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THE ADDITION FORMULA FOR JACOBI POLYNOMIALS AND THE THEORY OF ORTHOGONAL POLYNOMIALS IN TWO VARIABLES, A SURVEY

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THE ADDITION FORMULA FOR JACOBI POLYNOMIALS AND THE THEORY OF ORTHOGONAL POLYNOMIALS IN TWO VARIABLES, A SURVEY*)

by

T.H. KOORNWINDER

ABSTRACT

This report, which will be part of the author's thesis at the University of Amsterdam, gives an introduction to some research papers published by the author. It contains a survey of past and recent results related to the addition formula for Jacobi polynomials and the theory of orthogonal polynomials in two variables, and it opens some perspectives for further research. In particular, three different proofs of the addition formula for Jacobi polynomials are discussed, both of group theoretic and of analytic nature, and two new classes of orthogonal polynomials in two variables are described, which can be considered as two-variable analogues of the Jacobi polynomials.

*) This paper will be part of the author's thesis at the University of Amsterdam (1975), entitled "Jacobi polynomials and their two-variable analogues".

CONTENTS

INTRODUCTION I

PART I

The addition formula for Jacobi polynomials and the theory of orthogonal polynomials in two variables, a survey

1. Historical background of the addition formula 1
2. First group theoretic proof of the addition formula 2
3. Jacobi functions 4
4. Analytic proof of the addition formula 5
5. Second group theoretic proof of the addition formula 6
6. Further analytic results and open problems related to the addition formula 6
7. Further group theoretic results and open problems related to the addition formula 7
8. History of the theory of orthogonal polynomials in two variables 8
9. Orthogonal polynomials in two variables which are eigenfunctions of a Laplace-Beltrami operator 9
10. Further analytic results and open problems on orthogonal polynomials in two variables 11
11. Further group theoretic results and open problems on orthogonal polynomials in two variables 11

REFERENCES 13

INTRODUCTION

This paper will be the first part of the author's thesis at the University of Amsterdam. The second part of this thesis will consist of the references [1]-[5]. The purpose of the present report is to give an introduction to some work of the author, especially the papers [1]-[5], to present a survey of past and recent results related to the addition formula for Jacobi polynomials and the theory of orthogonal polynomials in two variables, and to open some perspectives for further research.

Both Jacobi polynomials and the polynomials considered in [4] and [5] are eigenfunctions of a second order differential operator. For particular values of the parameters and after a suitable transformation of variables this differential operator is the Laplace-Beltrami operator on a compact Riemannian symmetric space. In this way the two subjects of this report are connected with each other. A further correspondence is given by the fact that orthogonal polynomials on the disk naturally occur in the first proof of the addition formula and that the addition formula itself can be viewed as an expansion in terms of certain orthogonal polynomials in two variables (cf. §2).

Attempts to find an addition formula for Jacobi polynomials were originally motivated by the problem of proving the positivity of the generalized translation for Jacobi series (cf. §1). However, this formula is also important intrinsically, because it contains a large amount of analytic information in a rather compact form. It turns out that many other results for Jacobi polynomials, Laguerre polynomials and Bessel functions can be derived from this addition formula.

The polynomials studied in [4] and [5] provide examples of complete orthogonal systems of functions in two variables which cannot be factorized as products of functions in one variable. Both from a theoretical and a practical point of view, it seems useful to have available such examples. Furthermore, the results in [4] and [5] may be helpful in developing a theory of special functions associated with symmetric spaces of rank higher than one.

An important tool in the author's research is the group theoretic interpretation of special functions. We mention three recent books on this subject by VILENKIN [97], TALMAN [93] and MILLER [77]. It seems that the deepest results can be obtained by interpreting special functions as spherical functions on homogeneous spaces, in particular on symmetric spaces. Part of VILENKIN's book [97] follows this approach. The primary aim of the author is to obtain new results for special functions. The group theoretic method is only a tool, which must be combined with classical analytic methods in order to get satisfactory results. The connection between special functions and group theory should not be considered in a too dogmatic way. An interesting group theoretic result does not always have a counterpart for special functions which is worth mentioning, and not all deep formulas for special functions have group theoretic interpretations.

The derivation of explicit formulas for special orthogonal polynomials is the main subject of this report. Wherever possible, we have discussed the meaning of these formulas in connection with other fields of mathematics. Otherwise, this work would have been rather sterile. The following three branches of mathematics have interactions with the subject of this report:

a) The theory of Lie groups and homogeneous spaces. This was discussed in the previous paragraph.
b) The analysis of continuous orthogonal systems which are dual in a certain sense to the orthogonal systems of polynomials under consideration here. An example is given by the analysis of Jacobi functions in relation to Jacobi polynomials, cf. §3.
c) Harmonic analysis for orthogonal polynomial expansions. For instance, the problem of proving the positivity of the generalized translation for Jacobi series motivated research to derive an addition formula for Jacobi polynomials.

For particular values of the parameters the orthogonal systems considered in this report reduce to trigonometric orthogonal systems. Many of the identities obtained here are nontrivial generalisations of trigonometric identities. Conversely, reduction of the general case to the trigonometric case sometimes leads to new results.

The present survey paper consists of two parts. In sections 1-7 the addition formula for Jacobi polynomials is discussed and sections 8-11 deal with orthogonal polynomials in two variables. Each part opens with a historical survey (§1 and §8) and concludes with a list of further results and open problems of analytic nature (§6 and §10) and of group theoretic nature (§7 and §11). Corresponding to the papers [1], [2], [3], three different proofs of the addition formula are described in sections 2, 5 and 4, respectively. The important applications to Jacobi functions are the subject of §3. Finally, the papers [4], [5] on orthogonal polynomials in two variables are discussed in §9.

The following standard references may be helpful to the reader. For orthogonal polynomials in one variable we refer to SZEGO [92]. The Bateman project ERDELYI [33, Vol. 2, Chap. 10] contains a large collection of formulas on classical orthogonal polynomials. A pleasant introduction to spherical harmonics is given by MULLER [80]. A standard reference for the theory of symmetric spaces is HELGASON [52]. Symmetric spaces are defined in Chap. 4 and spherical functions are discussed in Chap. 10 of this reference. We shall mainly need the compact case. For an elementary discussion of this case see COIFMAN & WEISS [28, Chap. 2, §4].

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I am deeply indebted to Prof. Dr. Richard Askey, who was a visiting professor at the Mathematical Centre during the academic year 1969/1970. He introduced me to the subject of orthogonal polynomials and he suggested me the problem of finding an addition formula for Jacobi polynomials. His great enthusiasm, his encyclopedic knowledge of current and past research in classical analysis and his numerous contacts were of great value to me.

1. HISTORICAL BACKGROUND OF THE ADDITION FORMULA

The addition formula for ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)$ (cf. ERDELYI [33, Vol. 1, 3.15 (19)]) is due to GEGENBAUER [45]. For $\alpha = 0$ the formula was earlier obtained by LAPLACE [70] and LEGENDRE [72, p. 420], [73, p. 432] (see HEINE [51, pp. 2 and 313] for the question of priority). Other proofs of the addition formula were given by GEGENBAUER [46], HENRICI [54], MANOCHA [75] and CARLSON [22]. HENRICI [54] considerably generalized the addition formula to complex values of n and to Gegenbauer functions of the second kind.

If $\alpha = \frac{1}{2}q-3/2$ $(q = 3, 4, ...)$ then ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)$ (also called Gegenbauer polynomials) are the zonal spherical harmonics on the sphere S^{q-1} and the addition formula follows by using this group theoretic interpretation, cf. ERDELYI [33, Vol. 2, Chap. 11], MULLER [80] and VILENKIN [97, Chap. 9], see also KOORNWINDER [2, §2] for a sketch of this approach. In this context of spherical harmonics the nature of the addition formula can be best understood. It is quite usual that a group theoretic interpretation of a class of special functions only exists for discrete values of a parameter. In the present case, analytic continuation of the addition formula to general α is possible, since both sides of the identity are rational functions of α. In other cases analytic continuation may be justified by CARLSON's theorem (cf. TITCHMARSH [94, Theorem 5.81]).

The addition formula has a central position in the theory of Gegenbauer polynomials, since many other results for Gegenbauer polynomials, Hermite polynomials and Bessel functions can be derived from it as integrated forms or as limit cases. A very important corollary is the product formula ERDELYI [33, Vol. 1, 3.15 (20)] which expresses the product $P_n^{(\alpha,\alpha)}(x)\, P_n^{(\alpha,\alpha)}(y)$ as an integral of a single Gegenbauer polynomial of the same degree and order.

BOCHNER [16], [17] pointed out that the product formula implies the positivity of the generalized translation for Gegenbauer series ($\alpha \geqslant -\frac{1}{2}$). Hence, for Gegenbauer series there exists a positive convolution structure, which is an important tool for extending harmonic analysis of Fourier-cosine series to the ultraspherical case, see for instance MUCKENHOUPT & STEIN [78, §15]. WEINBERGER [100] obtained Bochner's positivity results in a different way, by using a maximum principle for hyperbolic differential equations.

Attemtps to extend BOCHNER's results to Jacobi series were the main impetus to find an addition formula for Jacobi polynomials. However, from an analytic point of view it was not clear at all how to generalize Gegenbauer's addition formula. WEINBERGER's approach to prove the positivity of generalized translation by using a maximum principle also failed in the Jacobi case, cf. ASKEY [9, §5]. In those special cases of the parameters α, β for which Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be interpreted as spherical functions on a compact Riemannian symmetric space of rank one (cf. [2, §3]) the positivity of convolution for Jacobi series follows by a group theoretic interpretation of the convolution product. Jacobi polynomials as spherical functions are discussed by CARTAN [23, Chap. 7, 8], HELGASON [53, §6], GANGOLLI [41], RAGOZIN [86]. GELFAND [47] pointed out that for a symmetric space G/K the convolution algebra $L^1(K\backslash G/K)$ is commutative. In the compact case the characters of this algebra are the spherical functions.

For general order, no group theoretic method was available to settle the positivity of convolution for Jacobi series. A weaker result than positivity is uniform boundedness. For $\alpha > \beta > -\frac{1}{2}$ the uniform boundedness of convolution for Jacobi series was proved by ASKEY & WAINGER [11]. Next, GASPER [42], [43] completely solved the positivity problem in the following way. Let the kernel $K(x, y, z)$ be defined by

$$(1.1)\quad P_n^{(\alpha,\beta)}(x)\,\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} = \int_{-1}^{1} P_n^{(\alpha,\beta)}(z)\; K(x,y,z).$$
$$.(1-z)^{\alpha}(1+z)^{\beta}\, dz\ ,\ n = 0, 1, 2, \ldots .$$

Then the integral defining the convolution product has K as a kernel and it has to be proved that K is nonnegative. By duality $K(x,y,z)$ can be expressed as a triple Jacobi series. GASPER [42] first rewrote this series as an integral of the product of three Bessel functions and next he expressed it as an integral of a nonnegative elementary function $(\alpha>\beta>-\frac{1}{2})$, where some formulas from WATSON [99] were applied. In [43] GASPER expressed $K(x, y, z)$ in terms of hypergeometric functions, and he proved that K is nonnegative if and only if $\alpha \geqslant \beta \geqslant -\frac{1}{2}$ or $\alpha \geqslant |\beta|, \beta > -1$. GASPER [43, §2] discussed several applications of this convolution structure. Applications to approximation theory are given by BAVINCK [13].

The feeling that an addition formula would exist for Jacobi polynomials was strengthened by the fact that Gasper succeeded in proving the positivity of the generalized translation for Jacobi series. It could be expected that such an addition formula would lead to a more elegant product formula than (1.1), where the explicit form of the kernel looks rather tedious. Such an elegant formula is suggested by the product formula

$$(1.2)\quad \phi(x)\,\phi(y) = \int_K \phi(x\,k\,y)\; d\,k\ \ (x, y \in G)$$

for spherical functions ϕ on a homogeneous space G/K (cf. HELGASON [52, Chap. 10, Prop. 3.2]).

2. FIRST GROUP THEORETIC PROOF OF THE ADDITION FORMULA

When the author got involved in the problem to find an addition formula for Jacobi polynomials, all results described in §1 were known. The desired addition formula for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ should reduce to the well-known addition formula for Gegenbauer polynomials for $\alpha = \beta$ and some integrated form of the addition formula should be a product formula equivalent to (1.1). A group theoretic approach seemed to be the most appropriate way to find such a formula.

For $\alpha = \frac{1}{2}p - 1, \beta = \frac{1}{2}q - 1$ (p, q integers $\geqslant 2$) Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be interpreted as spherical harmonics of degree $2n$ on the sphere S^{p+q-1} which are invariant with respect to the subgroup $SO(p) \times SO(q)$ of the rotation group $SO(p+q)$, cf. ZERNIKE & BRINKMAN [103]. Using this interpretation BRAAKSMA & MEULENBELD [21] derived a Laplace type integral representation which expresses $P_n^{(\alpha,\beta)}(x)$ as a double integral of a $(2n)^{\text{th}}$ power. Next, DIJKSMA & KOORNWINDER [32] expressed the product $P_n^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y)$ as a double integral of a Gegenbauer polynomial of degree $2n$ and of order $(\alpha + \beta + \frac{1}{2}, \alpha + \beta + \frac{1}{2})$. Finally, KOORNWINDER [2, (4.7)] obtained the corresponding addition formula, which expands a Gegenbauer polynomial of argument $\cos\theta_1 \cos\theta_2 \cos\phi + \sin\theta_1 \sin\theta_2 \cos\psi$ as a double Gegenbauer series in terms of ϕ and ψ with products of Jacobi polynomials depending on θ_1, and θ_2 as coefficients. These results reduced to the well-known formulas for the ultraspherical case by putting $\beta = -\frac{1}{2}$ and by performing a quadratic transformation. They did not lead to some product formula equivalent to (1.1). However, it was clear now that generalisations of ultraspherical results to the Jacobi case might involve the transition of single series to double series and of single integrals to double integrals.

Another group theoretic approach was suggested by the interpretation of Jacobi polynomials as spherical functions on compact symmetric spaces of rank one. Apart from spheres and real projective spaces, which lead to the addition formula for Gegenbauer polynomials, the most simple examples of such spaces are the complex projective spaces P^{q-1} (**C**) $= SU(q) / U(q-1)$, $q = 3, 4, \ldots$, cf. CARTAN [23, Chap. 7], [24, part 2, Chap. 5]. The spherical functions on P^{q-1} (**C**) are Jacobi polynomials of order $(q-2, 0)$. The analysis of "harmonics" on P^{q-1} (**C**) is contained in the theory of complex spherical harmonics on the sphere S^{2q-1} as homogeneous space $SU(q)/SU(q-1)$. This theory, which is a refinement of the classical theory of spherical harmonics, was initiated by IKEDA & KAYAMA [57], [58]. A somewhat different introduction to complex spherical harmonics was given by THE AUTHOR [63], [64], who applied the theory to derive the addition formula for Jacobi polynomials, cf. KOORNWINDER [64]. See KOORNWINDER [1] for a summary of the method and the results and KOORNWINDER [65] for the general philosophy of this approach.

The spherical functions on $SU(q) / SU(q-1)$ are the so-called disk polynomials $R^{(q-2)}_{m,n}(x+iy)$ of order $q-2$, cf. KOORNWINDER [63, pp. 18, 19]. These are orthogonal polynomials in two variables on the unit disk with respect to the weight function $(1-x^2-y^2)^{q-2}$. They can be expressed in terms of Jacobi polynomials. In particular, we have $R^{(q-2)}_{n,n}(r\, e^{i\phi}) = \text{const.}\ P_n^{(q-2,\,0)}(2r^2-1)$. Disk polynomials were already introduced by ZERNIKE & BRINKMAN [103] as an orthogonal system of spherical harmonics on S^{2q-1} which are invariant with respect to $SO(2q-2)$.

Using their properties as spherical functions, the author derived a Laplace type integral representation [63, (4.3)], a product formula [63, (4.10)] and an addition formula [64, (5.4)] for disk polynomials. These results implied similar formulas for Jacobi polynomials $P_n^{(q-2,0)}(x)$. By differentiation of the addition formula for $P_n^{(q-2,0)}(x)$ there followed an addition formula for $P_n^{(\alpha,\beta)}(x)$ in the case of integer $\alpha, \beta, \alpha > \beta \geqslant 0$. Finally, an addition formula for Jacobi polynomials of general order was obtained by analytic continuation with respect to α and β, cf. KOORNWINDER [64, §6], [1, (3)].

For $\alpha > \beta > -\frac{1}{2}$ this addition formula is an expansion in terms of an orthogonal system in two variables. The functions of this orthogonal system are products of Jacobi and Gegenbauer polynomials. In terms of suitable coordinates these functions are orthogonal polynomials in two variables on a region bounded by a straight line and a parabola, cf. KOORNWINDER [66, §3]. By integration of the addition formula there follows a product formula for Jacobi polynomials KOORNWINDER [1, (2)]. It is a double integral which turns out to be equivalent to GASPER's product formula (1.1), cf. KOORNWINDER [3, §5]. As a limit case of the product formula a Laplace type integral representation KOORNWINDER [1, (1)] is obtained. For $\alpha = \beta$ these three formulas degenerate to the corresponding results for Gegenbauer polynomials. For $\beta = -\frac{1}{2}$ the same is true after a quadratic transformation, cf. KOORNWINDER [66, §5, remark 3].

In §1 we remarked that a product formula for Jacobi polynomials would be the most important corollary of the addition formula. In the group theoretic case $(\alpha, \beta) = (q-2, 0)$ this product formula can be proved independently from the addition formula, cf. KOORNWINDER [63, §4]. However, to prove the product formula for $\beta \neq 0$ we need the complete addition formula for $(\alpha, \beta) = (q-2, 0)$.

After publication of [1] and [63] and after completion of the manuscript of [64] the author became aware of the papers [98], [87] by VILENKIN & SAPIRO. They developed the theory of complex spherical harmonics along the same lines as in the author's papers [63], [64], and in [87] SAPIRO derived the addition formula for disk polynomials and for Jacobi polynomials of order $(q-2, 0)$.

Complex spherical harmonics were also considered by DUNKL & RAMIREZ [31, Chap. 10], FOLLAND [39], BOYD [20] and ANNABI [104]. The homogeneous space $SU(q)/SU(q-1)$ is not a symmetric space, but still the convolution for radial functions on this space is commutative. This was first pointed out by DUNKL & RAMIREZ [31, §9. 4. 3]. ANNABI & TRIMECHE [7], [105] proved that convolution for series of disk polynomials of order $\alpha \geqslant 0$ is positive. Various applications of complex spherical harmonics occur in DUNKL [30], FOLLAND [40], BOYD [20, §2] and DUNKL & RAMIREZ [31, Chap. 10].

In the context of complex spherical harmonics GOETHALS obtained a new proof (unpublished) of the addition formula for Jacobi polynominals $P_n^{(q-2,0)}(x)$ by using combinatorial arguments and matrix methods. DELSARTE, GOETHALS & SEIDEL [109] applied the interpretation of Jacobi polynomials as complex spherical harmonics in order to obtain bounds for the cardinality of sets of complex lines having a prescribed number of angles.

3. JACOBI FUNCTIONS

Jacobi functions $\phi_\lambda^{(\alpha,\beta)}(t)$ form a continuous orthogonal system of functions on the interval $(0, \infty)$ with respect to the weight function $(\sinh t)^{2\alpha+1} (\cosh t)^{2\beta+1}$. They are the noncompact analogues of Jacobi polynomials of order (α, β) and they have the same analytic expression in terms of hypergeometric functions. Formally we have

$$\phi_\lambda^{(\alpha,\beta)}(t) = \text{const}\, P^{(\alpha,\beta)}_{-(\alpha+\beta+1)/2+i\lambda}(\cosh 2t).$$

For certain discrete α, β Jacobi functions are the spherical functions on noncompact symmetric spaces of rank one. FLENSTED-JENSEN [34] initiated a program to study harmonic analysis for Jacobi function expansions of general order analogous to the known results for the radial Fourier transform on noncompact rank one symmetric spaces. It turned out to be useful to carry over results for Jacobi polynomials to Jacobi functions and conversely.

By analytic continuation with respect to n the Laplace type integral representation and the product formula for Jacobi polynomials imply corresponding results KOORNWINDER [1, (4) and (5)] for Jacobi functions. Afterwards, independent proofs of these results were given, cf. FLENSTED-JENSEN [34, p. 150] and FLENSTED-JENSEN & KOORNWINDER [38, Theorem 4.1]. Using the integral representation KOORNWINDER [1, (4)] FLENSTED-JENSEN [34] could simplify the proofs of his estimates for Jacobi functions. These estimates enabled him to prove Paley-Wiener type theorems for the Jacobi function transform. In joint work FLENSTED-JENSEN

& THE AUTHOR [38] applied the product formula KOORNWINDER [1, (5)] in order to settle the positivity of convolution for Jacobi function expansions analogous to GASPER's results [42], [43] for Jacobi series. Using a maximum principle CHEBLI [25], [26] independently obtained positivity results similar to FLENSTED-JENSEN & KOORNWINDER [38] for a wider class of singular Sturm-Liouville problems. In a subsequent paper CHEBLI [27] extended the Paley-Wiener type theorem to this general class of integral transforms. THE AUTHOR [67] rewrote the Laplace type integral representation KOORNWINDER [1, (4)] as an integral representation of Mehler-Dirichlet type. This new formula enabled him to factorize the Jacobi function transform as the product of two different Weyl type fractional integral transforms and a Fourier-cosine transform. The Paley-Wiener type theorem is an immediate corollary of this factorisation. An analogous Mehler-Dirichlet formula for Jacobi polynomials similarly implied a Paley-Wiener theorem for Jacobi series, cf. KOORNWINDER [67, Theorem 5.1]. An equivalent form of this last formula was independently obtained by GASPER [44].

Further group theoretic interpretations of Jacobi functions were given by FLENSTED-JENSEN. Similarly to the interpretation of Jacobi polynomials as spherical harmonics due to ZERNIKE & BRINKMAN [103] (cf. §2), Jacobi functions of order $(\frac{1}{2}p-1, \frac{1}{2}q-1)$ $(p > q > 1)$ can be interpreted as "spherical" functions on the hyperboloid $SO(p,q) / SO(p,q-1)$, which is a pseudo-Riemannian symmetric space. Using this interpretation FLENSTED-JENSEN [35] obtained new group theoretic proofs of the integral representation, product formula and convolution for Jacobi functions. Finally, FLENSTED-JENSEN [36] studied harmonic analysis on the noncompact dual space of the homogeneous space $SU(q) / SU(q-1)$. Spherical functions on this space depend on two variables and can be expressed in terms of Jacobi functions.

4. ANALYTIC PROOF OF THE ADDITION FORMULA

After the discovery of the addition formula by group theoretic means it was tempting to furnish also an analytic proof. Such a proof was finally obtained in an exchange of letters between ASKEY, GASPER and THE AUTHOR. An essential point in finding this proof was the rediscovery of some old results of KOSCHMIEDER and BATEMAN. For the history of this proof and for more references the reader is referred to ASKEY [10].

The analytic proof of the addition formula consists of three steps. First, the Laplace type integral representation for Jacobi polynomials is derived from the corresponding formula for Gegenbauer polynomials by using a fractional integral for Jacobi polynomials, cf. ASKEY [10, §3]. Next, if BATEMAN's formula expressing $P_n^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(y)$ as a linear combination of terms $(x+y)^k\, P_k^{(\alpha,\beta)}((1+xy)/(x+y))$ is applied to the Laplace type integral representation then the product formula for Jacobi polynomials is obtained, cf. KOORNWINDER [3, §3]. Finally, the product formula can be integrated by parts and application of some new second order differential recurrence relations for Jacobi polynomials gives the addition formula, cf. KOORNWINDER [66, §4].

Many related results are discussed in KOORNWINDER [3] and [66]. In particular, we mention a new product formula [3, (3.8)] and addition formula [66, (4.15)] for Bessel functions. In KOORNWINDER [3, §5] it is explained why GASPER's proof in

[42] of the product formula (1.1) could work. The diagram in KOORNWINDER [66, §5, Remark 6] shows, how addition formulas, product formulas and integral representations are logically connected with each other. It is remarkable that the addition formula for any one specific order (α,β) implies the case of general order by differentiation and analytic continuation, cf. KOORNWINDER [66, §5, Remark 1].

As a final result related to this analytic proof we mention HORTON's work [56, Chap. 2]. He used BATEMAN's rediscovered bilinear sum in order to prove the variation diminishing property of the de la Vallée Poussin kernel for Jacobi series.

5. SECOND GROUP THEORETIC PROOF OF THE ADDITION FORMULA

In §2 it was pointed out that the interpretation of Jacobi polynomials as spherical harmonics did not lead to the desired type of addition formula. However, in the noncompact dual case FLENSTED-JENSEN [35] succeeded in proving the product formula for Jacobi functions by a similar interpretation, cf. §3. Flensted-Jensen's approach could be translated into the compact case (cf. KOORNWINDER [2, (5.5)]) and next THE AUTHOR [2, §5] extended this result to a new group theoretic proof of the addition formula. The idea of the proof is as follows. In the approach of KOORNWINDER [2, §2 and §4] an addition formula is obtained by expressing the reproducing kernel of a class of spherical harmonics of certain degree in terms of a canonical basis. However, in KOORNWINDER [2, §5] only those spherical harmonics of degree $2n$ on S^{q+p-1} $(p > q > 1)$ are considered which are invariant with respect to a certain rotation group isomorphic to $SO(q) \times SO(p-q)$. The reproducing kernel of this class of spherical harmonics is expressed in terms of a basis canonical with respect to the rotation group $SO(q) \times SO(p)$. Invariance of this reproducing kernel with respect to a one-parameter transformation group then leads to the addition formula for Jacobi polynomials of order $(\frac{1}{2}p-1, \frac{1}{2}q-1)$. In this way a group theoretic interpretation of the addition formula is given for all orders (α, β) such that $\alpha > \beta > -\frac{1}{2}$ and 2α, 2β are integers.

6. FURTHER ANALYTIC RESULTS AND OPEN PROBLEMS RELATED TO THE ADDITION FORMULA

THE AUTHOR [68, part I] obtained still another proof of the addition formula for Jacobi polynomials. It is shorter than the three other proofs described in the previous sections and it does not involve much computation. It is, however, improbable that anyone could have found this proof without already knowing the addition formula. The proof uses a class of orthogonal polynomials in three variables on a region bounded by a circular cone and a plane orthogonal to the axis of the cone. The reproducing kernel of the class of orthogonal polynomials of degree n is invariant with respect to the one-parameter rotation group of the orthogonality region. The addition formula follows from this invariance. In a certain sense, this proof gives a group theoretic interpretation of the addition formula in the case of arbitrary real α and β $(\alpha > \beta > -\frac{1}{2})$. For Gegenbauer polynomials a similar proof can be given by using orthogonal polynomials on the disk.

WEINBERGER's proof [100] of the positivity of generalized translation for Gegenbauer series by using a maximum principle can be extended to the Jacobi case ($\alpha > \beta > -\frac{1}{2}$). Part of this proof was given by ASKEY [9, §5]. THE AUTHOR [68, part III] completed ASKEY's argument.

For Laguerre series generalized translation is not positive, but still a convolution product can be defined on a subspace of the class of L^1-functions by using a product formula for Laguerre polynomials due to WATSON, cf. ASKEY [9, §5]. In the case of order zero there exists an addition formula due to BATEMAN [12, §9.31] which corresponds to WATSON's product formula. The limit case for Laguerre polynomials of the addition formula for Jacobi polynomials is different from BATEMAN's result, cf. [66, §5, remark 8]. Recently, THE AUTHOR [68, part II] succeeded in generalizing BATEMAN's addition formula to Laguerre polynomials of general order, such that WATSON's product formula follows by integration. The formula turns out to be a limit case of the addition formula for disk polynomials [64, (5.4)].

The Laplace type integral representation for Jacobi polynomials KOORNWINDER [1, (1)] may be applied in order to rewrite a Jacobi series as the double integral of a power series. Similarly, the product formula KOORNWINDER [1, (2)] may be used to rewrite a double Jacobi series $\Sigma\, c_n\, P_n^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(y)$ as the double integral of a single Jacobi series.

It is yet an open problem to find an addition formula corresponding to Gasper's product formula (1.1) in the case that $\max\left\{-1, -\alpha\right\} < \beta < -\frac{1}{2}$. Another problem is to find addition formulas, product formulas and integral representations for Jacobi functions of the second kind. An integral representation for the product $Q_n^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(y)$ is given in FLENSTED-JENSEN & KOORNWINDER [38, p. 255]. An integral representation for the product $Q_n^{(\alpha,\beta)}(x)\, Q_n^{(\alpha,\beta)}(y)$ would imply a Nicholson type formula for Jacobi polynomials. In the case $\alpha = \beta$ such a product formula was announced by DURAND (cf. Notices Amer. Math. Soc., April 1973, p. A-392).

7. FURTHER GROUP THEORETIC RESULTS AND OPEN PROBLEMS RELATED TO THE ADDITION FORMULA

Recent research in noncommutative harmonic analysis mainly concentrated on noncompact Riemannian symmetric spaces. It seems that many aspects of this theory can be extended to more general situations. First, there is FLENSTED-JENSEN's work [35] about "spherical" functions on pseudo-Riemannian symmetric spaces, cf. §3. Second, many results of the analysis on symmetric spaces also hold for nonsymmetric homogeneous spaces G/K for which the convolution algebra $L^1(K\backslash G/K)$ is commutative, cf. GODEMENT [49]. The homogeneous space $SU(q)/SU(q-1)$ is an example of this case. A more general example is given by the class of weakly symmetric spaces introduced by SELBERG [88]. See also the examples in FLENSTED-JENSEN [37].

The Laplace type integral representation for Jacobi functions corresponds to a well-known integral representation for spherical functions on noncompact symmetric spaces, cf. HELGASON [52, Chap. 10, Prop. 6.7]. SHERMAN [89] introduced the analogous formulas on compact symmetric spaces and he applied them to harmonic analysis on spheres.

UMEMURA & KÔNO [95] gave an infinite dimensional group theoretic interpretation of Hermite polynomials. TALMAN [93, Chap. 13] and PEETRE [84] discussed

Laguerre polynomials of zero order in connection with the Heisenberg group. VERE-JONES [96, Theorem 8] and DUNKL & RAMIREZ [106, Remark 6.6] pointed out that Krawtchouk polynomials are spherical functions on a certain homogeneous space G/K, where both G and K are finite groups. Using this interpretation DUNKL [108] derived an addition formula for Krawtchouk polynomials.

It might be of interest to formulate a general theorem about the existence of an addition formula for spherical functions on compact symmetric spaces, cf. KOORNWINDER [65].

The study of spherical functions on homogeneous spaces may lead to new interesting classes of special functions. The case of higher rank symmetric spaces will be discussed in §9. Another important case is given by spheres as homogeneous spaces, cf. the classification by BOREL [18], [19]. We already discussed the two classes $SO(q)/SO(q-1)$ and $SU(q)/SU(q-1)$. A third class is given by spheres S^{4q-1} as homogeneous spaces $Sp(q)/Sp(q-1)$. Study of this case would also give results for the quaternionic projective spaces $Sp(q)/Sp(q-1) \times Sp(1)$, which are compact symmetric spaces of rank one. Apart from these three classes there exist only a few other possibilities of spheres as homogeneous spaces. Finally, GODEMENT's theory [48] of spherical functions on G with respect to an irreducible representation of K (not necessarily the trivial representation) may be a source of new special functions.

8. HISTORY OF THE THEORY OF ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

A survey of the literature until 1953 is given in ERDELYI [33, Vol. 2, Chap. 12]. The best-known examples of orthogonal polynomials in two variables are the biorthogonal systems on circular and triangular regions, cf. APPELL & KAMPE DE FERIET [8]. However, we shall only consider orthogonal systems. Some general properties of such systems were given by JACKSON [59]. In contrast to the one-variable case the orthogonality region and weight function no longer uniquely determine the orthogonal system, but for each n they determine a $(n+1)$-dimensional class of orthogonal polynomials of degree n. The orthogonal basis depends on the sequence to which the Gram-Schmidt orthogonalisation process is applied.

There are known two distinct orthogonal systems of polynomials on the unit disk with respect to the weight function $(1-x^2-y^2)^\alpha$. Both systems can be expressed in terms of Jacobi polynomials. The first system is obtained by orthogonalisation of the sequence $1, x, y, x^2, xy, y^2, x^3, x^2y, \ldots$. It consists of the functions

$$P_{n-k}^{(\alpha+k+\frac{1}{2},\, \alpha+k+\frac{1}{2})}(x)\ (1-x^2)^{\frac{1}{2}k}\ P_k^{(\alpha,\alpha)}(y(1-x^2)^{-\frac{1}{2}}).$$

For $\alpha=0$ these polynomials were already considered by DIDON [29]. In KOORNWINDER [68, part I] this system was used for obtaining a new proof of the addition formula for Gegenbauer polynomials. The second orthogonal system consists of the disk polynomials $R_{m,n}^{(\alpha)}(z)$, which were already discussed in §2. Disk polynomials have the property that $R_{m,n}^{(\alpha)}(z)$ - const. $z^m \bar{z}^n$ has degree less than $m+n$. These polynomials were introduced by ZERNIKE & BRINKMAN [103]. ZERNIKE [102] used the case $\alpha=0$ for the study of diffraction problems. For further applications in optics see the references in MYRICK [83], cf. also MARR [76].

Orthogonal polynomials on a triangular region were introduced by PRORIOL [85]. These polynomials, which can be expressed in terms of Jacobi polynomials, were used to obtain solutions of the Schrödinger equation for the Helium atom, cf. MUNSCHY & PLUVINAGE [82], [81]. The same class of polynomials was independently obtained by KARLIN & McGREGOR [62] in view of applications to genetics.

LARCHER [71] and AGAHANOV [6] described a general method to construct certain orthogonal polynomials in two variables from given systems of orthogonal polynomials in one variable. They considered regions bounded by a circle, by a triangle or by a parabola and a straight line. Orthogonal polynomials in the last-mentioned region also occur in the addition formula for Jacobi polynomials, cf. KOORNWINDER [64, §3].

KRALL & SCHEFFER [69] classified all systems of orthogonal polynomials in two variables for which there exists a linear second order partial differential operator such that for each n the class of orthogonal polynomials of degree n is an eigenspace of the differential operator.

KARLIN & McGREGOR [61] considered antisymmetric orthogonal polynomials in two variables of the type $P_n(x)\, P_k(y) - P_k(x)\, P_n(y)$, where $P_n(x)$ is an orthogonal polynomial in one variable.

Finally, we mention the applications of orthogonal polynomials in two variables to cubature problems, i.e., the numerical evaluation of double integrals. See STROUD [91] for a survey and many references.

9. ORTHOGONAL POLYNOMIALS IN TWO VARIABLES WHICH ARE EIGENFUNCTIONS OF A LAPLACE-BELTRAMI OPERATOR

All orthogonal systems discussed in the previous section can be expressed in terms of orthogonal polynomials in one variable. Two deeper examples of orthogonal polynomials in two variables, for which no simple explicit expression seems to exist, are discussed in KOORNWINDER [4] and [5]. Let us briefly describe the way in which these examples originated.

Since Jacobi polynomials of certain orders are spherical functions on compact symmetric spaces of rank one, it may be conjectured that, in terms of suitable variables, the spherical functions on a compact symmetric space of rank r are orthogonal polynomials in r variables. The spherical functions are eigenfunctions of all invariant differential operators on the symmetric space. The algebra of invariant differential operators is commutative and has r generators, one of which is the Laplace-Beltrami operator, cf. GELFAND [47], SELBERG [88], HELGASON [52, Chap. 10, §2]. The radial part of the Laplace-Beltrami operator is an explicitly known second order operator (cf. HARISH-CHANDRA [50, §7]), the other generators of the algebra of invariant differential operators are unknown in the general case.

In the case of rank two the radial part of the Laplace-Beltrami operator is self-adjoint with respect to a given weight function on a triangular region on the boundary of which the operator becomes singular. The reflections in the edges of this triangle generate a discrete symmetry group of the plane which leaves the differential operator invariant. Corresponding to the three possible vector diagrams of the symmetric space three different triangles are possible with angles $\pi/2$, $\pi/4$, $\pi/4$ (see [4]) or $\pi/3$, $\pi/3$, $\pi/3$ (see [5]) or $\pi/2$, $\pi/3$, $\pi/6$ (not yet considered). The differential operator depends on some additional parameters (three parameters α, β, γ in KOORNWINDER [4, (2.3)] and one parameter α in KOORNWINDER [5, (5.13)]) which take certain discrete values depending on the multiplicities of the restricted roots for the symmetric space.

In KOORNWINDER [4] and [5] we consider these second order operators (denoted by D_1) for all real values of the parameters for which the weight function is absolutely integrable. In each case we prove that after a suitable transformation of variables there exists a complete orthogonal system of polynomials in two variables which are eigenfunctions of D_1. The group theoretic interpretation of these polynomials is not further examined. In particular, we do not prove the conjecture that these polynomials are spherical functions.

In KOORNWINDER [4] we obtain polynomials denoted by $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ which are orthogonal on a region bounded by two straight lines $1 - u + v = 0$, $1 + u + v = 0$ and a parabola $u^2 - 4v = 0$ touching these lines. The weight function equals $(1 - u + v)^{\alpha}$ $(1 + u + v)^{\beta}$ $(u^2 - 4v)^{\gamma}$. The polynomials $p_{n,k}^{\alpha,\beta,\gamma}$ (u, v) result from orthogonalisation of the sequence $1, u, v, u^2, u\,v, v^2, u^3, u^2\,v, \ldots$. The "highest" term of $p_{n,k}^{\alpha,\beta,\gamma}$ (u,v) is $u^{n-k}\,v^k$. For $u = x + y$, $v = x\,y$ we have

$$p_{n,k}^{\alpha,\beta,-\frac{1}{2}}(u, v) = \text{const.}\ (P_n^{(\alpha,\beta)}(x)\,P_k^{(\alpha,\beta)}(y) + P_k^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y))$$

and

$$p_{n,k}^{\alpha,\beta,\frac{1}{2}}(u, v) = \text{const.}\ (P_{n+1}^{(\alpha,\beta)}(x)\,P_k^{(\alpha,\beta)}(y) - P_k^{(\alpha,\beta)}(x)\,P_{n+1}^{(\alpha,\beta)}(y))\ /\ (x-y),$$

cf. KOORNWINDER [4, §3] and SPRINKHUIZEN [90, §3]. The choice of the variables u and v and of the orthogonalisation method in the case of general γ is suggested by the explicit expressions for $\gamma = -\frac{1}{2}$. In KOORNWINDER [4, §5] two second order differential operators D_- and D_+ are introduced such that $D_-\,p_{n,k}^{\alpha,\beta,\gamma} = \text{const.}\ p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma}$ and $D_+ p_{n-1,k-1}^{\alpha+1,\beta+1,\gamma} = \text{const.}\ p_{n,k}^{\alpha,\beta,\gamma}$. The explicit form of these operators is again suggested by the case that $\gamma = -\frac{1}{2}$. It follows that the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u, v)$ are also eigenfunctions of the fourth order differential operator $D_2 = D_+\,D_-$. This operator is algebraically independent of D_1.

The orthogonal polynomials considered in KOORNWINDER [5] are denoted by $p_{m,n}^{\alpha}$ $(z, \bar{z})$, where $z = x + i\,y$, $\bar{z} = x - i\,y$. They are orthogonal on a region bounded by Steiner's hypocycloid $\mu(z, \bar{z}) = 0$ with respect to the weight function $(\mu(z, \bar{z}))^{\alpha}$, where $\mu(z, \bar{z})$ is a certain polynomial of fourth degree. The polynomial $p_{m,n}^{\alpha}$ $(z, \bar{z})$ has degree $m + n$ in z and $\bar{z}$ such that $z^m\,\bar{z}^n$ is the only term of degree $m + n$. The choice of the variables $z, \bar{z}$ and the method of orthogonalisation is motivated by the special cases $\alpha = \pm\frac{1}{2}$. For $z = e^{i(s+t/\sqrt{3})} + e^{i(-s+t/\sqrt{3})} + e^{-2it/\sqrt{3}}$ the functions $p_{m,n}^{-\frac{1}{2}}(z, \bar{z})$ and $(\mu(z, \bar{z}))^{\frac{1}{2}}\,p_{m-1,n-1}^{\frac{1}{2}}(z, \bar{z})$ can be expressed as explicit trigonometric polynomials in s and t, cf. KOORNWINDER [5, §2, §3]. If $\alpha = \pm\frac{1}{2}$ then a third order differential operator D_2 for which the polynomials $p_{m,n}^{\alpha}$ $(z, \bar{z})$ are eigenfunctions is easily obtained. In KOORNWINDER [5, §6] this operator is generalized to the case of general α. The operators D_1 and D_2 are algebraically independent.

Finally, in KOORNWINDER [4, §6] and [5, §7] we consider the algebra of all differential operators for which the polynomials $p_{n,k}^{\alpha,\beta,\gamma}$ (u, v) or $p_{m,n}^{\alpha}$ $(z, \bar{z})$, respectively, are eigenfunctions. The study of this algebra is motivated by the known properties of the algebra of invariant differential operators on a symmetric space. In both cases KOORNWINDER [4] and [5] we prove that the algebra is commutative and that it is generated by the two algebraically independent operators D_1 and D_2.

10. FURTHER ANALYTIC RESULTS AND OPEN PROBLEMS ON ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

A further analysis of the polynomials $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ introduced in KOORNWINDER [4] was given by IDA SPRINKHUIZEN [90]. In addition to the operators D_- and D_+ (see §9) she introduced second order differential operators E_- and E_+ such that $E_- p_{n,k}^{\alpha,\beta,\gamma} = \text{const.}\ p_{n-1,k}^{\alpha,\beta,\gamma+1}$ and $E_+ p_{n-1,k}^{\alpha,\beta,\gamma+1} = \text{const.}\ p_{n,k}^{\alpha,\beta,\gamma}$. By using D_+ and E_+ there follows a Rodrigues type formula for these polynomials. She also proved that in the power series expansion of $p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ only those terms $u^{m-l}v^l$ occur for which both $m \leqslant n$ and $m+l \leqslant n+k$. As a corollary, two recurrence relations were obtained which express $u\,p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ and $v\,p_{n,k}^{\alpha,\beta,\gamma}(u,v)$ as linear combinations of five, respectively nine polynomials $p_{m,l}^{\alpha,\beta,\gamma}(u,v)$. It is still an open problem to find analogous results for the polynomials introduced in KOORNWINDER [5]. Of course, for both classes of polynomials many further problems can be formulated. For instance, standard formulas for special functions like power series expansions, integral representations and generating functions need to be found. It would also be of interest to obtain qualitative theorems about the zero curves and stationary points of these polynomials.

The polynomials $p_{m,o}^{-\frac{1}{2}}(z,\bar{z})$ and $p_{o,n}^{-\frac{1}{2}}(z,\bar{z})$ considered in KOORNWINDER [5] were independently introduced by EIER & LIDL [110] as a formal algebraic generalisation of the Chebyshev polynomials in one variable.

The orthogonal systems associated with the third type of vector diagram of rank two (cf. §9) are not yet studied. Further new classes of orthogonal polynomials in two variables being eigenfunctions of a second order differential operator may be obtained by considering spherical harmonics on S^2 which are invariant with respect to a discrete symmetry group of S^2 generated by reflections. The orthogonal polynomials on circular, triangular and parabolic domains discussed in §9 can thus be interpreted as spherical harmonics for special values of the parameters.

Having studied enough special cases one might be able to define "classical" orthogonal polynomials in two variables and to classify all polynomials satisfying this definition. A further step might be to develop a general theory for a much larger class of orthogonal polynomials in two variables. Finally, there is the transition from two to n variables, which will probably be easier than the extension from one to two variables.

11. FURTHER GROUP THEORETIC RESULTS AND OPEN PROBLEMS ON ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

In three special cases more results are known about the explicit form of spherical functions and invariant differential operators on compact symmetric spaces of rank two. First, on the compact groups $SO(5)$ ($\alpha,\beta,\gamma = \frac{1}{2}$in KOORNWINDER [4]) and $SU(3)$ ($\alpha = \frac{1}{2}$in KOORNWINDER [5]) the spherical functions are characters given by WEYL's character formula (cf. [101]) and the invariant differential operators follow from BEREZIN [14]. Second, spherical functions and invariant differential operators on complex Grassmann manifolds $SU(q+2)/S(U_q \times U_2)$ ($(\alpha,\beta,\gamma) = (q-2,0,\frac{1}{2})$ in [4]) were explicitly given by BEREZIN & KARPELEVIC [15]. Third, MAASS [74] expressed spherical functions on Grassmann manifolds $SO(q+2)/SO(q)\times SO(2)$ ($(\alpha,\beta,\gamma) = (\frac{1}{2}q-3/2, -\frac{1}{2}, 0)$ in [4]) as orthogonal polynomials in two variables. For general compact symmetric spaces of rank two it is an open problem to prove that the spherical functions can be expressed as the orthogonal polynomials introduced in KOORNWINDER [4] and [5] and that the radial part of some invariant differential operator coincides with D_2. In §9 we already mentioned the problem to prove that spherical functions on compact symmetric spaces of rank r are orthogonal polynomials in r variables.

HERZ [55] introduced hypergeometric functions of matrix argument. For extensions of this theory and for applications to multivariate statistical analysis see for instance JAMES [60] and MUIRHEAD [79]. The hypergeometric functions studied by HERZ include generalized Jacobi polynomials [55, (6.4^1)], which are related to the polynomials $p_{n,n}^{\alpha,\beta,0}$ introduced in [4].

JAMES & CONSTANTINE [107] considered so-called intertwining functions on $O(n)$ which belong to an irreducible representation of $O(n)$ and which are constant on the double cosets of the subgroups $O(m) \times O(n-m)$ and $O(q) \times O(n-q)$ in $O(n)$, thus generalizing the concept of spherical functions on Grassmann manifolds. They expanded these intertwining functions in terms of the spherical functions of $GL(m,\mathbf{R})/O(m)$ called zonal polynomials. For $m = 1$ these functions reduce to Jacobi polynomials considered as spherical harmonics, cf. ZERNIKE & BRINKMAN [103]. For $m = 2$ these functions correspond to the case $(\alpha, \beta, \gamma) = (\frac{1}{2}(q-3), \frac{1}{2}(n-q-3), 0)$ in KOORNWINDER [4]. JAMES & CONSTANTINE [107] also considered intertwining functions on complex Grassmann manifolds. These functions correspond to the case $(\alpha, \beta, \frac{1}{2})$ (α, β integers) in KOORNWINDER [4].

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