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THREE NOTES ON CLASSICAL ORTHOGONAL POLYNOMIALS

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Yet another proof of the addition formula for Jacobi polynomials *)

by

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ABSTRACT

Short proofs of the addition formulas for Gegenbauer polynomials and for Jacobi polynomials are given. The properties of certain special orthogonal polynomials in two, respectively three variables are used.

KEY WORDS & PHRASES : *addition formula for Gegenbauer polynomials, addition formula for Jacobi polynomials, polynomials in two variables orthogonal on the disk, polynomials in three variables orthogonal on a conical region.*

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*) AMS(MOS) subject classification scheme (1970): 33A65

1. INTRODUCTION

The addition formula for Jacobi polynomials was announced by the author in [2]. Afterwards three different proofs were published, cf. [3], [4], [5]. A special case was earlier obtained by ŠAPIRO [6]. The addition formula is a central result in the theory of Jacobi polynomials which implies many other important formulas. Therefore, it seems worthwhile to publish yet another proof of this addition formula. Compared to the earlier proofs the present proof is rather short and it does not involve many calculations. However, it would not have been easy to obtain this proof without knowing the addition formula already.

The idea of the proof is as follows. Consider the three-dimensional region bounded by the cone $z^2 - 2xy = 0$ and by the plane $x + y = 1$. Let $\mathcal{H}_n$ denote the class of all n^{th} degree orthogonal polynomials on this region with respect to the weight function $(1-x-y)^{\alpha-\beta-1}(2xy-z^2)^{\beta-\frac{1}{2}}$. Then an explicit orthogonal basis can be constructed for $\mathcal{H}_n$ in terms of products of certain Jacobi polynomials. The region and the weight function are invariant with respect to rotations around the axis of the cone. Therefore, the reproducing kernel of $\mathcal{H}_n$ is invariant under such rotations. The addition formula for Jacobi polynomials follows from this symmetry relation for the reproducing kernel. There exists a similar proof of the addition formula for Gegenbauer polynomials. It uses orthogonal polynomials in two variables on the unit disk.

It is of interest to compare the present proof of the addition formula for Jacobi polynomials with two earlier proofs by group theoretic methods (cf. [3], [4]). In these two references a much bigger symmetry group was used than the one-parameter group considered in the present paper. Furthermore, a restriction to integer or half integer values of the parameters α and β is not required here.

2. PRELIMINARIES

For $\alpha,\beta > -1$ Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal polynomials of degree n on the interval $(-1,1)$ with respect to the weight function

$(1-x)^{\alpha}(1+x)^{\beta}$ and with the normalization $P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n!$. The quadratic norm $h_n^{(\alpha,\beta)}$ of a Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is given by

$$(2.1)\qquad h_n^{(\alpha,\beta)} = \int_{-1}^{1} (P_n^{(\alpha,\beta)}(x))^2(1-x)^{\alpha}(1+x)^{\beta}dx = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)} .$$

For $\alpha = \beta$ Jacobi polynomials are called Gegenbauer polynomials. Note that

$$P_n^{(\alpha,\alpha)}(-x) = (-1)^n P_n^{(\alpha,\alpha)}(x).$$

Let R be a bounded region in the q-dimensional Euclidean space E_q and let $w(x) = w(x_1,x_2,\ldots,x_q)$ be a positive continuous integrable function on R. The class H_n of orthogonal polynomials of degree n on R with respect to the weight function w(x) consists of all polynomials $p(x_1,x_2,\ldots,x_q)$ of degree n such that

$$\int_R p(x)q(x)w(x)dx = 0$$

if q is a polynomial of degree less than n. There are infinitely many ways to choose an orthogonal basis of H_n. One possible method is to apply the Gram-Schmidt orthogonalization process to the monomials

$$x_1^{n_1-n_2}x_2^{n_2-n_3}\cdots x_{q-1}^{n_{q-1}-n_q}x_q^{n_q} \quad (n_1\geq n_2\geq\ldots\geq n_q\geq 0),$$

which are arranged by lexicographic ordering of the q-tuples $(n_1,n_2,\ldots,n_q)$.

Let $p_1,p_2,\ldots,p_N$ be an arbitrary orthogonal basis of H_n and let

$$\|p_k\|^2 = \int_R (p_k(x))^2w(x)dx.$$

The function

$$(2.2)\qquad K(x,y) = \sum_{k=1}^{N} \|p_k\|^{-2} p_k(x)p_k(y) \qquad (x,y \in R)$$

is called the reproducing kernel of H_n. Note that K(x,y) is independent of the choice of the orthogonal basis. In particular, if T is an isometric mapping of E_q onto itself such that $T(R) = R$ and $w(Tx) = w(x)$ $(x \in R)$ then

$$(2.3) \qquad K(Tx,Ty) = K(x,y).$$

3. THE ADDITION FORMULA FOR GEGENBAUER POLYNOMIALS

Let $\alpha > -\frac{1}{2}$. The formula

$$(3.1) \qquad P_n^{(\alpha,\alpha)}(\cos\theta\cos\tau + \sin\theta\sin\tau\cos\phi) =$$

$$= \sum_{k=0}^{n} c_{n,k}^{\alpha} (\sin\theta)^k P_{n-k}^{(\alpha+k,\alpha+k)}(\cos\theta) \cdot$$

$$\cdot (\sin\tau)^k P_{n-k}^{(\alpha+k,\alpha+k)}(\cos\tau) P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\cos\phi),$$

where

$$(3.2) \qquad c_{n,k} = \frac{(\alpha+k)(n+2\alpha+1)_k(2\alpha+1)_k(n-k)!}{2^{2k}(\alpha+\frac{1}{2}k)(\alpha+\frac{1}{2})_k(\alpha+1)_n}$$

is called the addition formula for Gegenbauer polynomials (cf. [1, 3.15(19)]). For fixed θ formula (3.1) can be considered as an expansion of the left hand side in terms of the functions

$$(\sin\tau)^k P_{n-k}^{(\alpha+k,\alpha+k)}(\cos\tau) P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\cos\phi).$$

Lemma 3.1 below states that these functions are orthogonal polynomials in the two variables $x = \cos\tau$ and $y = \sin\tau\cos\phi$. A new short proof of (3.1) then follows very easily.

<u>LEMMA 3.1</u>. *Let H_n be the class of orthogonal polynomials of degree* n *on the disk* $R = \{(x,y) \mid x^2 + y^2 < 1\}$ *with respect to the weight function* $(1-x^2-y^2)^{\alpha-\frac{1}{2}}$, $\alpha > -\frac{1}{2}$. *Then the functions*

$$(3.3) \qquad p_{n,k}(x,y) = P_{n-k}^{(\alpha+k,\alpha+k)}(x)(1-x^2)^{\frac{1}{2}k} P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(y(1-x^2)^{-\frac{1}{2}})$$

$(k=0,1,2,\ldots,n)$ *form an orthogonal basis of* H_n *which is obtained by orthogonalization of the sequence* $1,x,y,x^2,xy,y^2,x^3,x^2y,\ldots$.

PROOF. Clearly, $p_{n,k}(x,y)$ is a linear combination of the monomials $1,x,y,x^2,xy,y^2,\ldots,x^n,x^{n-1}y,\ldots,x^{n-k}y^k$, and the coefficient of $x^{n-k}y^k$ is non-zero. By substituting $u = x$, $v = y(1-x^2)^{-\frac{1}{2}}$ and by using the orthogonality properties of Jacobi polynomials it follows that

$$\iint_R p_{n,k}(x,y)p_{m,l}(x,y)(1-x^2-y^2)^{\alpha-\frac{1}{2}}dxdy =$$
$$= \delta_{n,m}\delta_{k,l}h_{n-k}^{(\alpha+k,\alpha+k)}h_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}. \qquad \square$$

Next we prove (3.1). Any rotation T around the origin maps the disk R onto itself and leaves the weight function $(1-x^2-y^2)^{\alpha-\frac{1}{2}}$ invariant. Let

$$K((x,y),(x',y')) = \sum_{k=0}^{n} \|p_{n,k}\|^{-2}\, p_{n,k}(x,y)p_{n,k}(x',y'). \tag{3.4}$$

Hence, it follows from (2.3) that

$$K((x,y),(\cos\theta,\ \sin\theta)) = K((x\cos\theta + y\sin\theta,\ -x\sin\theta + y\cos\theta),(1,0)). \tag{3.5}$$

Subsitution of (3.3) and (3.4) in (3.5) gives

$$\|p_{n,0}\|^{-2}\, P_n^{(\alpha,\alpha)}(1)P_n^{(\alpha,\alpha)}(x\cos\theta + y\sin\theta) = \tag{3.6}$$
$$= \sum_{k=0}^{n} \|p_{n,k}\|^{-2}\, P_{n-k}^{(\alpha+k,\alpha+k)}(x)(1-x^2)^{\frac{1}{2}k}\, P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(y(1-x^2)^{-\frac{1}{2}})\,\cdot$$
$$\cdot\, P_{n-k}^{(\alpha+k,\alpha+k)}(\cos\theta)(\sin\theta)^k\, P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(1).$$

Putting $x = \cos\tau$, $y = \sin\tau\cos\theta$ in (3.6) we obtain (3.1) with

$$c_{n,k}^{\alpha} = \frac{\|p_{n,0}\|^2\, P_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(1)}{\|p_{n,k}\|^2\, P_n^{(\alpha,\alpha)}(1)}.$$

By using that $\|p_{n,k}\|^2 = h_{n-k}^{(\alpha+k,\alpha+k)} h_k^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}$, a straightforward calculation gives back (3.2).

4. THE ADDITION FORMULA FOR JACOBI POLYNOMIALS

Let $\alpha > \beta > -\frac{1}{2}$. The formula

$$
\begin{aligned}
(4.1)\qquad & P_n^{(\alpha,\beta)}(2\cos^2\theta\cos^2\tau + 2\sin^2\theta\sin^2\tau\, r^2 + \\
& + \sin 2\theta \sin 2\tau\, r\cos\phi - 1) = \sum_{k=0}^{n}\sum_{l=0}^{n} c_{n,k,l}^{(\alpha,\beta)} \cdot \\
& \cdot (\sin\theta)^{2k-l}(\cos\theta)^l\, P_{n-k}^{(\alpha+2k-l,\beta+l)}(\cos 2\theta) \cdot \\
& \cdot (\sin\tau)^{2k-l}(\cos\tau)^l\, P_{n-k}^{(\alpha+2k-l,\beta+l)}(\cos 2\tau) \cdot \\
& \cdot r^l\, P_{k-l}^{(\alpha-\beta-1,\beta+l)}(2r^2-1)\, P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\phi),
\end{aligned}
$$

where

$$
(4.2)\qquad c_{n,k,l}^{(\alpha,\beta)} = \frac{(\alpha+2k-l)(\beta+l)(n+\alpha+\beta+1)_k(\beta+n-k+l+1)_{k-l}(2\beta+1)_l(n-k)!}{(\alpha+k)(\beta+\frac{1}{2}l)(\beta+1)_k(\beta+\frac{1}{2})_l(\alpha+k+1)_{n-l}}
$$

is called the addition formula for Jacobi polynomials (cf. KOORNWINDER [2, (3)]). It was pointed out in [5, §3] that for fixed θ and τ formula (4.1) can be considered as an expansion of the left hand side in terms of the functions

$$
r^l\, P_{k-l}^{(\alpha-\beta-1,\beta+l)}(2r^2-1)\, P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(\cos\phi),
$$

which are orthogonal polynomials in the two variables r^2 and $r\cos\phi$. However, for fixed θ formula (4.1) can also be considered as an expansion of the left hand side in terms of functions in τ, r, ϕ which are orthogonal polynomials in the three variables $x = \cos^2\tau$, $y = r^2\sin^2\tau$, $z = 2^{-\frac{1}{2}} r \sin 2\tau \cos\phi$. This will be proved in Lemma 4.1 below. Then the addition formula (4.1) follows in a similar way as the result in §3.

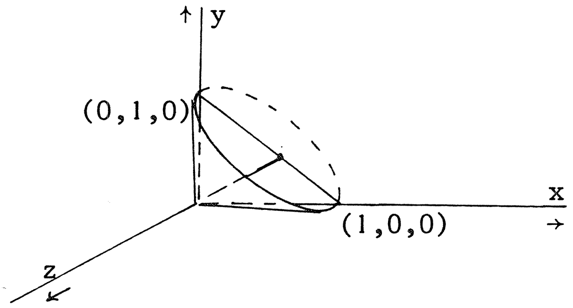

Figure 1

Let R be the three-dimensional region $\{(x,y,z) \mid 0 < x + y < 1, z^2 < 2xy\}$, which is bounded by the cone $z^2 = 2xy$ and by the plane $x + y = 1$ orthogonal to the axis of the cone (cf. Fig. 1). Let H_n be the class of orthogonal polynomials of degree n on the region R with respect to the weight function

$$(4.3) \qquad w(x,y,z) = (1-x-y)^{\alpha-\beta-1}(2xy-z^2)^{\beta-\frac{1}{2}}, \quad \alpha > \beta > -\tfrac{1}{2}.$$

LEMMA 4.1. *The functions*

$$(4.4) \qquad p_{n,k,l}(x,y,z) = P_{n-k}^{(\alpha+2k-1,\beta+1)}(2x-1)(1-x)^{k-l} \cdot$$

$$\cdot\, P_{k-l}^{(\alpha-\beta-1,\beta+1)}\left(\frac{x+2y-1}{1-x}\right)(xy)^{\frac{1}{2}l}\, P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}((2xy)^{-\frac{1}{2}}z)$$

$(n \geq k \geq l \geq 0)$ *form an orthogonal basis of* H_n, *which is obtained by orthogonalization of the sequence*

$$1, x, y, z, x^2, xy, xz, y^2, yz, z^2, x^3, \ldots .$$

PROOF. Clearly, the function $p_{n,k,l}(x,y,z)$ is a polynomial of degree n in x,y,z, of degree k in y,z and of degree l in z. Hence, $p_{n,k,l}(x,y,z)$ is a linear combination of the monomials $x^{m_1-m_2}y^{m_2-m_3}z^{m_3}$ with "highest" term const. $x^{n-k}y^{k-l}z^{l}$. Let $u = 2x - 1$, $v = (x+2y-1)/(1-x)$, $w = z(2xy)^{-\frac{1}{2}}$. The mapping $(x,y,z) \to (u,v,w)$ is a diffeomorphism from R onto the cubic region $\{(u,v,w) \mid -1 < u < 1, -1 < v < 1, -1 < w < 1\}$. By making this substitution and by using the orthogonality properties of Jacobi polynomials it follows that

$$\iiint_R p_{n,k,l}(x,y,z)p_{n',k',l'}(x,y,z)w(x,y,z)\,dxdydz =$$

$$= \delta_{n,n'}\delta_{k,k'}\delta_{l,l'}\, 2^{-2\alpha-2k-l-1}\, h_{n-k}^{(\alpha+2k-1,\beta+1)} \cdot$$

$$\cdot\, h_{k-l}^{(\alpha-\beta-1,\beta+l)} h_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}. \quad \square$$

Next we prove the addition formula (4.1). Let

$$\text{(4.5)} \qquad K((x,y,z),(x',y',z')) = \sum_{k=0}^{n} \sum_{l=0}^{n} \| p_{n,k,l}\|^{-2} \cdot$$

$$\cdot\, p_{n,k,l}(x,y,z)p_{n,k,l}(x',y',z').$$

It follows from (4.4) that $p_{n,k,l}(1,0,0) = 0$ if $(n,k,l) \neq (n,0,0)$. Hence

$$\text{(4.6)} \qquad K((x,y,z),(1,0,0)) = \| p_{n,0,0}\|^{-2}\, P_n^{(\alpha,\beta)}(1)P_n^{(\alpha,\beta)}(2x-1).$$

Any rotation around the axis $\{(x,y,z) \mid x = y,\ z = 0\}$ of the cone maps the region R onto itself and leaves the weight function $w(x,y,z)$ invariant. In particular, consider a rotation of this type over an angle -2θ. It maps the point $(\cos^2\theta, \sin^2\theta, 2^{-\frac{1}{2}}\sin 2\theta)$ onto $(1,0,0)$ and the point (x,y,z) onto a point (ξ,η,ζ) where $\xi = x\cos^2\theta + y\sin^2\theta + 2^{-\frac{1}{2}}z\sin 2\theta$. Hence, by (2,3), (4.5), (4.6) and (4.4) we have

$$\text{(4.7)} \qquad \| p_{n,0,0}\|^{-2}\, P_n^{(\alpha,\beta)}(1)P_n^{(\alpha,\beta)}(2(x\cos^2\theta + y\sin^2\theta + 2^{-\frac{1}{2}}z\sin 2\theta)-1) =$$

$$= K((x,y,z),(\cos^2\theta, \sin^2\theta, 2^{-\frac{1}{2}}\sin 2\theta)) =$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{k} \| p_{n,k,l}\|^{-2}\, P_{k-l}^{(\alpha-\beta-1,\beta+l)}(1)P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(1) \cdot$$

$$\cdot\, (\sin\theta)^{2k-l}(\cos\theta)^l\, P_{n-k}^{(\alpha+2k-1,\beta+1)}(\cos 2\theta) \cdot$$

$$\cdot\, P_{n-k}^{(\alpha+2k-1,\beta+1)}(2x-1)(1-x)^{k-l}\, P_{k-l}^{(\alpha-\beta-1,\beta+l)}\left(\frac{x+2y-1}{1-x}\right) \cdot$$

$$\cdot\, (xy)^{\frac{1}{2}l}\, P_l^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}((2xy)^{-\frac{1}{2}}z).$$

Substitution of $x = \cos^2 \tau$, $y = r \sin^2 \tau$, $z = 2^{-\frac{1}{2}} r \sin 2\tau \cos \phi$ gives (4.1) with

$$c_{n,k,1}^{(\alpha,\beta)} = \frac{\| p_{n,0,0} \|^2 \, P_{k-1}^{(\alpha-\beta-1,\beta+1)}(1) P_1^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(1)}{\| p_{n,k,1} \|^2 \, P_n^{(\alpha,\beta)}(1)} .$$

Using the expression for $\| p_{n,k,1} \|^2$ at the end of the proof of Lemma 4.1 we get back formula (4.2).

REFERENCES

[1] ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. & TRICOMI, F.G., *Higher Transcendental Functions, Vol. I.* McGraw-Hill, New York, 1953.

[2] KOORNWINDER, T.H., *The addition formula for Jacobi polynomials, I. Summary of results.* Nederl. Akad. Wetensch. Proc. A 75 = Indag. Math. 34 (1972) 188-191.

[3] KOORNWINDER, T.H., *The addition formula for Jacobi polynomials, II, III.* Math. Centrum Amsterdam Reports TW 133, 135 (1972).

[4] KOORNWINDER, T.H., *The addition formula for Jacobi polynomials and spherical harmonics.* SIAM J. Appl. Math. 25 (1973) 236-246.

[5] KOORNWINDER, T.H., *Jacobi polynomials, III. An analytic proof of the addition formula.* SIAM J. Math. Anal. 6 (1975) 533-543.

[6] ŠAPIRO, R.L., *Special functions related to representations of the group* SU(n), *of class I with respect to* SU(n-1) (n≥3) (Russian). Izv. Vysš. Učeb. Zaved. Mathematika (1968), no. 4(71) pp. 97-107.

The addition formula for Laguerre polynomials *)

by

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ABSTRACT

Bateman's addition formula for Laguerre polynomials of order zero is generalized to the case of order $\alpha > 0$. The result is obtained as a limit case of the addition formula for disk polynomials.

KEY WORDS & PHRASES: *Addition formula for disk polynomials, addition formula for Laguerre polynomials.*

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*) AMS (MOS) subject classification scheme (1970): 33 A65.

1. INTRODUCTION

This note answers a question posed by ASKEY [2, p.83]. An addition formula for Laguerre polynomials $L_n^\alpha(x)$ $(\alpha>0)$ will be derived which reduces to Bateman's addition formula [3, p.457] for $\alpha \downarrow 0$ and which leads by integration to Watson's integral representation [15] for the product $L_n^\alpha(x)L_n^\alpha(y)$ of two Laguerre polynomials.

This addition formula turns out to be a limit case of the addition formula for the so-called disk polynomials which are orthogonal polynomials in two variables on the unit disk. If r, ψ are polar coordinates on the unit disk then the addition formula is an orthogonal expansion of

$$L_n^\alpha(x^2 + y^2 - 2xy\, r \cos \psi) \quad \exp(xyre^{i\psi})$$

in terms of disk polynomials of order $\alpha - 1$ depending on r and ψ.

2. PRELIMINARIES

Let Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, Laguerre polynomials $L_n^\alpha(x)$ and Bessel functions $J_\alpha(x)$ be as defined in ERDÉLYI [6]. It will be convenient to use the slightly different functions $R_n^{(\alpha,\beta)}(x)$, $\mathcal{L}_n^\alpha(x)$ and $\mathcal{J}_\alpha(x)$, respectively, which are defined by

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1),$$

$$\mathcal{L}_n^\alpha(x) = e^{-\frac{1}{2}x}\, L_n^\alpha(x)/L_n^\alpha(0),$$

$$\mathcal{J}_\alpha(x) = \Gamma(\alpha+1)(\tfrac{1}{2}x)^{-\alpha}\, J_\alpha(x).$$

Laguerre polynomials are a confluent case of Jacobi polynomials by the limit formula

$$(2.1) \qquad e^{\frac{1}{2}x}\, \mathcal{L}_n^\alpha(x) = \lim_{\beta\to\infty} R_n^{(\alpha,\beta)}(1-2\beta^{-1}x),$$

which holds uniformly for x in bounded sets. The functions $L_n^\alpha(x)$ satisfy the inequality

(2.2) $$|L_n^\alpha(x)| \le 1 \quad (\alpha \ge 0,\ x \ge 0),$$

cf. ERDÉLYI [6, 10.18(14)].

Let $z = x + iy$, $\bar{z} = x - iy$, $x,y \in \mathbb{R}$. For $\alpha > -1$ and for nonnegative integers m,n the so-called disk polynomials $R_{m,n}^\alpha(z)$ are defined in terms of Jacobi polynomials by

(2.3) $$R_{m,n}^\alpha(z) = \begin{cases} R_n^{(\alpha,m-n)}(2z\bar{z}-1)z^{m-n} & \text{if } m \ge n, \\ R_m^{(\alpha,n-m)}(2z\bar{z}-1)\bar{z}^{n-m} & \text{if } m \le n. \end{cases}$$

It is easily proved that the polynomials $R_{m,n}^\alpha(z)$ are orthogonal polynomials of degree $m + n$ in x and y on the unit disk with respect to the weight function $(1-x^2-y^2)^\alpha$. In fact, disk polynomials are characterized by the following properties:

(i) $R_{m,n}^\alpha(z) = \text{const.}\ z^m\bar{z}^n$ + polynomial of degree less than $m + n$;

(ii) $\iint_{x^2+y^2<1} R_{m,n}^\alpha(x+iy)\ \overline{p(x,y)}\ (1-x^2-y^2)^\alpha dxdy = 0$ for every polynomial $p(x,y)$ of degree less than $m + n$;

(iii) $R_{m,n}^\alpha(1) = 1$.

These polynomials were first studied by ZERNIKE & BRINKMAN [16]. The notation $R_{m,n}^\alpha(z)$ was introduced by the author [7, p.18].

It can be proved that $|R_{m,n}^\alpha(z)| \le 1$ if $\alpha \ge 0$, $|z| \le 1$. However, we shall only need the estimate

(2.4) $$|R_{m,n}^\alpha(z)| = \mathcal{O}(m^n) \text{ for } m \to \infty,$$

uniformly for $|z| \le 1$, where $\alpha > -1$ and n are fixed. This estimate follows from SZEGÖ [13, (7.32.2)] by using (2.3).

3. THE ADDITION FORMULA FOR LAGUERRE POLYNOMIALS

Let $\alpha > 0$. The formula

(3.1) $$R^{\alpha}_{m,n}(\cos\theta_1\, e^{i\phi_1}\cos\theta_2\, e^{i\phi_2} + \sin\theta_1 \sin\theta_2\, r\, e^{i\psi}) =$$

$$= \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{\alpha}{\alpha+k+l}\binom{m}{k}\binom{n}{l}\frac{(\alpha+n+1)_k(\alpha+m+1)_l}{(\alpha+1)_k(\alpha+k)_l}\,\cdot$$

$$\cdot\,(\sin\theta_1)^{k+l}\, R^{\alpha+k+l}_{m-k,n-l}(\cos\theta_1\, e^{i\phi_1})\,\cdot$$

$$\cdot\,(\sin\theta_2)^{k+l}\, R^{\alpha+k+l}_{m-k,n-l}(\cos\theta_2\, e^{i\phi_2})\, R^{\alpha-1}_{k,l}(r\, e^{i\psi})$$

is called the addition formula for disk polynomials, cf. ŠAPIRO [12, (1,21)] and KOORNWINDER [8, (5.4)]. For $\alpha = 1,2,3,\ldots$ both authors independently obtained this formula by interpreting disk polynomials $R^{\alpha}_{m,n}(z)$ as spherical functions on the homogeneous space $SU(\alpha+2)/SU(\alpha+1)$. Since both sides of (3.1) are rational functions in α, the case of general α then follows by analytic continuation.

By putting $\phi_1 = \phi_2 = 0$, $x = \sin\theta_1$, $y = \sin\theta_2$ in (3.1) and by substituting (2.3) in (3.1) we obtain for $m \geq n$, $\alpha > 0$, $0 \leq x \leq 1$, $0 \leq y \leq 1$:

(3.2) $$R^{(\alpha,m-n)}_{n}(2(1-x^2)(1-y^2) + 2x^2y^2r^2 + 4xy(1-x^2)^{\frac12}(1-y^2)^{\frac12} r\cos\psi - 1)$$

$$((1-x^2)^{\frac12}(1-y^2)^{\frac12} + xyr\, e^{i\psi})^{m-n} =$$

$$= \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{\alpha}{\alpha+k+l}\binom{m}{k}\binom{n}{l}\frac{(\alpha+n+1)_k(\alpha+m+1)_l}{(\alpha+1)_k(\alpha+k)_l}\,\cdot$$

$$\cdot\, x^{k+l}\, R^{(\alpha+k+l,|m-n-k+l|)}_{(m-k)\wedge(n-l)}(1-2x^2)(1-x^2)^{\frac12|m-n-k+l|}\,\cdot$$

$$\cdot\, y^{k+l}\, R^{(\alpha+k+l,|m-n-k+l|)}_{(m-k)\wedge(n-l)}(1-2y^2)(1-y^2)^{\frac12|m-n-k+l|}\, R^{\alpha-1}_{k,l}(r\, e^{i\psi}).$$

Here $m \wedge n$ denotes the minimum of m and n. Both in (3.1) and (3.2) the right hand side is an orthogonal expansion of the left hand side in terms of disk polynomials $R^{\alpha-1}_{k,l}(r\, e^{i\psi})$.

Let us next replace x by $m^{-\frac12}x$ and y by $m^{-\frac12}y$ in (3.2). Denote this new formula by (3.2)' and let $m \to \infty$. First we calculate the formal limit case of (3.2)' by taking termwise limits. Using (2.1) we obtain

$$(3.3) \qquad L_n^{\alpha}(x^2+y^2-2xy\ r\cos\psi)\exp(ixy\ r\sin\psi) = \\ = \sum_{k=0}^{\infty}\sum_{l=0}^{n} \frac{\alpha}{\alpha+k+l}\binom{n}{l}\frac{(\alpha+n+1)_k}{k!(\alpha+1)_k(\alpha+k)_l} \cdot \\ \cdot x^{k+l} L_{n-l}^{\alpha+k+l}(x^2) y^{k+l} L_{n-l}^{\alpha+k+l}(y^2) R_{k,l}^{\alpha-1}(r\,e^{i\psi}),$$

where $x \ge 0$, $y \ge 0$, $0 \le r \le 1$, $0 \le \psi < 2\pi$, $\alpha > 0$, $n = 0,1,2,\ldots$. For fixed x, y, α, n the convergence of the left hand side of (3.2)' to the left hand side of (3.3) is uniform in r and ψ. Denote the right hand side by

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_{k,l}(x,y,\alpha,n) R_{k,l}^{\alpha-1}(r\,e^{i\psi}),$$

where $c_{k,l} \equiv 0$ if $l > n$. Then the coefficients $c_{k,l}$ denote the Fourier coefficients of the left hand side with respect to the orthogonal functions $R_{k,l}^{\alpha-1}(r\,e^{i\psi})$. We shall prove that this Fourier series uniformly converges in r and ψ. Then the identity (3.3) actually holds.

Let α and n be fixed and let x and y be in bounded sets. Then, by (2.2) and (2.4) there is a constant $M > 0$ such that

$$|c_{k,l}(x,y,\alpha,n) R_{k,l}^{\alpha-1}(r\,e^{i\psi})| \le M^k/k!,$$

uniformly in r and ψ. Hence the Fourier series is uniformly convergent in r and ψ.

Integration of (3.3) gives the product formula

$$(3.4) \qquad L_n^{\alpha}(x^2)L_n^{\alpha}(y^2) = 2\alpha\pi^{-1}\int_0^1\int_0^{\pi} L_n^{\alpha}(x^2+y^2+2xyr\cos\psi) \cdot \\ \cdot \cos(xyr\sin\psi)\, r(1-r^2)^{\alpha-1}\,dr\,d\psi \qquad (x,y \ge 0,\ \alpha > 0)$$

By putting $r\cos\psi = \cos\theta$, $r\sin\psi = \sin\theta\cos\phi$ in (3.4) and by substituting Poisson's integral representation for Bessel functions we obtain

(3.5) $$L_n^\alpha(x^2)L_n^\alpha(y^2) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^\pi L_n^\alpha(x^2+y^2+2xy\cos\theta)\,\cdot$$
$$\cdot\, J_{\alpha-\frac{1}{2}}(xy\sin\theta)(\sin\theta)^{2\alpha}d\theta\,(x,y \geq 0,\ \alpha > -\tfrac{1}{2}).$$

The case $-\frac{1}{2} < \alpha \leq 0$ follows by analytic continuation. This formula is due to WATSON [15]. ASKEY [2, pp.82,83] applied this product formula to define a convolution structure for Laguerre series, thus extending earlier results of McCULLY [10] for the case $\alpha = 0$. However, this convolution structure is not positive and it is not defined for all L^1-functions.

If we put $r = 1$ in (3.3) and let $\alpha \downarrow 0$ then we obtain the addition formula

(3.6) $$L_n^0(x^2+y^2-2xy\cos\psi)\exp(ixy\sin\psi) =$$
$$= \sum_{k=0}^{\infty}\frac{1}{k!}\binom{n+k}{k}x^k L_n^k(x^2)y^k L_n^k(y^2)\, e^{ik\psi} +$$
$$+ \sum_{l=1}^{n}\frac{1}{l!}\binom{n}{l}x^l L_{n-l}^l(x^2)y^l L_{n-l}^l(y^2)\, e^{-il\psi}.$$

This formula was stated without proof by BATEMAN [3]. Later two different proofs were given by BUCHHOLZ [4, p.144] and by CARLITZ [5].

4. REMARKS

4.1. For $x = y$, $r = 1$, $\psi = 0$ formula (3.3) implies the identity

$$1 = \sum_{k=0}^{\infty}\sum_{l=0}^{n}\frac{\alpha}{\alpha+k+l}\binom{n}{l}\frac{(\alpha+n+1)_k}{k!(\alpha+1)_k(\alpha+k)_l}\left(x^{k+l}L_{n-l}^{\alpha+k+l}(x^2)\right)^2.$$

Inequality (2.2) is contained in this identity. Expressions for $L_n^\alpha((x+y)^2)$ and $L_n^\alpha((x-y)^2)$ follow from (3.3) by putting $r = 1$ and $\psi = 0$ or π.

4.2. The addition formula (3.3) for Laguerre polynomials cannot be obtained as a limit case of the addition formula for Jacobi polynomials, cf. KOORNWINDER [9, §5, remark 8]. The addition formula (3.1) for disk polynomials is a more general result which implies both the addition formula for Jacobi polynomials (m=n) and for Laguerre polynomials ($m\to\infty$).

4.3. The convolution structure for Laguerre series of order zero has a group theoretic interpretation on the Heisenberg group, cf. PEETRE [11]. Integration of (3.1) gives a product formula for disk polynomials. This product formula implies a positive convolution structure for disk polynomial expansions which has a group theoretic interpretation on the homogeneous space $SU(\alpha+2)/SU(\alpha+1)$, $\alpha = 1,2,3,\ldots$ (cf. ANNABI & TRIMÈCHE [1], TRIMÈCHE [14]). It would be of interest to study the relation between both convolution structures.

REFERENCES

[1] ANNABI, H. & TRIMÈCHE, K., *Convolution généralisée sur le disque unité*. C.R. Acad. Sci. Paris Sér. A 278 (1974) 21-24.

[2] ASKEY, R., *Orthogonal polynomials and positivity*. Studies in Applied Mathematics, Vol 6, pp.64-85, Society for Industrial and Applied Mathematics, Philadelphia, 1970.

[3] BATEMAN, H., *Partial differential equations of mathematical physics*. Cambridge University Press, 1932.

[4] BUCHHOLZ, H., *Die konfluente hypergeometrische Funktion*. Springer-Verlag, Berlin, 1953.

[5] CARLITZ, L., *A formula of Bateman*. Proc. Glasgow Math. Ass. 3 (1957), 99-101.

[6] ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. & TRICOMI, F.G., *Higher Transcendental Functions, Vol. II*. McGraw-Hill, New York, 1953.

[7] KOORNWINDER, T.H., *The addition formula for Jacobi polynomials, II. The Laplace type integral representation and the product formula*. Math. Centrum Amsterdam Report TW 133 (1972).

[8] KOORNWINDER, T.H., *The addition formula for Jacobi polynomials, III. Completion of the proof*. Math. Centrum Amsterdam Report TW 135 (1972).

[9] KOORNWINDER, T.H., *Jacobi polynomials, III. An analytic proof of the addition formula*. SIAM J. Math. Anal. 6 (1975) 533-543.

[10] McCULLY, J., *The Laguerre transform*. SIAM Review 2 (1960) 185-191.

[11] PEETRE, J., *The Weyl transform and Laguerre polynomials*. Le Matematiche 27 (1972) 301-323.

[12] ŠAPIRO, R.L., *Special functions related to representations of the group* SU(n), *of class I with respect to* SU(n-1) (n≥3) (Russian). Izv. Vysš. Učeb. Zaved. Matematika (1968), no. 4 (71), pp.97-107.

[13] SZEGÖ, G., *Orthogonal polynomials*. Amer. Math. Soc. Colloquium Publications, American Mathematical Society, Providence (R.I.), Third edition, 1967.

[14] TRIMÈCHE, K., *Convolution généralisée sur le disque unité*. Thèse, Université de Tunis, 1974.

[15] WATSON, G.N., *Another note in Laguerre polynomials*, J. London Math. Soc. 14 (1939) 19-22.

[16] ZERNIKE, F. & BRINKMAN, H.C., *Hypersphärische Funktionen und die in sphärischen Bereichen orthogonalen Polynome*. Proc. Kon. Akad. v. Wet., Amsterdam 38 (1935) 161-170.

New proof of the positivity of generalized translation for Jacobi series *)

by

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ABSTRACT

Weinberger's maximum theorem for hyperbolic differential equations is applied to obtain a new proof of Gasper's result concerning the positivity of generalized translation for Jacobi series.

KEY WORDS & PHRASES: *Positivity of generalized translation for Jacobi series, Weinberger's maximum theorem for hyperbolic differential equations.*

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*) AMS (MOS) subject classification scheme (1970): 42 A56.

1. INTRODUCTION

Let $\alpha,\beta > -1$. Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$ are orthogonal polynomials of degree n on the interval (-1,1) with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ and with the normalization $R_n^{(\alpha,\beta)}(1) = 1$.

Let us consider functions u defined on $[0,\pi/2] \times [0,\pi/2]$ of the form

$$u(s,t) = \sum_{k=0}^{n} c_k \, R_k^{(\alpha,\beta)}(\cos 2s) \, R_k^{(\alpha,\beta)}(\cos 2t), \tag{1.1}$$

where $n = 0,1,2,\ldots$ and $c_0,c_1,\ldots,c_n$ are arbitrary real constants. The function $u(.,t)$ is called the generalized translate of the function $u(.,0)$. The purpose of the present paper is to give a new proof of the following theorem of GASPER [5], [6].

<u>THEOREM 1.1</u>. *Let* $\alpha \geq \beta \geq -\frac{1}{2}$. *If* $u(s,t)$ *has the form* (1.1) *and if* $u(s,0) \geq 0$ *for each* $s \in [0,\pi/2]$ *then* $u(s,t) \geq 0$ *for each* $s,t \in [0,\pi/2]$.

We mention three possible approaches to prove Theorem 1.1.

A. BOCHNER [2] pointed out that in the case $\alpha = \beta \geq -\frac{1}{2}$ the positivity result follows from the product formula for Gegenbauer polynomials. The product formula for Jacobi polynomials (cf. KOORNWINDER [7]) has a similar corollary in the case $\alpha \geq \beta \geq -\frac{1}{2}$.

B. GASPER [5], [6] explicitly calculated the kernel of generalized translation and he expressed the kernel in terms of hypergeometric functions. This enabled him to prove that generalized translation for Jacobi series is positive if and only if $\alpha \geq \beta \geq -\frac{1}{2}$ or $\alpha \geq |\beta|$, $\beta > -1$.

C. Functions of the form (1.1) are solutions of a hyperbolic differential equation. A maximum property of such solutions implies that if $\alpha \geq \beta > -1$, $\alpha + \beta + 1 \geq 0$ and if $u(s,0) \geq 0$ for each $s \in [0,\pi/2]$ then $u(s,t) \geq 0$ for $0 \leq t \leq s \leq \pi/2 - t$. This result is due to WEINBERGER [8] in the case $\alpha = \beta \geq -\frac{1}{2}$ and to ASKEY [1, p.81] in the general case. If $\alpha = \beta \geq -\frac{1}{2}$ then the positivity of generalized translation follows from the case $0 \leq t \leq s \leq \pi/2 - t$ by the identities $u(s,t) = u(t,s)$ and $u(s,t) = u(\pi/2-s,\pi/2-t)$. However, if $\alpha > \beta$ then this method fails since the second identity no longer holds.

In the present note we shall prove Theorem 1.1 by using the approach described in C. The proof is a refinement of Askey's argument used in [1, p.81].

It is of interest to compare these results with the corresponding case of Jacobi functions $\phi_\lambda^{(\alpha,\beta)}(t)$. FLENSTED-JENSEN & KOORNWINDER [4] proved that the generalized translation for Jacobi function expansions is positive in the case $\alpha \geq \beta \geq -\frac{1}{2}$ by using approach A. CHÉBLI [3] independently obtained the positivity result for $\alpha \geq \beta \geq -\alpha - 1$ by using approach C. In the case of Jacobi functions one needs only to consider the region $\{(s,t) \mid 0 \leq t \leq s\}$. Hence, Chébli could obtain his result without using the more intricate argument we shall need.

2. THE POSITIVITY OF GENERALIZED TRANSLATION FOR $\alpha \geq \beta \geq -\frac{1}{2}$

Let us write

$$w(s) = w_{\alpha,\beta}(s) = (\sin s)^{2\alpha+1}(\cos s)^{2\beta+1} \qquad (0 < s < \pi/2)$$

and

$$a(s,t) = a_{\alpha,\beta}(s,t) = w_{\alpha,\beta}(s)w_{\alpha,\beta}(t).$$

Jacobi polynomials satisfy the differential equation

$$(w_{\alpha,\beta}(s))^{-1} \frac{d}{ds}\left[w_{\alpha,\beta}(s) \frac{d}{ds} R_n^{(\alpha,\beta)}(\cos 2s)\right] =$$
$$= -4n(n+\alpha+\beta+1)\, R_n^{(\alpha,\beta)}(\cos 2s).$$

Hence, any function u of the form (1.1) is a solution of the hyperbolic differential equation

$$(2.1) \qquad (a\, u_s)_s - (a\, u_t)_t = 0.$$

Using Bateman's integral for Jacobi polynomials ASKEY [1, p.82] proved:

<u>LEMMA 2.1</u>. *Let* $\alpha \geq \beta > -1$ *and let* $u(s,t)$ *have the form* (1.1). *If* $u(s,0) \geq 0$ *for each* $s \in [0,\pi/2]$ *then* $u(\pi/2,t) \geq 0$ *for each* $t \in [0,\pi/2]$.

Using approach C we shall prove:

<u>LEMMA 2.2</u>. *Let* $\alpha \geq \beta \geq -\frac{1}{2}$ *and let* $u(s,t)$ *have the form* (1.1). *If* $u(s,0) > 0$ *for each* $s \in [0,\pi/2]$ *and if* $u(\pi/2,t) > 0$ *for each* $t \in [0,\pi/2]$ *then* $u(s,t) > 0$ *for each* (s,t) *such that* $0 \leq t \leq s \leq \pi/2$.

Lemma 2.1 and Lemma 2.2 together imply theorem 1.1.

<u>PROOF OF LEMMA 2.2</u>. Let $O = (0,0)$, $C = (\pi/2,0)$, $D = (\pi/2,\pi/2)$, $E = (0,\pi/2)$. Choose a point P in the closed triangular region OCD and let A and B be points on AC or CD such that the slopes of AP and BP are 1 and -1, respectively.

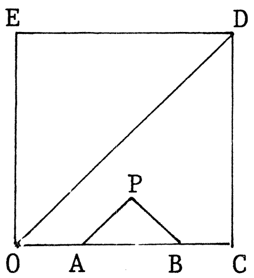

Figure 1

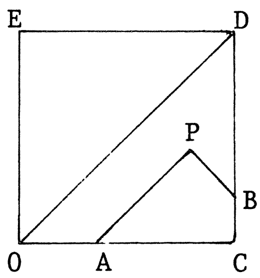

Figure 2

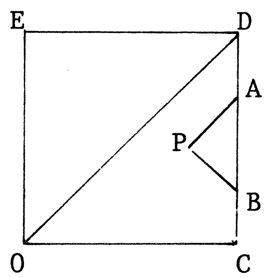

Figure 3

For any possible contour ABP (cf. figure 1,2,3) we have by (2.1) and by Gauss's theorem:

$$0 = \iint_{ABP} [(a\, u_s)_s - (a\, u_t)_t]\, ds\, dt$$

$$= \pm \oint_{ABPA} (a\, u_s\, dt + a\, u_t\, ds) = \mp \left(\int_{AP} + \int_{BP} \right) a\, du.$$

If u is positive on OC and CD then $a(A)u(A) + a(B)u(B) \geq 0$. Hence, integration by parts gives

$$\text{(2.2)} \qquad 2a(P)u(P) \geq \int_{AP} u(a_s + a_t)dt + \int_{BP} u(-a_s + a_t)\, dt.$$

It follows by a simple calculation that

$$\pm a_s(s,t) + a_t(s,t) = a(s,t)(\operatorname{cotg} t \pm \operatorname{cotg} s) \cdot$$
$$\cdot ((2\alpha+1) \mp (2\beta+1)\operatorname{tg}s \operatorname{tg}t).$$

Hence, $-a_s + a_t > 0$ if $0 < t < s < \pi/2$. Let Γ denote the curve $\{(s,t) \mid 2\alpha + 1 - (2\beta+1)\operatorname{tg}s \operatorname{tg}t = 0\}$, cf. figure 2.

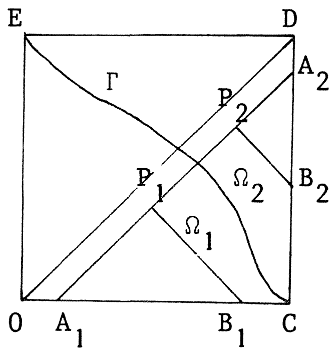

Figure 2

Any line inside OCDE with slope 1 intersects Γ in one and only one point. The curve Γ separates the region OCDE in two connected regions Ω_1 and Ω_2 on which $a_s + a_t$ is positive, respectively negative. Suppose now that u is positive on OC and CD but that $u(s,t) \le 0$ for some (s,t), $0 < t \le s < \pi/2$. Then, by continuity, there is a line A_1A_2 (A_1 on OC and A_2 on CD) with slope 1 and there are points P_1 and P_2 on A_1A_2 (cf. figure 2) such that $u(P_1) = 0 = u(P_2)$ and $u(s,t) > 0$ on the open region A_1CA_2 and on the open line segments A_1P_1 and A_2P_2. Let B_i $(i=1,2)$ be on OC or CD such that P_iB_i has slope -1. In at least one of the two cases $i = 1,2$ the open line segment A_iP_i is contained in the region Ω_i. For this choice of A_i, B_i, P_i formula (2.2) gives a contradiction. □

REFERENCES

[1] ASKEY, R., *Orthogonal polynomials and positivity*, <u>in</u> Studies in Applied Mathematics, Vol. 6, Society for Industrial and Applied Mathematics, Philadelphia, 1970, pp.64-85.

[2] BOCHNER, S., *Positive zonal functions on spheres*, Proc. Nat. Acad. USA, 40 (1954) 1141-1147.

[3] CHÉBLI, H., *Sur la positivité des opérateurs de "translation généralisée" associés à un opérateur de Sturm-Liouville sur* [0,∞[, C.R. Acad. Sc. Paris Sér. A, 275 (1972) 601-604.

[4] FLENSTED-JENSEN, M. & KOORNWINDER, T.H., *The convolution structure for Jacobi function expansions*, Ark. Mat. 11 (1973) 245-262.

[5] GASPER, G., *Positivity and the convolution structure for Jacobi series*, Ann. of Math. 93 (1971) 112-118.

[6] GASPER, G., *Banach algebras for Jacobi series and positivity of a kernel*, Ann. of Math. 95 (1972) 261-280.

[7] KOORNWINDER, T.H., *The addition formula for Jacobi polynomials, I. Summary of results*, Nederl. Akad. Wetensch. Proc. Ser. A, 75 = Indag. Math. 34 (1972) 188-191.

[8] WEINBERGER, H., *A maximum property of Cauchy's problem*, Ann. of Math. 64 (1956) 505-513.