2

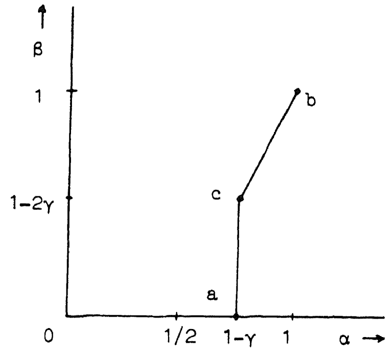

Fig. 3

4. THE PARABOLIC BOUNDARY LAYER

For $\gamma = 1/2$ the ordinary boundary layer vanishes and the intermediate layer transforms into

(d) *The parabolic boundary layer*

The parabolic boundary layer solution has to satisfy the limiting equation

$$(15)\qquad \frac{\partial^2 Z}{\partial \xi^2} + a_{\frac{1}{2}} \frac{\partial Z}{\partial \xi} - b \frac{\partial Z}{\partial y} - cZ = 0, \qquad (\alpha = 1/2,\ \beta = 0)$$

and the boundary conditions

$$(16)\qquad Z(\xi,0) = 0, \qquad Z(0,y) = f(y),$$

which yields

$$(17)\qquad Z(\xi,y) = \frac{\xi\sqrt{b}\, e^{-\frac{a_{\frac{1}{2}}}{2}\xi}}{2\sqrt{\pi}} \int_0^y \frac{f(p)\, e^{\frac{-\xi^2 b}{4(y-p)} - (\frac{1}{4}a_{\frac{1}{2}}^2 + c)(y-p)}}{(y-p)^{3/2}}\, dp.$$

Since $Z(\xi,y)$ remains regular for $a_{\frac{1}{2}} \to 0$, we may conclude that the parabolic boundary layer solution is represented by (17) in all cases where the boundary coincides with the subcharacteristic of L_1 with an accuracy of $O(\sqrt{\varepsilon})$.

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