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A NEW PROOF OF A PALEY-WIENER TYPE THEOREM FOR THE JACOBI TRANSFORM

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A new proof of a Paley-Wiener type theorem for the Jacobi transform *)

by

T.H. Koornwinder

ABSTRACT

A new integral representation for Jacobi functions is derived, containing the Mehler-Dirichlet formula for Legendre functions as a special case. As a result, the Fourier-Jacobi transform, which generalizes the Mehler-Fok transform, can be factorized as the product of two Weyl type fractional integral transforms and a Fourier-cosine transform. There follow new short proofs of a Paley-Wiener type theorem and the inversion formula for the Jacobi transform. By analytic continuation these results hold for all complex values of the parameters.

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1. INTRODUCTION

Jacobi functions $\phi_\lambda(t)$ of order (α,β) are the eigenfunctions of the differential operator $(\Delta(t))^{-1}(d/dt)(\Delta(t)d/dt)$, $\Delta(t) = (e^t-e^{-t})^{2\alpha+1}(e^t+e^{-t})^{2\beta+1}$, such that $\phi_\lambda(0) = 1$, $\phi_\lambda{}'(0) = 0$. The Jacobi transform

$$(1.1)\quad f^\wedge(\lambda) = (2^{\frac{1}{2}}/\Gamma(\alpha+1)) \int_0^\infty f(t)\phi_\lambda(t)\Delta(t)dt,$$

which generalizes the Mehler-Fok transform, was studied by TITCHMARSH [23,§4.17], OLEVSKIĬ [21], BRAAKSMA and MEULENBELD [2], FLENSTED-JENSEN [9], [11,§2 and §12] and FLENSTED-JENSEN and KOORNWINDER [12]. Some papers by CHÉBLI [3], [4], [5] deal with a larger class of integral transforms which includes the Jacobi transform. An even more general class was considered by BRAAKSMA and DE SNOO [24].

In the present paper short proofs will be given of a Paley-Wiener type theorem and the inversion formula for the Jacobi transform. The L^2-theory, i.e. the Plancherel theorem, is then an easy consequence. These results were earlier obtained by FLENSTED-JENSEN [9], [11,§2] and by CHÉBLI [5]. However, to prove the Paley-Wiener theorem these two authors needed the L^2-theory, which can be obtained as a corollary of the Weyl-Stone-Titchmarsh-Kodaira theorem about the spectral decomposition of a singular Sturm-Liouville operator (cf. for instance DUNFORD and SCHWARTZ [6,Chap.13,§5]). The proofs presented here exploit the properties of Jacobi functions as hypergeometric functions and no general theorem needs to be invoked. Furthermore, it turns out that the Paley-Wiener theorem, which was proved by FLENSTED-JENSEN [11,§2] for real α,β, $\alpha > -1$, holds for all complex values of α and β.

The key formula in this paper is a generalized Mehler formula

$$(1.2)\quad (\Gamma(\alpha+1))^{-1}\Delta(t)\phi_\lambda(t) = \pi^{-\frac{1}{2}} \int_0^t \cos \lambda s\ A(s,t)\ ds,$$

where for $\mathrm{Re}\ \alpha > \mathrm{Re}\ \beta > -\frac{1}{2}$ $A(s,t)$ is given as an integral of elementary functions. Substituting (1.2) in (1.1) we can write the Jacobi transform

$f^\wedge$ as the Fourier-cosine transform of F(f), where the mapping F consists of two successive Weyl type fractional integral transforms. Thus the Jacobi transform is factorized as the product of three integral transforms with elementary kernels and the Paley-Wiener theorem follows from the mapping properties of these elementary transforms.

For certain discrete values of α and β the mapping F has a geometric and group-theoretic interpretation as a Radon transform on rank one symmetric spaces (cf. HELGASON [16,Chap.1,2]). For integer or half integer values of α and β such that $\alpha \geq \beta \geq -\frac{1}{2}$ a similar interpretation was given by FLENSTED-JENSEN [10] on certain pseudo-Riemannian symmetric spaces. A large class of integral transforms for which the corresponding mapping F is positive was examined by CHÉBLI [5]. Finally, FLENSTED-JENSEN and RAGOZIN [13] wrote a note on the analogue of (1.2) for spherical functions on non-compact symmetric spaces of arbitrary rank.

In section 2 of this paper some properties and formulas for Jacobi functions are given. Section 3 contains the proof of the Paley-Wiener theorem for all complex α and β. Formula (1.2) is the only result on Jacobi functions which is needed there. In section 4 the inversion formula is derived by using the Paley-Wiener theorem, some estimates for Jacobi functions and a formula for Jacobi functions of the second kind which is dual to (1.2). The paper concludes with some remarks, in particular about the Plancherel theorem and about Paley-Wiener type theorems for the Hankel transform and for Jacobi series.

Notation. This is mainly similar to the notation used in [12]. For reasons of elegance and in order to avoid singularities if $\alpha=-1,-2,\ldots$, some constant factors have been changed. If no confusion is possible the indices α,β denoting the order may be deleted.

2. JACOBI FUNCTIONS

Consider for $\alpha,\beta,\lambda \in \mathbb{C}$ and $0 < t < \infty$ the differential equation

$$(2.1)\quad (\Delta_{\alpha,\beta}(t))^{-1}\frac{d}{dt}(\Delta_{\alpha,\beta}(t)\frac{du(t)}{dt}) = -(\lambda^2+\rho^2)u(t),$$

where $\rho = \alpha + \beta + 1$ and

$$(2.2)\quad \Delta_{\alpha,\beta}(t) = (e^t-e^{-t})^{2\alpha+1}(e^t+e^{-t})^{2\beta+1}\ .$$

By substituting $z = -(\sinh t)^2$ in (2.1) a hypergeometric differential equation is obtained (cf.[7,2.1(1)]) with parameters $\frac{1}{2}(\rho+i\lambda)$, $\frac{1}{2}(\rho-i\lambda)$, $\alpha + 1$. Hence, if $\alpha \neq -1,-2,-3,\ldots$ then the function

$$(2.3)\quad \phi_\lambda^{(\alpha,\beta)}(t) = F(\tfrac{1}{2}(\rho+i\lambda),\ \tfrac{1}{2}(\rho-i\lambda);\ \alpha + 1;\ -(\sinh t)^2)$$

is the solution of (2.1) which satisfies $\phi_\lambda(0) = 1$, $\phi_\lambda'(0) = 0$. Here the hypergeometric function $F(a,b;c;z)$ denotes the unique analytic continuation for $z \notin [1,\infty)$ of the power series

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n!} z^n, \qquad |z| < 1.$$

Note that $(\Gamma(\alpha+1))^{-1}\phi_\lambda^{(\alpha,\beta)}(t)$ is an entire function of α,β and λ (also for $\alpha=-1,-2,\ldots$).

For $\lambda \neq -i,-2i,-3i,\ldots$ another solution of (2.1) (cf.[7,2.9(9)]) is given by the function

$$(2.4)\quad \Phi_\lambda^{(\alpha,\beta)}(t) = (e^t-e^{-t})^{i\lambda-\rho}$$
$$\cdot\ F(\tfrac{1}{2}(-\alpha+\beta+1-i\lambda),\ \tfrac{1}{2}(\alpha+\beta+1-i\lambda);\ 1 - i\lambda;\ -(\sinh t)^{-2}).$$

This solution is characterized by the property that $\Phi_\lambda(t) = e^{(i\lambda-\rho)t}(1+o(1))$ for $t \to \infty$. The functions $\phi_\lambda(t)$ and $\Phi_\lambda(t)$ are called Jacobi functions of the first and second kind, respectively.

Using [7,2.10(2) and 2.10(5)] we obtain for non-integer λ the identity

(2.5) $$\pi^{\frac{1}{2}}(\Gamma(\alpha+1))^{-1}\phi_\lambda(t) = \tfrac{1}{2}c(\lambda)\Phi_\lambda(t) + \tfrac{1}{2}c(-\lambda)\Phi_{-\lambda}(t),$$

where

(2.6) $$c_{\alpha,\beta}(\lambda) = \frac{2^{\alpha+\beta+1}\ \Gamma(\frac{1}{2}i\lambda)\ \Gamma(\frac{1}{2}(1+i\lambda))}{\Gamma(\frac{1}{2}(\alpha+\beta+1+i\lambda))\ \Gamma(\frac{1}{2}(\alpha-\beta+1+i\lambda))}\ .$$

Note that for real λ,α,β $\overline{c(\lambda)} = c(-\lambda)$.

It follows easily from (2.1) and the definitions of $\phi_\lambda(t)$, $\Phi_\lambda(t)$, $\Delta(t)$ and $c(\lambda)$ that

(2.7) $$\begin{cases} \phi_\lambda^{(-\frac{1}{2},-\frac{1}{2})}(t) = \cos\lambda t \quad , & \Phi_\lambda^{(-\frac{1}{2},-\frac{1}{2})}(t) = e^{i\lambda t}\ , \\ \Delta_{-\frac{1}{2},-\frac{1}{2}}(t) = 1 \quad , & c_{-\frac{1}{2},-\frac{1}{2}}(\lambda) = 1\ , \end{cases}$$

and

(2.8) $$\begin{cases} \phi_{2\lambda}^{(\alpha,\alpha)}(t) = \phi_\lambda^{(\alpha,-\frac{1}{2})}(2t) \quad , & \Phi_{2\lambda}^{(\alpha,\alpha)}(t) = \Phi_\lambda^{(\alpha,-\frac{1}{2})}(2t), \\ \Delta_{\alpha,\alpha}(t) = \Delta_{\alpha,-\frac{1}{2}}(2t) \quad , & c_{\alpha,\alpha}(2\lambda) = c_{\alpha,-\frac{1}{2}}(\lambda). \end{cases}$$

The first two formulas of (2.8) can also be interpreted as quadratic transformations for hypergeometric functions, cf. [7,2.11(2) and 2.11(26)].

Application of [7,2.8(20) and 2.8(27)] gives the differentiation formulas

(2.9) $$(\Gamma(\alpha+1))^{-1}\ \frac{d\phi_\lambda^{(\alpha,\beta)}(t)}{dt} =$$

$$= -\tfrac{1}{4}((\alpha+\beta+1)^2+\lambda^2)(\Gamma(\alpha+2))^{-1}\ \sinh 2t\ \ \phi_\lambda^{(\alpha+1,\beta+1)}(t)$$

and

$$(2.10)\quad (\Gamma(\alpha+2))^{-1}\frac{d}{dt}\left[(\sinh 2t)^{-1}\Delta_{\alpha+1,\beta+1}(t)\phi_\lambda^{(\alpha+1,\beta+1)}(t)\right] =$$

$$= 16(\Gamma(\alpha+1))^{-1}\Delta_{\alpha,\beta}(t)\phi_\lambda^{(\alpha,\beta)}(t).$$

Next we derive some useful integration formulas for Jacobi functions. It follows from Bateman's integral [7,2.4(2)] and the identity

$$(2.11)\quad F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z)$$

(cf.[7,2.1(23)]) that for $y > 0$, $\mathrm{Re}\,\mu > 0$, $\mathrm{Re}\,c > 0$

$$(2.12)\quad (\Gamma(c+\mu))^{-1}y^{c+\mu-1}(1+y)^{a+b-c+\mu}F(a+\mu,b+\mu;c+\mu;-y) =$$

$$= \frac{1}{\Gamma(c)\Gamma(\mu)}\int_0^y x^{c-1}(1+x)^{a+b-c}F(a,b;c;-x)(y-x)^{\mu-1}dx.$$

It follows from ASKEY and FITCH [1,(2.10)] that for $x > 0$, $\mathrm{Re}\,\mu > 0$, $\mathrm{Re}\,b > 0$

$$(2.13)\quad \Gamma(b)x^{-b}F(a,b;c;-x^{-1}) =$$

$$= \frac{\Gamma(b+\mu)}{\Gamma(\mu)}\int_x^\infty y^{-b-\mu}F(a,b+\mu;c;-y^{-1})(y-x)^{\mu-1}dy.$$

Translating (2.12) and (2.13) in terms of Jacobi functions we obtain

$$(2.14)\quad (\Gamma(\alpha+\mu+1))^{-1}\Delta_{\alpha+\mu,\beta+\mu}(t)\phi_\lambda^{(\alpha+\mu,\beta+\mu)}(t) =$$

$$= \frac{2^{3\mu+1}\sinh 2t}{\Gamma(\alpha+1)\Gamma(\mu)}\int_0^t \Delta_{\alpha,\beta}(s)\phi_\lambda^{(\alpha,\beta)}(s)(\cosh 2t-\cosh 2s)^{\mu-1}ds,$$

where $t > 0$, $\mathrm{Re}\,\mu > 0$, $\mathrm{Re}\,\alpha > -1$, and

$$(2.15)\quad (c_{\alpha,\beta}(-\lambda))^{-1}\Phi_\lambda^{(\alpha,\beta)}(s) = \frac{2^{3\mu+1}}{c_{\alpha+\mu,\beta+\mu}(-\lambda)\Gamma(\mu)}$$

$$\cdot\int_s^\infty \Phi_\lambda^{(\alpha+\mu,\beta+\mu)}(t)(\cosh 2t-\cosh 2s)^{\mu-1}\sinh 2t\,dt,$$

where $s > 0$, $\mathrm{Re}\ \mu > 0$, $\mathrm{Im}\ \lambda > -\mathrm{Re}(\alpha+\beta+1)$.

The integrals (2.14) and (2.15) connect Jacobi functions of order (α,β) with functions of order $(\alpha-\beta-\frac{1}{2},-\frac{1}{2})$ and Jacobi functions of order $(\alpha-\beta-\frac{1}{2},\alpha-\beta-\frac{1}{2})$ with functions of order $(-\frac{1}{2},-\frac{1}{2})$. Hence, by (2.7),(2.8), (2.14) and (2.15) we conclude that for $\mathrm{Re}\ \alpha > \mathrm{Re}\ \beta > -\frac{1}{2}$

$$(2.16)\quad (\Gamma(\alpha+1))^{-1}\Delta(t)\phi_\lambda(t) = \pi^{-\frac{1}{2}}\int_0^t \cos\lambda s\ A(s,t)\ ds$$

and

$$(2.17)\quad e^{i\lambda s} = (c(-\lambda))^{-1}\int_s^\infty \Phi_\lambda(t)\ A(s,t)\ dt\quad,\qquad \mathrm{Im}\ \lambda > 0,$$

where the kernel is given by

$$(2.18)\quad A_{\alpha,\beta}(s,t) = \frac{2^{3\alpha+5/2}\sinh 2t}{\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}$$

$$\cdot\int_s^t (\cosh 2t-\cosh 2w)^{\beta-\frac{1}{2}}(\cosh w-\cosh s)^{\alpha-\beta-1}\sinh w\ dw.$$

By substituting $\tau = (\cosh t-\cosh w)/(\cosh t - \cosh s)$ in (2.18) and using Euler's integral [7,2.1(10)] we obtain

$$(2.19)\quad A_{\alpha,\beta}(s,t) = 2^{3\alpha+2\beta+3/2}(\Gamma(\alpha+\tfrac{1}{2}))^{-1}\sinh 2t(\cosh t)^{\beta-\frac{1}{2}}$$

$$\cdot(\cosh t-\cosh s)^{\alpha-\frac{1}{2}}F(\tfrac{1}{2}+\beta,\tfrac{1}{2}-\beta;\alpha+\tfrac{1}{2};\ \frac{\cosh t-\cosh s}{2\cosh t}).$$

Combination of (2.19) and (2.11) gives

$$(2.20)\quad A_{\alpha,\beta}(s,t) = 2^{\alpha+2\beta+5/2}(\Gamma(\alpha+\tfrac{1}{2}))^{-1}\sinh 2t(\cosh t)^{\beta-\alpha}$$

$$\cdot(\cosh 2t-\cosh 2s)^{\alpha-\frac{1}{2}}F(\alpha+\beta,\alpha-\beta;\alpha+\tfrac{1}{2};\ \frac{\cosh t-\cosh s}{2\cosh t}).$$

Note that for $0 \le s < t$ the argument of the hypergeometric functions in (2.19) and (2.20) has its value between 0 and $\frac{1}{2}$. Hence these hypergeometric functions are bounded functions in s and t. By analytic continuation with respect to α and β and by using the expressions (2.19) or (2.20) for the kernel it follows that formula (2.16) is valid for Re $\alpha > -\frac{1}{2}$ and formula (2.17) holds if Re $\alpha > -\frac{1}{2}$, Im $\lambda > 0$. It is clear from (2.19) and (2.20) that $A_{\alpha,\beta}(s,t) > 0$ if $0 \le s < t$, $\alpha > -\frac{1}{2}$ and $|\beta| \le \max(\frac{1}{2},\alpha)$.

From (2.16) and (2.20) we have the integral representation

$$(2.21)\quad \phi_\lambda^{(\alpha,\beta)}(t) = 2^{-\alpha+3/2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\frac{1}{(\sinh t)^{2\alpha}(\cosh t)^{\alpha+\beta}}$$

$$\cdot\int_0^t \cos\lambda s(\cosh 2t-\cosh 2s)^{\alpha-\frac{1}{2}}F(\alpha+\beta,\alpha-\beta;\alpha+\tfrac{1}{2};\frac{\cosh t-\cosh s}{2\cosh t})ds,$$

valid for Re $\alpha > -\frac{1}{2}$. In view of (2.9) formula (2.21) in the case of order $(\alpha+1,\beta+1)$ gives in integral representation for $d\phi_\lambda^{(\alpha,\beta)}(t)/dt$. This last integral can be rewritten by using integration by parts and by [7,2.8(27)]. Thus we obtain the integral representation

$$(2.22)\quad \frac{d\phi_\lambda^{(\alpha,\beta)}(t)}{dt} = -2^{-\alpha+3/2}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\frac{(\alpha+\beta+1)^2+\lambda^2}{\lambda}$$

$$\cdot\frac{1}{(\sinh t)^{2\alpha+1}(\cosh t)^{\alpha+\beta}}\int_0^t \sin\lambda s\sinh s(\cosh 2t-\cosh 2s)^{\alpha-\frac{1}{2}}$$

$$\cdot F(\alpha+\beta+1,\alpha-\beta-1;\alpha+\tfrac{1}{2};\frac{\cosh t-\cosh s}{2\cosh t})ds,$$

which is also valid for Re $\alpha > -\frac{1}{2}$.

We shall need some estimates which are essentially due to FLENSTED-JENSEN [9,Theorem 2],[11,§2], but which will be stated here for all $\alpha,\beta \in \mathbb{C}$. The proof of lemma 2.3 below is different from the proof given in [9].

LEMMA 2.1. *For each* $\alpha,\beta \in \mathbb{C}$ *and* $\delta > 0$ *there exists a positive constant* K *such that for all* $t \ge \delta$ *and all* $\lambda \in \mathbb{C}$ *with* Im $\lambda \ge 0$

$$|\phi_\lambda^{(\alpha,\beta)}(t)| \le K\ e^{-(\mathrm{Im}\,\lambda+\mathrm{Re}\,\rho)t}.$$

LEMMA 2.2. *For each* $\alpha,\beta \in \mathbb{C}$ *there exists a positive constant* K *such that if* $\lambda \in \mathbb{C}$, Im $\lambda \ge 0$ *and* λ *is outside arbitrary small neighborhoods of the poles of* $(c_{\alpha,\beta}(-\lambda))^{-1}$ *then*

$$|c_{\alpha,\beta}(-\lambda)|^{-1} \le K(1+|\lambda|)^{\alpha+\frac{1}{2}}.$$

Lemma 2.1 follows by extending the proof of [9,Lemma 7] to the case of complex α and β. Lemma 2.2 follows from (2.6) and Stirling's formula.

LEMMA 2.3. *For each* $\alpha,\beta \in \mathbb{C}$ *and for each non-negative integer* n *there exists a positive constant* K *such that for all* $t \ge 0$ *and all* $\lambda \in \mathbb{C}$

$$|(\Gamma(\alpha+1))^{-1} \frac{d^n}{dt^n} \phi_\lambda^{(\alpha,\beta)}(t)| \le K(1+|\lambda|)^{n+k}(1+t)e^{(|\mathrm{Im}\ \lambda|-\mathrm{Re}\ \rho)t},$$

where $k = 0$ *if* $\mathrm{Re}\ \alpha > -\frac{1}{2}$ *and* $k = [\frac{1}{2}-\mathrm{Re}\ \alpha]$ *if* $\mathrm{Re}\ \alpha \le -\frac{1}{2}$.

Proof. Consider first the case that $n = 0$ and $\mathrm{Re}\ \alpha > -\frac{1}{2}$. It follows from (2.21) that

$$|\phi_\lambda^{(\alpha,\beta)}(t)| \le \text{const.}\ e^{(|\mathrm{Im}\ \lambda|+\mathrm{Re}(\alpha-\beta))t}$$

$$\cdot\ (\sinh t \cosh t)^{-2\mathrm{Re}\ \alpha} \int_0^t (\cosh 2t-\cosh 2s)^{\mathrm{Re}\ \alpha-\frac{1}{2}} ds$$

$$= \text{const.}\ e^{(|\mathrm{Im}\ \lambda|+\mathrm{Re}(\alpha-\beta))t} \phi_0^{(\mathrm{Re}\ \alpha,\mathrm{Re}\ \alpha)}(t).$$

Applying [7,2.10(7)] we have the estimate

$$\phi_0^{(\mathrm{Re}\ \alpha,\mathrm{Re}\ \alpha)}(t) \le \text{const.}(1+t)\ e^{-(2\mathrm{Re}\ \alpha+1)t}.$$

By combining the last two equalities the lemma is proved for $n = 0$. The estimate in the case that $n = 1$, $\mathrm{Re}\ \alpha > -\frac{1}{2}$ and $|\lambda| < 1$ follows from (2.9) and the estimate for $\phi_\lambda^{(\alpha+1,\beta+1)}(t)$. If $n = 1$, $\mathrm{Re}\ \alpha > -\frac{1}{2}$ and $|\lambda| \ge 1$ then we conclude from (2.22) that

$$\left|\frac{d}{dt} \phi_\lambda^{(\alpha,\beta)}(t)\right| \le \text{const.}(1+|\lambda|)e^{(|\mathrm{Im}\ \lambda|+\mathrm{Re}(\alpha-\beta))t}\ \phi_0^{(\mathrm{Re}\ \alpha,\mathrm{Re}\ \alpha)}(t) \le$$

$$\le \text{const.}(1+|\lambda|)(1+t)\ e^{(|\mathrm{Im}\ \lambda|-\mathrm{Re}\ \rho)t}.$$

The case that $n = 0,1$ and $\mathrm{Re}\ \alpha \le -\frac{1}{2}$ follows by complete induction with respect to $k = [\frac{1}{2}-\mathrm{Re}\ \alpha]$, where formulas (2.9) and (2.10) are used. Finally we prove the case that $n=2,3,\ldots$ by complete induction with respect to n using the formula

$$(\Gamma(\alpha+1))^{-1} \frac{d^n}{dt^n} \phi_\lambda^{(\alpha,\beta)}(t) = -(\rho^2+\lambda^2)(\Gamma(\alpha+1))^{-1} \frac{d^{n-2}}{dt^{n-2}} \phi_\lambda^{(\alpha,\beta)}(t) +$$

$$+ \tfrac{1}{2}(\rho^2+\lambda^2)(\Gamma(\alpha+2))^{-1} \frac{d^{n-2}}{dt^{n-2}} \left[(\rho\cosh 2t + \alpha - \beta)\phi_\lambda^{(\alpha+1,\beta+1)}(t)\right] .$$

This formula follows by differentiating the formula

$$\frac{d^2}{dt^2} \phi_\lambda^{(\alpha,\beta)}(t) = (\rho^2+\lambda^2) \left[\frac{\rho\cosh 2t + \alpha - \beta}{2(\alpha+1)} \phi_\lambda^{(\alpha+1,\beta+1)}(t) - \phi_\lambda^{(\alpha,\beta)}(t)\right] ,$$

which is a consequence of (2.1) and (2.9). □

3. A PALEY-WIENER TYPE THEOREM

Let C_0^∞ be the class of all even infinitely differentiable functions on $\mathbb{R}$ with compact support. Let H be the class of even, entire, rapidly decreasing functions of exponential type, i.e., $g \in H$ if and only if g is an even and entire analytic function on $\mathbb{C}$ and there exist positive constants A and K_n $(n=0,1,2,\ldots)$ such that for all $\lambda \in \mathbb{C}$ and for all $n=0,1,2,\ldots$

$$(3.1) \qquad |g(\lambda)| \le K_n(1+|\lambda|)^{-n} e^{A|\mathrm{Im}\,\lambda|}.$$

Let for $f \in C_0^\infty$ and $\mathrm{Re}\,\alpha > -1$ the Fourier-Jacobi transform $f \to \hat{f}_{\alpha,\beta}$ be defined by

$$(3.2) \qquad \hat{f}_{\alpha,\beta}(\lambda) = (2^{\frac{1}{2}}/\Gamma(\alpha+1)) \int_0^\infty f(t)\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)dt.$$

Clearly $\hat{f}_{\alpha,\beta}(\lambda)$ is analytic in $\alpha,\beta,\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > -1$. Substitution of (2.10) in (3.2) and repeated integration by parts gives

$$(3.3) \qquad \hat{f}_{\alpha,\beta}(\lambda) = \frac{(-1)^n}{2^{4n}\Gamma(\alpha+n+1)} \int_0^\infty \left(\left(\frac{1}{\sinh 2t}\frac{d}{dt}\right)^n f(t)\right) \cdot \phi_\lambda^{(\alpha+n,\beta+n)}(t)\Delta_{\alpha+n,\beta+n}(t)dt, \qquad n=0,1,2,\ldots .$$

This formula defines the analytic continuation of $\hat{f}_{\alpha,\beta}(\lambda)$ for $\mathrm{Re}\,\alpha > -n-1$. Hence $\hat{f}_{\alpha,\beta}(\lambda)$ is an entire function of α,β,λ.

If $\alpha = \beta = -\frac{1}{2}$ then (3.2) reduces to the Fourier-cosine transform

$$(3.4) \qquad \hat{f}_{-\frac{1}{2},-\frac{1}{2}}(\lambda) = (2/\pi)^{\frac{1}{2}} \int_0^\infty f(t)\cos\lambda t\,dt.$$

<u>THEOREM 3.1.</u> (Paley and Wiener). *The Fourier-cosine transform is a bijection from* C_0^∞ *onto* H.

For a proof see for instance HÖRMANDER [17,Theorem 1.7.7]. In this section we shall generalize theorem 3.1 to general complex values of α and β.

Let for $f \in C_0^\infty$ and $\mathrm{Re}\,\alpha > -\frac{1}{2}$ the mapping $f \to F_{\alpha,\beta}(f)$ be defined by

$$(3.5)\qquad (F_{\alpha,\beta}(f))(s) = \int_s^\infty f(t)A_{\alpha,\beta}(s,t)dt\ , \qquad s > 0.$$

Note that $(F_{\alpha,\beta}(f))(s)$ is analytic in α and β. In particular, if $\mathrm{Re}\,\alpha > \mathrm{Re}\,\beta > -\frac{1}{2}$ then by (2.18) we have

$$(3.6)\qquad (F_{\alpha,\beta}(f))(s) = \frac{2^{3\alpha+3/2}}{\Gamma(\alpha-\beta)} \int_s^\infty \left[\frac{1}{\Gamma(\beta+\frac{1}{2})} \int_w^\infty f(t)(\cosh 2t-\cosh 2w)^{\beta-\frac{1}{2}} d(\cosh 2t)\right] \cdot (\cosh w-\cosh s)^{\alpha-\beta-1} d(\cosh w).$$

Combining (2.16), (3.2) and (3.5) we obtain that for $f \in C_0^\infty$ and $\mathrm{Re}\,\alpha > -\frac{1}{2}$

$$(3.7)\qquad f^\wedge_{\alpha,\beta}(\lambda) = (2/\pi)^{\frac{1}{2}} \int_0^\infty (F_{\alpha,\beta}(f))(s)\cos\lambda s\, ds.$$

This means that the Jacobi transform of order (α,β) of f is the cosine transform of $F_{\alpha,\beta}(f)$.

To analyze the transform $F_{\alpha,\beta}$ consider the Weyl fractional integral transform W_μ which is for $a \in \mathbb{R}$, $g \in C_0^\infty([a,\infty))$ and $\mathrm{Re}\,\mu > 0$ defined by

$$(3.8)\qquad (W_\mu(g))(y) = (\Gamma(\mu))^{-1} \int_y^\infty g(x)(x-y)^{\mu-1}dx$$

(cf.[8,Chap.13]). Here $C_0^\infty([a,\infty))$ denotes the class of infinitely differentiable functions on the interval $[a,\infty)$ (right differentiable in a) with compact support. Repeated integration by parts in (3.8) gives

$$(3.9)\qquad (W_\mu(g))(y) = \frac{(-1)^n}{\Gamma(\mu+n)} \int_y^\infty \left(\frac{d^n}{dx^n} g(x)\right)(x-y)^{\mu+n-1}dx, \qquad n=0,1,2,\dots .$$

By (3.8) and (3.9) $(W_\mu(g))(y)$ is defined as an entire function in μ, continuous in $(\mu,y) \in \mathbb{C} \times [a,\infty)$. Clearly, the function $W_\mu(g)$ has also compact support and, since $(W_\mu(g))' = W_\mu(g')$ we conclude that $W_\mu(g) \in C_0^\infty([a,\infty))$. It is an easy exercise to prove that $W_0 = \mathrm{id}$, $W_{-1}(g) = -g'$, $W_\mu \circ W_\nu = W_{\mu+\nu}$. In particular, $W_\mu \circ W_{-\mu} = \mathrm{id} = W_{-\mu} \circ W_\mu$. This proves the following theorem.

THEOREM 3.2. *For all* $a \in \mathbb{R}$ *and* $\mu \in \mathbb{C}$ *the mapping* $\mathcal{W}_\mu$, *defined by* (3.9), *is bijective from* $C_0^\infty([a,\infty))$ *onto itself.*

Let us next define for $f \in C_0^\infty$, $\mathrm{Re}\,\mu > 0$, $\sigma > 0$, $s \geq 0$

$$(3.10)\quad (\mathcal{W}_\mu^\sigma(f))(s) = (\Gamma(\mu))^{-1} \int_s^\infty f(t)(\cosh \sigma t - \cosh \sigma s)^{\mu-1} d(\cosh \sigma t).$$

Again we can extend $(\mathcal{W}_\mu^\sigma(f))(s)$ to an entire function of μ by

$$(3.11)\quad (\mathcal{W}_\mu^\sigma(f))(s) = \frac{(-1)^n}{(\mu+n)} \int_s^\infty \left(\frac{d^n}{d(\cosh \sigma t)^n} f(t)\right) \cdot (\cosh \sigma t - \cosh \sigma s)^{\mu+n-1} d(\cosh \sigma t),$$

$$n=0,1,2,\ldots \quad , \qquad \mathrm{Re}\,\mu > -n.$$

Let $f(t) = g(\cosh \sigma t)$. Then $f \in C_0^\infty$ if and only if $g \in C_0^\infty([1,\infty))$. Hence it follows from theorem 3.2 that for each $\mu \in \mathbb{C}$ the mapping $\mathcal{W}_\mu^\sigma$ is bijective from C_0^∞ onto itself. The inverse mapping of $\mathcal{W}_\mu^\sigma$ is $\mathcal{W}_{-\mu}^\sigma$. Applying this result to (3.6) we obtain

COROLLARY 3.3. *If* $f \in C_0^\infty$ *then* $(F_{\alpha,\beta}(f))(s)$ *has an analytic continuation to an entire function in* α *and* β *which is given by*

$$(3.12)\quad F_{\alpha,\beta}(f) = 2^{3\alpha+3/2} \mathcal{W}_{\alpha-\beta}^1 \circ \mathcal{W}_{\beta+\frac{1}{2}}^2 (f).$$

For all $\alpha,\beta \in \mathbb{C}$ *the mapping* $F_{\alpha,\beta}$ *is bijective from* C_0^∞ *onto itself. The inverse mapping is given by*

$$(3.13)\quad f = 2^{-3\alpha-3/2} \mathcal{W}_{-\beta-\frac{1}{2}}^2 \circ \mathcal{W}_{\beta-\alpha}^1 \circ F_{\alpha,\beta}(f).$$

Combination of Theorem 3.1, corollary 3.3 and formula (3.7) gives the Paley-Wiener type theorem for the Jacobi-transform.

THEOREM 3.4. *For all* $\alpha,\beta \in \mathbb{C}$ *the mapping* $f \to \hat{f}_{\alpha,\beta}$ *is bijective from* C_0^∞ *onto* $\mathcal{H}$.

4. THE INVERSION FORMULA

It is well-known that the inversion formula for the cosine transform is given by

$$(4.1)\qquad f(t) = (2/\pi)^{\frac{1}{2}} \int_0^\infty \hat{f}_{-\frac{1}{2},-\frac{1}{2}}(\lambda) \cos \lambda t \, d\lambda,$$

where $f \in C_0^\infty$ and $\hat{f}_{-\frac{1}{2},-\frac{1}{2}}(\lambda)$ is defined by (3.4). Substituting $\cos \lambda t = \frac{1}{2}e^{i\lambda t}+\frac{1}{2}e^{-i\lambda t}$ and changing the path of integration in (4.1) we also have

$$(4.2)\qquad f(t) = (2\pi)^{-\frac{1}{2}} \int_{i\eta-\infty}^{i\eta+\infty} \hat{f}_{-\frac{1}{2},-\frac{1}{2}}(\lambda) e^{i\lambda t} d\lambda,$$

where η is an arbitrary real number. In this section we shall generalize (4.1) and (4.2) to inversion formulas for the Jacobi transform.

Let for $g \in H$, $t > 0$ and $\alpha,\beta \in \mathbb{C}$

$$(4.3)\qquad \check{g}_{\alpha,\beta}(t) = (2\pi)^{-\frac{1}{2}} \int_{i\eta-\infty}^{i\eta+\infty} g(\lambda)\, \Phi_\lambda^{(\alpha,\beta)}(t)(c_{\alpha,\beta}(-\lambda))^{-1} d\lambda,$$

where $\eta \geq 0$, $\eta > -\mathrm{Re}(\alpha+\beta+1)$, $\eta > -\mathrm{Re}(\alpha-\beta+1)$, i.e., $(c_{\alpha,\beta}(-\lambda))^{-1}$ is a regular function of λ for $\mathrm{Im}\,\lambda \geq \eta$. Let for $g \in H$ A be a positive constant such that the estimates (3.1) hold and choose $\delta > 0$. Then by lemmas 2.1 and 2.2 there is a positive constant K such that for all $t \geq \delta$ and all $\lambda \in \mathbb{C}$ with $\mathrm{Im}\,\lambda \geq 0$ and λ outside arbitrary small neighborhoods of the poles of $(c_{\alpha,\beta}(-\lambda))^{-1}$ we have

$$(4.4)\qquad |g(\lambda)\Phi_\lambda^{(\alpha,\beta)}(t)(c_{\alpha,\beta}(-\lambda))^{-1}| \leq K\, e^{-\rho t}(1+|\lambda|)^{-2} e^{(A-t)\mathrm{Im}\,\lambda}.$$

It follows that the integral in (4.3) absolutely converges and that its value does not depend on the choice of η. In particular, if $|\mathrm{Re}\,\beta| < \mathrm{Re}(\alpha+1)$ then we can put $\eta = 0$ in (4.3) and by (2.5) we obtain

$$(4.5)\qquad \check{g}_{\alpha,\beta}(t) = \frac{\sqrt{2}}{\Gamma(\alpha+1)} \int_0^\infty \frac{g(\lambda)\phi_\lambda^{(\alpha,\beta)}(t)}{c_{\alpha,\beta}(\lambda)c_{\alpha,\beta}(-\lambda)} d\lambda.$$

LEMMA 4.1. *Let* Re $\alpha > -\frac{1}{2}$ *and* $|\mathrm{Re}\,\beta| < \mathrm{Re}(\alpha+1)$. *If* $g \in H$ *then* $\check{g}_{\alpha,\beta} \in C_0^\infty$ *and* $(\check{g}_{\alpha,\beta})^\wedge_{\alpha,\beta} = g$.

Proof. It follows from (4.3) and (4.4) by letting $\eta \to \infty$ that $\check{g}_{\alpha,\beta}(t) = 0$ if $t > A$. It is clear from (4.5) that $\check{g}_{\alpha,\beta}$ is even. The estimates from lemmas 2.2 and 2.3 and formula (3.1) show that

$$\left|g(\lambda)\left(\frac{d^n}{dt^n}\,\phi_\lambda^{(\alpha,\beta)}(t)\right)(c_{\alpha,\beta}(\lambda)c_{\alpha,\beta}(-\lambda))^{-1}\right| \le$$

$$\le \text{const.}(1+t)e^{-(\mathrm{Re}\,\rho)t}(1+\lambda)^{-2} ,$$

uniformly if $\lambda, t \ge 0$. Hence, by (4.5), $\check{g}_{\alpha,\beta} \in C_0^\infty$. To prove the second part of the theorem observe that for $\eta > 0$ and $s > 0$

$$(F_{\alpha,\beta}(\check{g}_{\alpha,\beta}))(s) =$$

$$= (2\pi)^{-\frac{1}{2}} \int_s^\infty A_{\alpha,\beta}(s,t)dt \int_{i\eta-\infty}^{i\eta+\infty} g(\lambda)\Phi_\lambda^{(\alpha,\beta)}(t)(c_{\alpha,\beta}(-\lambda))^{-1}d\lambda =$$

$$= (2\pi)^{-\frac{1}{2}} \int_{i\eta-\infty}^{i\eta+\infty} \left[\int_s^\infty \Phi_\lambda^{(\alpha,\beta)}(t)(c_{\alpha,\beta}(-\lambda))^{-1}A_{\alpha,\beta}(s,t)dt\right]g(\lambda)d\lambda,$$

where the interchanging of integrals is allowed by Fubini's theorem, in view of (4.4) and the estimate

$$|A_{\alpha,\beta}(s,t)| \le \text{const.}e^{\rho t}(t-s)^{\alpha-\frac{1}{2}} , \qquad t > s > 0,$$

which is evident from (2.20). Inserting (2.17) we find that

$$(F_{\alpha,\beta}(\check{g}_{\alpha,\beta}))(s) = (2\pi)^{-\frac{1}{2}} \int_{i\eta-\infty}^{i\eta+\infty} g(\lambda)e^{i\lambda s}d\lambda.$$

By inverting this formula it follows that

$$(\check{g}_{\alpha,\beta})^\wedge_{\alpha,\beta}(\lambda) = (2/\pi)^{\frac{1}{2}} \int_0^\infty (F_{\alpha,\beta}(\check{g}_{\alpha,\beta}))(s)\cos\lambda s\, ds = g(\lambda). \ \square$$

<u>THEOREM 4.2.</u> *Let* $\alpha,\beta \in \mathbb{C}$. *Then* $f \in C_0^\infty$ *and* $g = f^\wedge_{\alpha,\beta}$ *if and only if* $g \in H$ *and* $f = g^\vee_{\alpha,\beta}$.

Proof. In view of theorem 3.4 it is sufficient to prove that $(f^\wedge_{\alpha,\beta})^\vee_{\alpha,\beta}(t)=f(t)$ if $f \in C_0^\infty$, $t > 0$ and $\alpha,\beta \in \mathbb{C}$. By theorem 4.1 this is true for $\mathrm{Re}\,\alpha > -\frac{1}{2}$, $|\mathrm{Re}\,\beta| < \mathrm{Re}(\alpha+1)$. By (3.3) and (4.3) $(f^\wedge_{\alpha,\beta})^\vee_{\alpha,\beta}(t)$ is an entire function of α and β. Hence the theorem follows by analytic continuation. □

5. SOME REMARKS

Remark 1. Suppose that $(c_{\alpha,\beta}(-\lambda))^{-1}$ has N poles $\lambda_1,\lambda_2,\ldots,\lambda_N$ such that $\mathrm{Im}\,\lambda_n > 0$. Then a formula similar to (4.5) can be derived with additional terms of the type $c_n^{(\alpha,\beta)}g(\lambda_n)\phi_{\lambda_n}^{(\alpha,\beta)}(t)$, $n=1,2,\ldots,N$ (cf. FLENSTED-JENSEN [11,§2]). Complications arise if some pole of $(c_{\alpha,\beta}(-\lambda))^{-1}$ is not simple or lies on the real axis or coincides with a pole of $(c_{\alpha,\beta}(\lambda))^{-1}$.

Remark 2. Let $f \in C_0^\infty$ and $g \in H$. Suppose for convenience that $(c_{\alpha,\beta}(-\lambda))^{-1}$ has no poles for $\mathrm{Im}\,\lambda \geq 0$, i.e., $|\mathrm{Re}\,\beta| < \mathrm{Re}(\alpha+1)$. Then it is clear from (3.2) and (4.5) that

$$\int_0^\infty f(t)g^\vee(t)\Delta(t)dt = \int_0^\infty f^\wedge(\lambda)g(\lambda)(c(\lambda)(c(-\lambda))^{-1}d\lambda.$$

Here Fubini's theorem is used together with the estimates of lemmas 2.2 and 2.3 and formula (3.1). It follows by theorem 4.2 that for $f_1,\ f_2 \in C_0^\infty$

$$(5.1)\qquad \int_0^\infty f_1(t)f_2(t)\Delta(t)dt = \int_0^\infty f_1^\wedge(\lambda)f_2^\wedge(\lambda)(c(\lambda)c(-\lambda))^{-1}d\lambda.$$

Remark 3. For real α and β, $|\beta| < \alpha+1$, formula (5.1) implies Parseval's formula

$$(5.2)\qquad \int_0^\infty f_1(t)\overline{f_2(t)}\Delta(t)dt = \int_0^\infty f_1^\wedge(\lambda)\overline{f_2^\wedge(\lambda)}|c(\lambda)|^{-2}d\lambda,$$

where $f_1,f_2 \in C_0^\infty$. Hence, since C_0^∞ is dense in $L^2(\Delta)$ and H is dense in $L^2(|c(\lambda)|^{-2})$, the Jacobi transform can be extended to an isometric mapping from $L^2(\Delta)$ onto $L^2(|c(\lambda)|^{-2})$. This gives an alternative proof for the Plancherel theorem obtained by FLENSTED-JENSEN [9,Prop.3]. If $(c(-\lambda))^{-1}$ has poles for $\mathrm{Im}\,\lambda > 0$ then a discrete spectrum must be added (cf. [11,§2]).

Remark 4. A Paley-Wiener type theorem for the Hankel transform can be proved by similar methods as in section 3. Let $\mathcal{J}_\alpha(t)$ be a solution of the differential equation $u''(t)+(2\alpha+1)t^{-1}u'(t)+u(t) = 0$, $\alpha \neq -1,-2,\ldots$, such that $\mathcal{J}_\alpha(0) = 1, \mathcal{J}_\alpha'(0) = 0$. Then $\mathcal{J}_\alpha(t) = 2^\alpha\Gamma(\alpha+1)t^{-\alpha}J_\alpha(t)$, where $J_\alpha(t)$ is a Bessel function. If $\mathrm{Re}\,\alpha > -\frac{1}{2}$ then it follows from the Poisson integral representation

$$J_\alpha(t) = \frac{\Gamma(\alpha+1)}{\pi^{\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi e^{it\cos\phi}(\sin\phi)^{2\alpha} d\phi$$

that

$$(5.3) \qquad t^{2\alpha} J_\alpha(\lambda t) = \frac{2\Gamma(\alpha+1)}{\pi^{\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})} \int_0^t \cos\lambda s (t^2-s^2)^{\alpha-\frac{1}{2}} ds.$$

Define for $f \in C_0^\infty$ and Re $\alpha > -1$ the Hankel transform by

$$(5.4) \qquad f^\wedge(\lambda) = \frac{1}{2^\alpha\Gamma(\alpha+1)} \int_0^\infty f(t) J_\alpha(\lambda t) t^{2\alpha+1} dt.$$

Then

$$(5.5) \qquad f^\wedge(\lambda) = (2/\pi)^{\frac{1}{2}} \int_0^\infty \cos\lambda s \, ds \, \frac{1}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})} \cdot \int_s^\infty f(t)(t^2-s^2)^{\alpha-\frac{1}{2}} d(t^2), \qquad \text{Re } \alpha > -\tfrac{1}{2}.$$

Formula (5.5) is analogous to (3.6) and (3.7) and it can be used in a similar way.

Remark 5. For certain discrete values of α and β Jacobi functions are the spherical functions on non-compact symmetric spaces of rank one. In this context many formulas and results of [9] and the present paper were earlier obtained. Formula (3.7) corresponds to HELGASON [15,(9)]. The function $e^{-\rho s}(F_{\alpha,\beta}(f))(s)$ has a geometric interpretation as a Radon transform, where f is a radial function on the symmetric space (cf. HELGASON [16,Chap.1,2]). The Paley-Wiener theorem for the spherical Fourier transform on non-compact symmetric spaces of rank one was first proved by HELGASON [15].

REMARK 6. Formulas (2.16) and (2.18) generalize the classical Mehler-Dirichlet formula (cf.MEHLER [20])

$$P_\nu(\cos\theta) = \frac{2^{\frac{1}{2}}}{\pi} \int_0^\theta \frac{\cos(\nu+\frac{1}{2})\phi}{(\cos\phi-\cos\theta)^{\frac{1}{2}}} d\phi,$$

where $P_\nu(x)$ is a Legendre function. These formulas can also be obtained from the Laplace type integral representation

$$(5.6)\quad \phi_\lambda^{(\alpha,\beta)}(t) = \frac{2\Gamma(\alpha+1)}{\pi^{\frac{1}{2}}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} \int_0^1 \int_0^\pi |\cosh t+\sinh t\, re^{i\psi}|^{i\lambda-\rho}$$
$$\cdot (1-r^2)^{\alpha-\beta-1} r^{2\beta+1} (\sin\psi)^{2\beta} dr\, d\psi\,, \quad t > 0,\ \mathrm{Re}\,\alpha > \mathrm{Re}\,\beta > -\tfrac{1}{2}$$

(cf. [18,(4)],[9,(3.5)]) by substituting first $\cosh t+\sinh t\, re^{i\psi} = e^s e^{i\chi}$ and next $\cosh w = \cos\chi \cosh t$. A general method of transforming integrals of type (5.6) into integrals of type (2.16) is discussed in [19,§5].

Remark 7. Let $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x) \,/\, P_n^{(\alpha,\beta)}(1)$, where $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial. Then

$$R_n^{(\alpha,\beta)}(\cos\theta) = F(-n,n+\alpha+\beta+1;\alpha+1;\sin^2\tfrac{1}{2}\theta) = \phi^{(\alpha,\beta)}_{(2n+\alpha+\beta+1)i}(\tfrac{1}{2}i\theta).$$

Analogous to (2.16), (2.18) and (2.19 we obtain

$$(5.7)\quad R_n^{(\alpha,\beta)}(\cos\theta) = \frac{2^{\alpha-2\beta-\frac{1}{2}}\Gamma(\alpha+1)}{\pi^{\frac{1}{2}}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} (\sin\tfrac{1}{2}\theta)^{-2\alpha}(\cos\tfrac{1}{2}\theta)^{-2\beta}$$
$$\cdot \int_0^\theta (\cos\psi-\cos\theta)^{\beta-\frac{1}{2}} \sin\tfrac{1}{2}\psi\, d\psi \int_0^\psi \cos(n+\tfrac{1}{2}(\alpha+\beta+1))\phi$$
$$\cdot (\cos\tfrac{1}{2}\phi-\cos\tfrac{1}{2}\psi)^{\alpha-\beta-1} d\phi\,, \quad \mathrm{Re}\,\alpha > \mathrm{Re}\,\beta > -\tfrac{1}{2},\ 0 < \theta < \pi,$$

$$(5.8)\quad R_n^{(\alpha,\beta)}(\cos\theta) = \frac{2^{\alpha-\frac{1}{2}}\Gamma(\alpha+1)}{\pi^{\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})} (\sin\tfrac{1}{2}\theta)^{-2\alpha}(\cos\tfrac{1}{2}\theta)^{-\beta-\frac{1}{2}}$$
$$\cdot \int_0^\theta \cos(n+\tfrac{1}{2}(\alpha+\beta+1))\phi\ F(\tfrac{1}{2}+\beta,\tfrac{1}{2}-\beta;\alpha+\tfrac{1}{2};\ \frac{\cos\frac{1}{2}\theta-\cos\frac{1}{2}\phi}{2\cos\frac{1}{2}\theta}) d\phi,$$
$$\mathrm{Re}\,\alpha > -\tfrac{1}{2},\ 0 < \theta < \pi.$$

Quadratic transformation of the hypergeometric function in (5.8) by means of [7,2.11(22)] gives another integral representation for $R_n^{(\alpha,\beta)}(\cos\theta)$, which was independently obtained by GASPER [14] in a quite different way.

Remark 8. Suppose that f is an even C^∞-function on $(-\pi,\pi)$ with compact support. If f is expanded in a Fourier-Jacobi series with respect to $R_n^{(\alpha,\beta)}(\cos\theta)(\alpha>\beta>-\frac{1}{2})$ then the Fourier coefficients are given by

$$(5.9)\qquad f^{\wedge}(n) = (\Gamma(\alpha+1))^{-1}\int_0^{\pi} f(\theta)R_n^{(\alpha,\beta)}(\cos\theta)(\sin\tfrac{1}{2}\theta)^{2\alpha+1}(\cos\tfrac{1}{2}\theta)^{2\beta+1}d\theta,\quad n=0,1,2,\ldots\ .$$

Substitution of (5.7) in (5.9) gives

$$(5.10)\qquad f^{\wedge}(n) = \frac{2^{\alpha-2\beta-3/2}}{\pi^{\frac{1}{2}}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}\int_0^{\pi}\cos(n+\tfrac{1}{2}(\alpha+\beta+1))\phi\ d\phi$$

$$\cdot\int_{\psi=\phi}^{\pi}(\cos\tfrac{1}{2}\phi-\cos\tfrac{1}{2}\psi)^{\alpha-\beta-1}d(\cos\tfrac{1}{2}\psi)\int_{\theta=\psi}^{\pi} f(\theta)(\cos\psi-\cos\theta)^{\beta-\frac{1}{2}}d(\cos\theta).$$

In the same way as in section 3 we can write

$$(5.11)\qquad f^{\wedge}(n) = \int_0^{\pi}\cos(n+\tfrac{1}{2}(\alpha+\beta+1))\phi\ (F(f))(\phi)d\phi,$$

where the mapping F is a bijection from the class of even C^{∞}-functions on $(-\pi,\pi)$ with compact support onto itself. Then the function $f^{\wedge}$ is well-defined and analytic for all complex values of its argument. Now the classical Paley-Wiener theorem implies a Paley-Wiener type theorem for Jacobi series.

THEOREM 5.1. *Let* $\alpha > \beta > -\frac{1}{2}$. *The function* $f^{\wedge}$ *is the Fourier-Jacobi transform of an even* C^{∞}*-function on* $(-\pi,\pi)$ *with compact support if and only if there is a function* $g \in H$ *such that* $A < \pi$ *in* (3.1) *and* $f^{\wedge}(n) = g(n+\frac{1}{2}(\alpha+\beta+1))$, $n=0,1,2,\ldots$.

Since g is of exponential type less than π an application of Carlson's theorem (cf. TITCHMARSH [23,§5.81]) shows that g is uniquely determined by $f^{\wedge}_{\alpha,\beta}(n)$, $n=0,1,2,\ldots$. Just as in section 3 theorem 5.1 remains valid for all $\alpha,\beta \in \mathbb{C}$. R.ASKEY informed me that in the case $\alpha = \beta = 0$ this theorem is due to BEURLING (unpublished).

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