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JACOBI POLYNOMIALS
III. AN ANALYTIC PROOF OF THE ADDITION FORMULA

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Jacobi polynomials
III. An analytic proof of the addition formula *)

T.H. Koornwinder

Abstract

The addition formula for Jacobi polynomials is derived from the integral representation for the product $P_n^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(y)$ of two Jacobi polynomials. The proof uses integration by parts and some new differentiation formulas for Jacobi polynomials. Several formulas related to the addition formula are also discussed.

*) This paper is not for review; it is meant for publication in a journal.

1. Introduction

This paper completes the analytic proof of the addition formula for Jacobi polynomials, which was initiated in [1] and [8]. In [1] Askey gave an elementary proof of the Laplace type integral representation for Jacobi polynomials. The author [8] derived from this formula the integral representation for the product $P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)$ of two Jacobi polynomials. The present paper contains the derivation of the addition formula from this product formula.

For the proof we need some new second order differential recurrence relations for Jacobi polynomials. These are obtained in §2. The addition formula can be considered as an orthogonal expansion in terms of certain functions in two variables. It can be rewritten as an expansion in orthogonal polynomials in two variables. This is discussed in §3. The addition formula is equivalent to a number of integration formulas which represent the respective terms of the orthogonal expansion. In §4 these integration formulas are derived from the product formula which was proved in [8]. This is done by repeated integration by parts and by applying the differentiation formulas obtained in §2. In a similar way the degenerate addition formula for Jacobi polynomials and a generalized addition formula for Bessel functions are obtained. Several related results are finally discussed in §5.

Three different proofs of the addition formula for Jacobi polynomials have now been published. The first two proofs applied group theoretic methods. In [4], [5], [6] it was used that certain Jacobi polynomials are spherical functions on the homogeneous space SU(q) / SU(q-1). The proof given in [7] was based on the interpretation of Jacobi polynomials as spherical harmonics. The present proof uses only analytic methods. A slightly different analytic proof by Gasper is unpublished (cf. §5, Remark 2). The author can announce yet another proof of the addition formula which is rather short and involves a certain class of orthogonal polynomials in three variables.

Remark. In the following some elementary formulas for gamma, hypergeometric and Bessel functions and for orthogonal polynomials will be used without reference. For these formulas the reader is referred to the chapters 1, 2, 7 and 10, respectively, in Erdélyi [3].

2. Some new differentiation formulas for Jacobi polynomials

Let the hypergeometric function be defined by

$$(2.1)\qquad {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n!}\, x^n, \quad |x| < 1.$$

There are a number of well-known first order differential recurrence relations for hypergeometric functions (cf.[3, §2.8, (20)-(27)] with n=1). In this section some second order differential recurrence relations for hypergeometric functions and for Jacobi polynomials will be derived, which are probably new.

Replacement of x by x^2 in (2.1) and termwise differentiation gives

$$(2.2)\qquad \left(\frac{d^2}{dx^2} + \frac{2c-1}{x}\frac{d}{dx}\right) {}_2F_1(a,b;c;x^2) = 4ab\ {}_2F_1(a+1,b+1;c;x^2).$$

Using the identity ${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\ {}_2F_1(c-a,c-b;c;x)$ we derive from (2.2) that

$$(2.3)\qquad \left(\frac{d^2}{dx^2} + \frac{2c-1}{x}\frac{d}{dx}\right) [(1-x^2)^{a+b-c+2}\ {}_2F_1(a+1,b+1;c;x^2)] =$$

$$= 4(c-a-1)(c-b-1)\ (1-x^2)^{a+b-c}\ {}_2F_1(a,b;c;x^2).$$

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be expressed as hypergeometric functions by the formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(\beta+1)_n}{n!}\ {}_2F_1(-n,n+\alpha+\beta+1;\beta+1;\tfrac{1+x}{2}).$$

Substituting this in (2.2) and (2.3) we obtain the pair of differential recurrence relations

$$(2.4)\qquad \left(\frac{d^2}{dx^2} + \frac{2\beta+1}{x}\frac{d}{dx}\right) P_n^{(\alpha,\beta)}(2x^2-1) =$$

$$= 4(n+\alpha+\beta+1)(n+\beta)\ P_{n-1}^{(\alpha+2,\beta)}(2x^2-1),$$

(2.5) $$\left(\frac{d^2}{dx^2} + \frac{2\beta+1}{x}\frac{d}{dx}\right)[(1-x^2)^{\alpha+2}\, P_{n-1}^{(\alpha+2,\beta)}(2x^2-1)] =$$

$$= 4n(n+\alpha+1)\ (1-x^2)^{\alpha}\ P_n^{(\alpha,\beta)}(2x^2-1).$$

Repeated application of (2.5) gives a Rodrigues type formula

(2.6) $$2^{2n}\ n!\ (n+\alpha+1)_n\ (1-x^2)^{\alpha}\ P_n^{(\alpha,\beta)}(2x^2-1) =$$

$$= \left(\frac{d^2}{dx^2} + \frac{2\beta+1}{x}\frac{d}{dx}\right)^n (1-x^2)^{2n+\alpha}\ .$$

If the variables x, y are expressed in the variables r, ϕ by $x = r\cos\phi$, $y = r\sin\phi$ then

$$\frac{\partial^2}{\partial r^2} + \frac{2\beta+1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{2\beta\ \mathrm{cotg}\ \phi}{r^2}\frac{\partial}{\partial\phi} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\beta}{y}\frac{\partial}{\partial y}\ .$$

Hence formula (2.6) can be rewritten as

(2.7) $$2^{2n}\ n!\ (n+\alpha+1)_n\ (1-x^2-y^2)^{\alpha}\ P_n^{(\alpha,\beta)}(2(x^2+y^2)-1) =$$

$$= (D_\beta)^n\ (1-x^2-y^2)^{2n+\alpha}\ ,$$

where D_β denotes the partial differential operator

(2.8) $$D_\beta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\beta}{y}\frac{\partial}{\partial y}\ .$$

Let the region $R = \{(x,y) \mid x^2 + y^2 < 1, y > 0\}$ denote the upper half unit disk.

LEMMA 2.1. Let f be a C^∞-function on the closed unit disk $\{(x,y) \mid x^2 + y^2 \le 1\}$ such that $f(x,y) = f(x,-y)$. Then the same holds for $D_\beta f$. Furthermore, if $\alpha > -1$ and $\beta > -\frac{1}{2}$ then

(2.9) $$\iint_R f(x,y)\, P_n^{(\alpha,\beta)}(2(x^2+y^2)-1)\, (1-x^2-y^2)^{\alpha}\, y^{2\beta}\, dx\, dy =$$
$$= \frac{1}{2^{2n}\, n!\, (n+\alpha+1)_n} \iint_R ((D_\beta)^n f(x,y))\, (1-x^2-y^2)^{2n+\alpha}\, y^{2\beta}\, dx\, dy\,.$$

Proof. It follows from (2.8) that $D_\beta f$ is a C^∞-function in x and y which is even in y. Let α be fixed and larger than -1. Both sides of (2.9) are well-defined and analytic in β if Re $\beta > -\frac{1}{2}$. Since by (2.8)

$$D_\beta = y^{-2\beta}\left(\frac{\partial}{\partial x}\left(y^{2\beta}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^{2\beta}\frac{\partial}{\partial y}\right)\right),$$

it follows by repeated integration by parts and by application of Gauss's theorem that for $k = 0,1,\ldots, n-1$ and $\beta > 0$ we have

$$\iint_R ((D_\beta)^{n-k}(1-x^2-y^2)^{2n+\alpha})\, ((D_\beta)^k f(x,y))\, y^{2\beta}\, dx\, dy =$$
$$= \iint_R ((D_\beta)^{n-k-1}(1-x^2-y^2)^{2n+\alpha})\, ((D_\beta)^{k+1} f(x,y))\, y^{2\beta}\, dx\, dy\,.$$

By these equalities and by (2.7) formula (2.9) is proved if $\alpha > -1$, $\beta > 0$. The case of general β follows by analytic continuation with respect to β.

Q.e.d.

We mention two other second order differential recurrence formulas for Jacobi polynomials, although we do not need these formulas in the following sections. If the identity

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\left(\frac{1+x}{2}\right)^n {}_2F_1\left(-n,-n-\beta;\alpha+1;\frac{x-1}{x+1}\right)$$

is substituted in (2.2) and (2.3) then we obtain the formulas

(2.10) $$\left(\frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}\right)\left((1+x^2)^n P_n^{(\alpha,\beta)}\left(\frac{1-x^2}{1+x^2}\right)\right) =$$
$$= -4(n+\alpha)(n+\beta)\, (1+x^2)^{n-1} P_{n-1}^{(\alpha,\beta)}\left(\frac{1-x^2}{1+x^2}\right),$$

$$(2.11)\qquad \left(\frac{d^2}{dx^2}+\frac{2\alpha+1}{x}\frac{d}{dx}\right)\left((1+x^2)^{-n-\alpha-\beta}\,P_{n-1}^{(\alpha,\beta)}\left(\frac{1-x^2}{1+x^2}\right)\right)=$$

$$=-4n(n+\alpha+\beta)\;(1+x^2)^{-n-\alpha-\beta-1}\;P_n^{(\alpha,\beta)}\left(\frac{1-x^2}{1+x^2}\right).$$

Repeated application of (2.11) gives a Rodrigues type formula

$$(2.12)\qquad (-1)^n\,2^{2n}\,n!\;(\alpha+\beta+1)_n\;(1+x^2)^{-n-\alpha-\beta-1}\;P_n^{(\alpha,\beta)}\left(\frac{1-x^2}{1+x^2}\right)=$$

$$=\left(\frac{d^2}{dx^2}+\frac{2\alpha+1}{x}\frac{d}{dx}\right)^n(1+x^2)^{-\alpha-\beta-1}\,.$$

This formula is particularly nice, since it expresses for fixed α and β Jacobi polynomials $P_n^{(\alpha,\beta)}$ (n=0,1,2,...) as functions which are obtained by n-fold application of a second order differential operator to an elementary function not depending on n. Tricomi obtained a simpler formula of this type for Gegenbauer polynomials C_n^λ (n=0,1,2,...), where λ is fixed. His formula [3, §10.9 (37)] involves the first order operator d/dx. There does not exist a straightforward generalization of Tricomi's formula to general Jacobi polynomials, because they cannot be written as a solution of Truesdell's F-equation (cf. Truesdell [11], Miller [9, §6.2]. However, formula (2.12) may be considered as a substitute.

3. A class of orthogonal polynomials in two variables

The addition formula for Gegenbauer polynomials (cf.[3, §3.15.1 (19)] or (5.1)) can be considered as an expansion of the function $P_n^{(\alpha,\alpha)}(xy+\sqrt{1-x^2}\,\sqrt{1-y^2}\,t)$ (x,y fixed, $\alpha > -\frac{1}{2}$) in terms of the orthogonal polynomials $P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(t)$ (k=0,1,2,...), i.e., with respect to the weight function $(1-t^2)^{\alpha-\frac{1}{2}}$ on the interval (-1,1).

Similarly, the addition formula for Jacobi polynomials (cf. [4, (3)] or (4.14)) can be considered as an orthogonal expansion of the function

$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(1+x)(1+y)+\tfrac{1}{2}(1-x)(1-y)r^2+\sqrt{1-x^2}\,\sqrt{1-y^2}\;r\cos\phi-1)$$

(x,y fixed and $\alpha > \beta > -\frac{1}{2}$) in terms of the functions $\mathcal{P}_{k,l}^{(\alpha,\beta)}(r,\phi)$ $(k\geq l\geq 0)$ defined by

$$(3.1) \qquad \mathcal{P}_{k,l}^{(\alpha,\beta)}(r,\phi) = P_l^{(\alpha-\beta-1,\beta+k-1)}(2r^2-1)\ r^{k-1}\ P_{k-l}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\phi),$$

which are orthogonal on the region $\{(r,\phi)\ |\ 0 < r < 1,\ 0 < \phi < \pi\}$ with respect to the measure $(1-r^2)^{\alpha-\beta-1}\ r^{2\beta+1}\ (\sin\phi)^{2\beta}\ dr\ d\phi$. We shall prove that in terms of suitable coordinates the functions $\mathcal{P}_{k,l}^{(\alpha,\beta)}$ are orthogonal polynomials.

Let us define the functions $P_{n,k}^{(\alpha,\beta)}(u,v)$ $(n\geq k\geq 0)$ in terms of Jacobi polynomials by

$$(3.2) \qquad P_{n,k}^{(\alpha,\beta)}(u,v) = P_k^{(\alpha,\beta+n-k+\frac{1}{2})}(2v-1)\ v^{(n-k)/2}\ P_{n-k}^{(\beta,\beta)}(v^{-\frac{1}{2}}u).$$

Since a Gegenbauer polynomial of degree n is even or odd according to whether n is even or odd it follows that $P_{n,k}^{(\alpha,\beta)}(u,v)$ is a polynomial in u and v. Comparing (3.1) and (3.2) we obtain that

$$(3.3) \qquad \mathcal{P}_{k,l}^{(\alpha,\beta)}(r,\phi) = P_{k,l}^{(\alpha-\beta-1,\beta-\frac{1}{2})}(r\cos\phi,\ r^2).$$

Let S denote the region $\{(u,v)\ |\ u^2 < v < 1\}$, bounded by the straight line $v = 1$ and by the parabola $v = u^2$ (cf. Fig. 1). The mapping $(x,y) \to (u,v)$ defined by $u = x$, $v = x^2 + y^2$ is a diffeomorphism from the upper half unit disk R onto the region S. If r, ϕ are polar coordinates on R such that $x = r\cos\phi$, $y = r\sin\phi$, then $u = r\cos\phi$, $v = r^2$ and $\partial(u,v)\ /\ \partial(r,\phi) = 2r^2\sin\phi$.

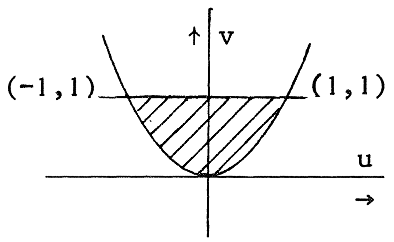

Fig. 1

THEOREM 3.1. Let α, $\beta > -1$. Then the polynomials $P_{n,k}^{(\alpha,\beta)}(u,v)$ satisfy the following properties:

$$\text{(i)} \qquad P_{n,k}^{(\alpha,\beta)}(u,v) - \frac{(n+\alpha+\beta+3/2)_k\ (n-k+2\beta+1)_{n-k}}{2^{n-k}\ k!\ (n-k)!}\ u^{n-k}\ v^k$$

is a polynomial of degree less than n.

(ii) $$\iint_S P_{n,k}^{(\alpha,\beta)}(u,v)\; P_{m,l}^{(\alpha,\beta)}(u,v)\; (1-v)^{\alpha}\; (v-u^2)^{\beta}\; du\, dv = 0$$

if $(n,k) \neq (m,l)$.

Furthermore, conditions (i) and (ii) define the polynomials $P_{n,k}^{(\alpha,\beta)}(u,v)$ uniquely.

Proof. It is clear from (3.2) that for some constant c the polynomial $P_{n,k}^{(\alpha,\beta)}(u,v) - c\, u^{n-k}\, v^{k}$ has degree less than n. The value of c follows from [3, §10.8 (5)].

To prove (ii) note that if $u = r \cos \phi$, $v = r^2$ then $(1-v)^{\alpha}\, (v-u^2)^{\beta}\, du\, dv =$ $= 2(1-r^2)^{\alpha}\, r^{2\beta+2}\, (\sin \phi)^{2\beta+1}\, dr\, d\phi$. Hence part (ii) follows by using (3.2) and the orthogonality relations for Jacobi polynomials. It is clear from (i) and (ii) that

(ii)' $$\iint_S P_{n,k}^{(\alpha,\beta)}(u,v)\; q(u,v)\; (1-v)^{\alpha}\; (v-u^2)^{\beta}\; du\, dv = 0$$

for each polynomial q of degree less than n.

Conditions (i) and (ii)' uniquely determine the polynomials $P_{n,k}^{(\alpha,\beta)}(u,v)$.

Q.e.d.

Since the region S is bounded, it follows that the polynomials $P_{n,k}^{(\alpha,\beta)}(u,v)$ form a complete orthogonal system on S with respect to the weight function $(1-v)^{\alpha}\, (v-u^2)^{\beta}$. Hence the functions $\mathcal{P}_{k,l}^{(\alpha,\beta)}(r,\phi)$ form a complete orthogonal system with respect to the weight function $(1-r^2)^{\alpha-\beta-1}\, r^{2\beta+1}\, (\sin \phi)^{2\beta}$, $0 < r < 1$, $0 < \phi < \pi$.

The author is preparing a paper in which the orthogonal polynomials $P_{n,k}^{(\alpha,\beta)}(u,v)$ and the related classes of orthogonal polynomials inside the circle and inside the triangle are discussed in more details.

4. The proof of the addition formula

It was pointed out in [8, §5] that the integral representations for a Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, for the product $P_n^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(y)$ of two Jacobi polynomials and for the product $J_\beta(x)\, J_\alpha(y)$ of two Bessel functions (cf. [8, (3.1), (3.7), (3.8)]) all have the form

$$(4.1)\qquad \int_0^1\int_0^\pi f(a^2r^2 + 2abr\cos\phi + b^2)\, dm_{\alpha,\beta}(r,\phi),$$

where f is a C^∞-function, a and b are positive real numbers, and $dm_{\alpha,\beta}(r,\phi)$ $(\alpha>\beta>-\frac{1}{2})$ denotes the measure

$$(4.2)\qquad dm_{\alpha,\beta}(r,\phi) = \mu_{\alpha,\beta}\,(1-r^2)^{\alpha-\beta-1}\, r^{2\beta+1}\,(\sin\phi)^{2\beta}\, dr\, d\phi,$$

with the constant $\mu_{\alpha,\beta}$ such that

$$\int_0^1\int_0^\pi dm_{\alpha,\beta}(r,\phi) = 1, \text{ i.e.,}$$

$$\mu_{\alpha,\beta} = \frac{2\,\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\,\Gamma(\beta+\frac{1}{2})}\,.$$

The functions $P_{k,l}^{(\alpha,\beta)}(r,\phi)$, defined by (3.1), are orthogonal with respect to the measure $dm_{\alpha,\beta}(r,\phi)$. Hence the integral (4.1) can be considered as the first term of the orthogonal expansion of $f(a^2r^2+2abr\cos\phi+b^2)$ in terms of the functions $P_{k,l}^{(\alpha,\beta)}(r,\phi)$. For the three cases mentioned above we shall derive this expansion in an explicit way and thus obtain three different addition formulas.

If f is a C^∞-function on $[0,\infty]$ then let $f^{(n)}$ denote the n^{th} derivative of f and define the function $f_{k,l}^{\beta}$ $(k\geq l\geq 0)$ on $[0,\infty)$ by

$$(4.3)\qquad f_{k,l}^{\beta}(t^2) = \left(\frac{d^2}{dt^2} + \frac{2(\beta+k-l)+1}{t}\frac{d}{dt}\right)^{l} f^{(k-l)}(t^2)\,.$$

LEMMA 4.1. Let $\alpha > \beta > -\frac{1}{2}$ and $k \geq l \geq 0$. Then for all C^∞-functions f and positive real numbers a, b there is the identity

$$(4.4)\qquad \int_0^1\int_0^\pi f(a^2r^2 + 2abr\cos\phi + b^2)\, P_{k,l}^{(\alpha,\beta)}(r,\phi)\, dm_{\alpha,\beta}(r,\phi) =$$

$$= \frac{(\alpha-\beta)_l\,(\beta+\frac{1}{2})_{k-l}\, a^{k+l}\, b^{k-l}}{2^{2l}\, l!\,(k-l)!\,(\alpha+1)_{k+l}}\,\cdot$$

$$\cdot \int_0^1\int_0^\pi f_{k,l}^{\beta}(a^2r^2 + 2abr\cos\phi + b^2)\, dm_{\alpha+k+l,\beta+k-l}(r,\phi).$$

Proof. The idea of the proof is to substitute Rodrigues' formula

$$(4.5)\qquad (-1)^n\, 2^n\, n!\, (1-x)^{\alpha}(1+x)^{\beta}\, P_n^{(\alpha,\beta)}(x) = \left(\frac{d}{dx}\right)^n [(1-x)^{n+\alpha}(1+x)^{n+\beta}]$$

and the Rodrigues type formula (2.6) in the explicit expression (3.1) for $P_{k,1}^{(\alpha,\beta)}(r,\phi)$ and then to perform repeated integration by parts. Let I be such that $\mu_{\alpha,\beta}$ I equals the left hand side of (4.4). Then

$$I = \int_0^1 \left[\int_0^{\pi} f(a^2r^2 + 2abr\cos\phi + b^2)\, P_{k-1}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\phi)\cdot\right.$$

$$\left.\cdot(\sin\phi)^{2\beta}\, d\phi\right] P_1^{(\alpha-\beta-1,\beta+k-1)}(2r^2-1)\, (1-r^2)^{\alpha-\beta-1}\, r^{2\beta+k-1+1}\, dr.$$

By using (4.5) and by repeated integration by parts it follows that

$$I = \frac{(ab)^{k-1}}{(k-1)!}\int_0^1\int_0^{\pi} f^{(k-1)}(a^2r^2 + 2abr\cos\phi + b^2)\cdot$$

$$\cdot P_1^{(\alpha-\beta-1,\beta+k-1)}(2r^2-1)\, (1-r^2)^{\alpha-\beta-1}\, (r\sin\phi)^{2(\beta+k-1)}\, r\, dr\, d\phi =$$

$$= \frac{(ab)^{k-1}}{(k-1)!}\iint_R f^{(k-1)}((ax+b)^2 + (ay)^2)\cdot$$

$$\cdot P_1^{(\alpha-\beta-1,\beta+k-1)}(2(x^2+y^2)-1)\, (1-x^2-y^2)^{\alpha-\beta-1}\, (y^2)^{\beta+k-1}\, dx\, dy,$$

where R denotes the upper half unit disk. Then, by Lemma 2.1,

$$I = \frac{(ab)^{k-1}}{2^{2\ell}\, 1!\, (k-1)!\, (1+\alpha-\beta)_1}\cdot$$

$$\cdot\iint_R \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2(\beta+k-1)}{y}\frac{\partial}{\partial y}\right)^1 f^{(k-1)}((ax+b)^2 + (ay)^2)\cdot$$

$$\cdot (1-x^2-y^2)^{\alpha-\beta+2\ell-1}\, (y^2)^{\beta+k-1}\, dx\, dy.$$

Note that if $ax + b = t\cos\psi$, $ay = t\sin\psi$ then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2(\beta+k-1)}{y}\frac{\partial}{\partial y}\right)^1 f^{(k-1)}((ax+b)^2 + (ay)^2) =$$

$$= a^{2\ell}\left(\frac{\partial^2}{\partial t^2} + \frac{2(\beta+k-1)+1}{t}\frac{\partial}{\partial t}\right)^1 f^{(k-1)}(t^2) = a^{2\ell} f^{\beta}_{k,1}(t^2).$$

Hence, by substituting $x = r\cos\phi$ and $y = r\sin\phi$ in the last expression for I, it follows that $\mu_{\alpha,\beta}$ I is equal to the right hand side of (4.4).

Q.e.d.

For the three choices of the function f in which we are interested the functions $f^{\beta}_{k,1}$ can easily be evaluated. We have

$$(4.6)\quad \begin{cases} f(t^2) = t^{2n}, \\ f^{\beta}_{k,1}(t^2) = \dfrac{2^{2\ell}\, n!\, (n-1+\beta+1)_1}{(n-k)!}\, t^{2n-2k}, \quad k \le n, \end{cases}$$

$$(4.7)\quad \begin{cases} f(t^2) = P^{(\alpha,\beta)}_n(2t^2-1), \\ f^{\beta}_{k,1}(t^2) = 2^{2\ell}(n+\alpha+\beta+1)_k\,(n-1+\beta+1)_1\, P^{(\alpha+k+1,\beta+k-1)}_{n-k}(2t^2-1), k\le n, \end{cases}$$

$$(4.8)\quad \begin{cases} f(t^2) = t^{-\beta} J_\beta(t), \\ f^{\beta}_{k,1}(t^2) = (-1)^k\, 2^{-k+1}\, t^{-\beta-k+1}\, J_{\beta+k-1}(t). \end{cases}$$

In (4.6) and (4.7) $f^{\beta}_{k,1} = 0$ if $k > n$. Formula (4.6) is evident. (4.7) follows by using (2.4) and the formula $(d/dx)\, P^{(\alpha,\beta)}_n(2x-1) = (n+\alpha+\beta+1)\, P^{(\alpha+1,\beta+1)}_{n-1}(2x-1)$. To prove (4.8) we need the formulas $(d/dt)(t^{-\beta} J_\beta(t)) = -\, t^{-\beta} J_{\beta+1}(t)$ and $((d/dt)^2 + (2\beta+1)t^{-1}(d/dt))(t^{-\beta} J_\beta(t)) = -t^{-\beta} J_\beta(t)$.

Using (4.4), (4.6), (4.7), (4.8) and [8, (3.1), (3.7), (3.8)] we obtain

(4.9) $$\int_0^1\int_0^\pi (\tfrac{1}{2}(x+1) + \tfrac{1}{2}(x-1)r^2 + \sqrt{x^2-1}\ r\cos\phi)^n\ \mathcal{P}_{k,1}^{(\alpha,\beta)}(r,\phi)\ dm_{\alpha,\beta}(r,\phi) =$$

$$= \frac{n!\ (\alpha-\beta)_1\ (n-1+\beta+1)_1\ (\beta+\tfrac{1}{2})_{k-1}}{2^k\ 1!\ (k-1)!\ (\alpha+1)_{n+1}} \cdot$$

$$\cdot (x-1)^{(k+1)/2}\ (x+1)^{(k-1)/2}\ P_{n-k}^{(\alpha+k+1,\beta+k-1)}(x) \text{ if } k \le n,$$

(4.10) $$\int_0^1\int_0^\pi P_n^{(\alpha,\beta)}(\tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)r^2 +$$

$$+ \sqrt{1-x^2}\ \sqrt{1-y^2}\ r\cos\phi - 1)\ \mathcal{P}_{k,1}^{(\alpha,\beta)}(r,\phi)\ dm_{\alpha,\beta}(r,\phi) =$$

$$= \frac{(n-k)!\ (n+\alpha+\beta+1)_k\ (\alpha-\beta)_1\ (n-1+\beta+1)_1\ (\beta+\tfrac{1}{2})_{k-1}}{2^{2k}\ 1!\ (k-1)!\ (\alpha+1)_{n+1}} \cdot$$

$$\cdot (1-x)^{(k+1)/2}\ (1+x)^{(k-1)/2}\ P_{n-k}^{(\alpha+k+1,\beta+k-1)}(x) \cdot$$

$$\cdot (1-y)^{(k+1)/2}\ (1+y)^{(k-1)/2}\ P_{n-k}^{(\alpha+k+1,\beta+k-1)}(y) \text{ if } k \le n,$$

(4.11) $$\int_0^1\int_0^\pi (x^2 + y^2r^2 + 2xyr\cos\phi)^{-\beta/2} \cdot$$

$$\cdot J_\beta((x^2 + y^2r^2 + 2xyr\cos\phi)^{\frac{1}{2}})\ \mathcal{P}_{k,1}^{(\alpha,\beta)}(r,\phi)\ dm_{\alpha,\beta}(r,\phi) =$$

$$= \frac{2^\alpha\ \Gamma(\alpha+1)\ (-1)^k\ (\alpha-\beta)_1\ (\beta+\tfrac{1}{2})_{k-1}}{1!\ (k-1)!}\ x^{-\beta}\ J_{\beta+k-1}(x)\ y^{-\alpha}\ J_{\alpha+k-1}(y).$$

The left hand sides of (4.9) and (4.10) are zero if $k > n$.

By using (3.1), (4.2) and [3, §10.8 (4)] it follows that

(4.12) $$\int_0^1\int_0^\pi (\mathcal{P}_{k,1}^{(\alpha,\beta)}(r,\phi))^2\ dm_{\alpha,\beta}(r,\phi) =$$

$$= \frac{(k+\alpha)\ ((k-1)/2+\beta)\ (\beta+\tfrac{1}{2})_{k-1}\ (\beta+\tfrac{1}{2})_{k-1}\ (\beta+1)_k\ (\alpha-\beta)_1}{(k+1+\alpha)\ (k-1+\beta)\ (2\beta+1)_{k-1}\ (k-1)!\ (\alpha+1)_k\ 1!} \cdot$$

Hence the expansions corresponding to (4.9) and (4.10) are

$$(4.13)\qquad (\tfrac{1}{2}(x+1) + \tfrac{1}{2}(x-1)r^2 + \sqrt{x^2-1}\ r\cos\phi)^n =$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(k+l+\alpha)\ (k-l+\beta)\ (2\beta+1)_{k-l}\ (n-l+\beta+1)_l\ n!}{2^k\ (k+\alpha)\ ((k-l)/2+\beta)\ (\beta+\tfrac{1}{2})_{k-l}\ (\beta+1)_k\ (\alpha+k+l)_{n-k+l}}\ \cdot$$

$$\cdot\ (x-1)^{(k+l)/2}\ (x+1)^{(k-l)/2}\ P_{n-k}^{(\alpha+k+l,\beta+k-l)}(x)$$

$$\cdot\ P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)\ r^{k-l}\ P_{k-l}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\phi)\ ,$$

$$(4.14)\qquad P_n^{(\alpha,\beta)}(\tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)r^2 + \sqrt{1-x^2}\ \sqrt{1-y^2}\ r\cos\phi - 1) =$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(k+l+\alpha)\ (k-l+\beta)\ (n+\alpha+\beta+1)_k\ (2\beta+1)_{k-l}\ (n-l+\beta+1)_l\ (n-k)!}{2^{2k}\ (k+\alpha)\ ((k-l)/2+\beta)\ (\beta+1)_k\ (\beta+\tfrac{1}{2})_{k-l}\ (k+\alpha+1)_{n-k+l}}\ \cdot$$

$$\cdot\ (1-x)^{(k+l)/2}\ (1+x)^{(k-l)/2}\ P_{n-k}^{(\alpha+k+l,\beta+k-l)}(x)\ \cdot$$

$$\cdot\ (1-y)^{(k+l)/2}\ (1+y)^{(k-l)/2}\ P_{n-k}^{(\alpha+k+l,\beta+k-l)}(y)\ \cdot$$

$$\cdot\ P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)\ r^{k-l}\ P_{k-l}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\phi).$$

Formula (4.14) is the addition formula for Jacobi polynomials (cf.[4,(3)]). We call (4.13) the degenerate addition formula for Jacobi polynomials (cf. §5, Remark 4).

The formal expansion corresponding to (4.11) is

$$(4.15)\qquad (x^2 + y^2r^2 + 2xyr\cos\phi)^{-\beta/2}\ J_\beta((x^2 + y^2r^2 + 2xyr\cos\phi)^{\frac{1}{2}}) =$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{2^\alpha\ \Gamma(\alpha+1)\ (-1)^k\ (k+l+\alpha)\ (k-l+\beta)\ (2\beta+1)_{k-l}\ (\alpha+1)_k}{(k+\alpha)\ ((k-l)/2+\beta)\ (\beta+\tfrac{1}{2})_{k-l}\ (\beta+1)_k}\ \cdot$$

$$\cdot\ x^{-\beta}\ J_{\beta+k-l}(x)\ y^{-\alpha}\ J_{\alpha+k-l}(y)\ \cdot$$

$$\cdot\ P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)\ r^{k-l}\ P_{k-l}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\phi)\ .$$

By rough asymptotic estimates it follows that in the right hand side of (4.15) the term of index (k,l) is of order $(\Gamma(k-c))^{-1}$ if $k \to \infty$, where c is some real constant. This estimate is uniform in $l (0 \le l \le k)$. Hence the series in (4.15) converges absolutely and the identity holds.

5. Discussion of the results

We conclude this paper with some remarks about the addition formulas (4.13), (4.14), (4.15). No proofs will be given in this section.

Remark 1. If both sides of (4.13) or (4.14) are differentiated once with respect to ϕ then the same formula is obtained with n, α, β replaced by $n-1$, $\alpha+1$, $\beta+1$ respectively. If the partial differential operator

$$\frac{\partial^2}{\partial r^2} + \frac{2\beta+1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + 2\beta\,\frac{\text{cotg}\,\phi}{r^2}\frac{\partial}{\partial \phi}$$

is applied on both sides of (4.13) or (4.14) then the same formula is obtained with n, α, β replaced by $n-1$, $\alpha+2$, β, respectively. The same is true for (4.15) except that the parameter n does not occur here. Both sides of (4.13) and (4.14) are rational functions in α and β. It follows that if these two formulas are known in one specific case (α_0,β_0) then they can be proved in the case of general (α,β) by repeated differentiation and by analytic continuation with respect to α and β.

Remark 2. Using the results in [8] Gasper obtained another analytic proof of the addition formula (4.14) (personal communication to the author). He first proved (4.9) by reducing the left hand side of (4.9) to a multiple summation and by manipulating this sum, and next he derived (4.10) from (4.9) by using Bateman's formula [8, (2.19)].

Remark 3. If either $\alpha = \beta > -\frac{1}{2}$ or $\alpha > \beta = -\frac{1}{2}$ then (4.13), (4.14) and (4.15) degenerate to orthogonal expansions in terms of functions of one variable. For instance, putting $\alpha = \beta$ and $r = 1$ in (4.14) we obtain Gegenbauer's addition formula

$$(5.1)\qquad P_n^{(\alpha,\alpha)}(xy + \sqrt{1-x^2}\,\sqrt{1-y^2}\,\cos\phi) =$$

$$= \sum_{k=0}^{n} \frac{(k+\alpha)\,(n+2\alpha+1)_k\,(2\alpha+1)_k\,(n-k)!}{2^{2k}\,(k/2+\alpha)\,(\alpha+\frac{1}{2})_k\,(\alpha+1)_n}\,\cdot$$

$$\cdot\,(1-x^2)^{k/2}\,P_{n-k}^{(\alpha+k,\alpha+k)}(x)\,(1-y^2)^{k/2}\,P_{n-k}^{(\alpha+k,\alpha+k)}(y)\,P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\cos\phi).$$

The same formula with n, x, y, cos ϕ replaced by 2n, $((1+x)/2)^{\frac{1}{2}}$, $((1+y)/2)^{\frac{1}{2}}$, r, respectively, is obtained by putting $\beta = -\frac{1}{2}$ and $\phi = 0$ in (4.14) and by substituting the quadratic transformation formulas for Gegenbauer polynomials.

Remark 4. We call (4.13) the degenerate addition formula for Jacobi polynomials, since it can be derived from the addition formula (4.14) by dividing both sides of (4.14) by y^n and then letting $y \to \infty$.

Remark 5. The generalized addition formula (4.15) for Bessel functions is also a limit case of (4.14). The formula is obtained by dividing both sides of (4.14) by $P_n^{(\alpha,\beta)}(-1)$ and then letting $n \to \infty$, where the formulas [8, (3.9), (3.10)] are applied.

Remark 6. In the following diagram it is indicated how several related results concerning the addition formula for Jacobi polynomials follow from each other. Here an arrow denotes a direction of proof.

Laplace representation
for $\alpha = \beta$
↕

general α and β:

Laplace representation ↔ degenerate addition formula
↕ (Laplace representation ↔ product formula)
↕ (degenerate addition formula ↔ addition formula)
product formula ↔ addition formula
↕
addition formula
for any one specific (α,β)

In the approach used in [4], [5], [6] the author started at the bottom of the diagram ($\alpha=1,2,\ldots$ and $\beta=0$). In the approach used in the present series of papers we start at the top of the diagram.

Remark 7. The addition formula (4.14) in the case that $\beta = 0$ is also a special case of the addition formula for the so-called disk polynomials (cf. Šapiro [10, (1.20)] and Koornwinder [6, (5.4)]. In these two references the addition formula for disk polynomials was proved by group theoretic methods. An analytic proof of this formula might be given by using the methods of the present paper, starting from the product formula [5, (4.10)] for disk polynomials.

Remark 8. There is yet another limit case of the addition formula (4.14). Replacing the variables x, y, r in (4.14) by $2\alpha^{-1}x-1$, $2y-1$, $\alpha^{-\frac{1}{2}}r$, respectively, letting $\alpha \to \infty$ and using that $\lim_{\alpha\to\infty} P_n^{(\alpha,\beta)}(2\alpha^{-1}x-1) = (-1)^n L_n^\beta(x)$ and $\lim_{\alpha\to\infty} \alpha^{-n} P_n^{(\alpha,\beta)}(2x-1) = x^n/n!$ we obtain that

$$(5.2)\qquad L_n^\beta(xy + (1-y)r^2 + 2\sqrt{xy(1-y)}\, r\cos\phi) =$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(-1)^{k+l}\,(k-l+\beta)\,(2\beta+1)_{k-l}\,(n-l+\beta+1)_l}{((k-l)/2+\beta)\,(\beta+1)_k\,(\beta+\frac{1}{2})_{k-l}}\,.$$

$$\cdot\, x^{(k-l)/2}\, L_{n-k}^{\beta+k-l}(x)\, y^{n-(k+l)/2}\,(1-y)^{(k+l)/2}$$

$$\cdot\, L_l^{\beta+k-l}(r^2)\, r^{k-l}\, P_{k-l}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\phi).$$

This is a kind of addition formula for Laguerre polynomials $L_n^\beta(x)$. Integration of (5.2) gives

$$(5.3)\qquad L_n^\beta(x)\, y^n = \frac{2}{\sqrt{\pi}\,\Gamma(\beta+\frac{1}{2})}\,.$$

$$\cdot \int_0^\infty\int_0^\pi L_n^\beta(xy + (1-y)r^2 + 2\sqrt{xy(1-y)}\, r\cos\phi)$$

$$\cdot\, e^{-r^2}\, r^{2\beta+1}\,(\sin\phi)^{2\beta}\, dr\, d\phi\ ,\ \beta > -\tfrac{1}{2}\ .$$

Dividing both sides of (5.3) by y^n and letting $y \to \infty$ we finally obtain

$$(5.4) \qquad L_n^\beta(x) = \frac{2(-1)^n}{\sqrt{\pi}\,\Gamma(\beta+\frac{1}{2})\,n!} \cdot$$

$$\cdot \int_0^\infty \int_0^\pi (x-r^2+2i\sqrt{x}\,r\cos\phi)^n\, e^{-r^2}\, r^{2\beta+1}(\sin\phi)^{2\beta}\, dr\, d\phi, \quad \beta > -\tfrac{1}{2}.$$

It was pointed out by Askey (personal communication) that the integral representation (5.4) can also be proved from the Laplace type integral representation for Gegenbauer polynomials by using Askey and Fitch [2,(3.29)].

References

[1] R. Askey, Jacobi polynomials, I. New proofs of Koornwinder's Laplace type integral representation and Bateman's bilinear sum, SIAM J. Math. Anal., 5 (1974), to appear.

[2] R. Askey, and J. Fitch, Integral representations for Jacobi polynomials and some applications, J. Math. Anal. Appl., 26 (1969), pp. 411-437.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Higher Transcendental Functions, vols. I,II, McGraw-Hill, New York, 1953

[4] T.H. Koornwinder, The addition formula for Jacobi polynomials, I. Summary of results, Indag. Math., 34 (1972), pp. 188-191.

[5] ————————, The addition formula for Jacobi polynomials, II. The Laplace type integral representation and the product formula, Math. Centrum Amsterdam,Afd. Toegepaste Wiskunde, Rep. TW 133 (1972).

[6] ————————, The addition formula for Jacobi polynomials, III. Completion of the proof, Ibid., Rep. TW 135 (1972).

[7] ————————, The addition formula for Jacobi polynomials and spherical harmonics, SIAM J. Appl. Math., 25 (1973), pp. 236-246.

[8] ————————, Jacobi polynomials, II. An analytic proof of the product formula, SIAM J. Math. Anal., 5 (1974), to appear.

[9] W. Miller, Lie Theory and Special Functions, Academic Press, New York, 1968.

[10] R.L. Šapiro, The special functions connected with the representations of the group SU(n) of class one with respect to SU(n-1) (n≥3), Izv. Vysš. Učebn. Zaved. Matematika, 62 (1967), pp. 9-20 (Russian).

[11] C. Truesdell, An Essay towards a Unified Theory of Special Functions, Princeton University Press, Princeton, 1948.