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AFDELING TOEGEPASTE WISKUNDE TW 152/75 SEPTEMBER

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ON THE NUMERICAL EVALUATION OF THE ORDINARY BESSEL FUNCTION OF THE SECOND KIND

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 33A40, 65B10, 65Q05

On the numerical evaluation of the ordinary Bessel Function of the second kind *)

by

N.M.Temme**)

ABSTRACT

An algorithm is given for the numerical computation of the Bessel function $Y_\nu(z)$ for general ν and z. For small $|z|$ the Taylor expansion of the Bessel function $J_\nu(z)$ is used, whereas for the remaining values the computation is based upon a combination of algorithms due to J.C.P.Miller, W. Gautschi and F.W.J. Olver. In both cases the function $Y_{\nu+1}$ is obtained as well. ALGOL 60 procedures are given for ν and z real.

KEY WORDS & PHRASES: *ordinary Bessel function, Miller algorithm, ALGOL 60.*

*) This paper is not for review; it is meant for publication in a journal.

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1. INTRODUCTION

1.1. *Definitions and relevant properties.* The ordinary Bessel function of the first kind

$$(1.1)\qquad J_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{\Gamma(\nu+k+1)k!}$$

and the ordinary Bessel function of the second kind

$$(1.2)\qquad Y_\nu(z) = [\cos \nu\pi\ J_\nu(z) - J_{-\nu}(z)]/\sin \nu\pi$$

are two linearly independent solutions of the difference equation

$$(1.3)\qquad f_{\nu+1} - (2\nu/z)\ f_\nu + f_{\nu-1} = 0.$$

This equation can be used to compute $Y_{\nu+n}$ for $n = 2,3,\ldots$ when Y_ν and $Y_{\nu+1}$ are given. In the forward direction the recurrence formula (1.3) for Y_ν is numerically stable, whereas it is unstable for J_ν (see GAUTSCHI [1]).

The ordinary Bessel functions of the third kind are the Hankel functions

$$(1.4)\qquad H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z),\ H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z).$$

Important for the representation of the Hankel functions for large $|z|$ are the function $P(\nu,z)$ and $Q(\nu,z)$ defined by

$$(1.5)\qquad H_\nu^{(1,2)}(z) = [2/(\pi z)]^{\frac{1}{2}} e^{\pm i\chi}\ [P(\nu,z) \pm i\ Q(\nu,z)],$$

where the + is for $H_\nu^{(1)}$ and

$$(1.6)\qquad \chi = z - \pi(2\nu + 1)/4.$$

For large $|z|$, P and Q are slowly varying and the oscillatory behaviour of $H_\nu^{(1)}$ and $H_\nu^{(2)}$ is contained in the exponential function in (1.5). From (1.4) and (1.5) we obtain

$$(1.7)\qquad \begin{cases} Y_\nu(z) = [2/(\pi z)]^{\frac{1}{2}}\ [P(\nu,z)\ \sin\chi + Q(\nu,z)\ \cos\chi] \\ J_\nu(z) = [2/(\pi z)]^{\frac{1}{2}}\ [P(\nu,z)\ \cos\chi - Q(\nu,z)\ \sin\chi]. \end{cases}$$

Again, the oscillatory behaviour of J_ν and Y_ν is fully described by the circular functions in (1.7).

The connection between the ordinary Bessel functions and the modified Bessel functions follow from

$$(1.8)\qquad H_\nu^{(1)}(z) = -2i\pi^{-1}\ e^{-\frac{1}{2}\nu\pi i}\ K_\nu(ze^{-i\pi/2}).$$

From the Wronskian

$$J_{\nu+1}(z)\ J_\nu(z) - J_\nu(z)\ Y_{\nu+1}(z) = 2/(\pi z)$$

and (1.7) it easily follows that

$$(1.9)\qquad P(\nu,z)\ P(\nu+1,z) + Q(\nu,z)\ Q(\nu+1,z) = 1.$$

I.2. *Contents of the paper.* We give algorithims for the computation of Y_ν and $Y_{\nu+1}$ and we use the methods of our previous paper on the computation of K_ν and $K_{\nu+1}$ (see TEMME [6]). Our results in [6] can be used for complex values of z. Here we give the explicit results for Y_ν and $Y_{\nu+1}$ and these results follow immediately from [6] by using (1.8).

For the computation of J_ν the reader is referred to GAUTSCHI [1], where an algorithm is given for the computation of $J_{\nu+n}(z)$, $n = 0,1,2,\dots,N$. See also GAUTSCHI [2]. In LUKE [4] rational approximations for J_ν and Y_ν are given based on Padé-representations for large $|z|$. In LUKE [5] a double series of Chebyshev polynomials and values of the coefficients are given for both Y_ν J_ν for $z \geq 5$. In GOLDSTEIN & THALER [3] the computations of Y_ν is based on series expansions in ordinary Bessel functions of the first kind, but the treatment of small $|\nu|$-values is not satisfactory.

II. THE COMPUTATION FOR SMALL $|z|$.

In order to obtain a more symmetric representation in (1.2) we write

$$\cos \nu\pi\, J_\nu(z) - J_{-\nu}(z) = J_\nu(z) - J_{-\nu}(z) - 2 \sin^2(\nu\pi/2)\, J_\nu(z). \tag{2.1}$$

Furthermore we introduce the following notation

$$c_k = (-z^2/4)^k/k!,$$

$$p_k = (\nu/\sin \nu\pi)\ (z/2)^{-\nu}/\Gamma(k+1-\nu),$$

$$q_k = (\nu/\sin \nu\pi)\ (z/2)^{\nu}/\Gamma(k+1+\nu),$$

$$f_h = (p_k - q_k)/\nu,$$

$$g_k = f_k + 2\nu^{-1} \sin^2(\nu\pi/2)\ q_k,$$

$$h_k = -kg_k + p_k,$$

where $k = 0,1,\ldots$. We have for $k = 1,2,\ldots$ the recurrence relations

$$p_k = p_{k-1}/(k-\nu),\quad q_k = q_{k-1}/(k+\nu),$$

$$f_k = (k\, f_{k-1} + p_{k-1} + q_{k-1})/(k^2 - \nu^2).$$

Substitution of (1.1) in (1.2) and using (2.1) yields

$$Y_\nu(z) = -\sum_{k=0}^{\infty} c_k\, g_k. \tag{2.2}$$

Considering (1.2) with ν replaced by $\nu+1$ and using (1.3) we have

$$\cos\, (\nu+1)\pi\ J_{\nu+1}(z) - J_{-\nu-1}(z) =$$

$$- [J_{\nu+1}(z) - J_{-\nu+1}(z)] + (2\nu)/z\ J_{-\nu}(z) + 2 \sin^2(\nu\pi/2)\ J_{\nu+1}(z).$$

We obtain by substitution of (1.1)

$$(2.3) \qquad Y_{\nu+1}(z) = -\ (2/z) \sum_{k=0}^{\infty} c_k\, h_k.$$

As in [6], f_0 can be represented in such a way that it can be computed with a satisfactorily small relative error.

For small values of $|z|$ the series in (2.2) and (2.3) converge rapidly. But cancellation may occur in summing the series numerically. A strict error analysis as for the modified Bessel function can not easily be given, but from numerical experiments it turns out that for $|z| < 3$ the computation is stable.

III. THE COMPUTATION FOR $|z| \geq 3$.

For $|z| \geq 3$ we compute $P(\nu,z)$, $P(\nu+1,z)$, $Q(\nu,z)$ and $Q(\nu+1,z)$, by using the functions $k_n(z)$ introduced in our previous paper [6]. For K_ν and $K_{\nu+1}$ we needed $k_0(z)$ and $k_1(z)$. From (1.8) it turns out that for the P and Q-functions the functions $k_0(-iz)$ and $k_1(-iz)$ can be used. The application of the method in [6] is straightforward. However, the determination of the starting index N for the Miller algorithm caused some trouble, since our error analysis in [6] was based on the case of real variables. But trying out the results of [6] for the P and Q-functions we noticed that the determination of the starting index N can indeed be based upon the estimations given in [6].

IV. ALGOL 60 PROCEDURES

The algorithms for the computation of $Y_\nu(z)$ and $Y_{\nu+1}(z)$ are given as an ALGOL 60 procedure for the case of real values of ν and z, $z > 0$. For convenience we write $\nu = a$ and $z = x$.

The procedure *bessya* computes for $x > 0$ and $a \in \mathbb{R}$ the functions $Y_a(x)$ and $Y_{a+1}(x)$; *bessya* calls for three nonlocal procedures *sinh*, *recip gamma* and *besspqa*. For the text of *sinh*, and *recip gamma* the reader is referred to [6]. In *besspqa* the functions P(a,x), P(a+1,x), Q(a,x) and Q(a+1,x) are computed. We supply *besspqa* as a separate procedure since it can also be used for the computation of the Bessel functions $J_a(x)$ and $J_{a+1}(x)$ (see (1.7)). In *bessya* the procedure *besspqa* is called for $x \geq 3$ and $|a| < .5$, but the algorithm in *besspqa* converges for all x and a $(x > 0)$. It is recommended however, to use not too small x and / or not to large $|a|$. For large values of $|a|$ the recurrence relations

$$P(a+1,x) = P(a-1,x) - 2a/x\ Q(a,x)$$
$$Q(a+1,x) = Q(a-1,x) + 2a/x\ P(a,x)$$

can be used. These relations are valid for real a and x. They can be derived by substitution of (1.5) in (1.3).

The precision in the procedures *bessya* and *besspqa* can be controlled by using the variable eps. For *besspqa* its entry value corresponds to the desired relative accuracy in pa, pa1, qa and qa1. Also in *bessya* it corresponds to relative accuracy, except in the neighbourhoods of zeros of $Y_a(x)$ or $Y_{a+1}(x)$. In that case ya or ya1 are given with absolute accuracy eps.

The procedures *bessya* and *besspqa* are tested on the CD CYBER 73 of SARA, Amsterdam. For a = 0, 0.2, 0.4, x = .5, 1, 2, 3, 5, 7, 10, 20, 50, 100 and eps = 10^{-15} we checked relation (1.9). The output of |pa.pa1 + qa qa1 - 1| is given in TABLE I. The procedure *bessya* is also tested in the neighbourhood of x = 3. For $x^{\pm} = 3 \pm 2^{-46}$ we computed the numerical values of the expressions

$$d_0 = \{Y_a(x^-) - Y_a(x^+)\},$$

$$d_1 = \{Y_{a+1}(x^-) - Y_{a+1}(x^+)\}.$$

In TABLE II we give d_0, d_1, the maximum number of terms (n) used in (2.1), and the starting index N for the Miller algorithm.

```
procedure bessya(a,x,eps,ya,ya1); value a,x,eps; real a,x,eps,ya,ya1;
begin real b,c,d,e,f,g,h,p,pi,q,r,s; integer n,na; boolean rec,rev;
    pi:= 4 × arctan(1); na:= entier(a+.5); rec:= a ≥ .5;
    rev:= a < -.5; if rev ∨ rec then a:= a-na;
    if a = -.5 then
    begin p:= sqrt(2/pi/x); f:= p × sin(x); g:= -p × cos(x) end else
    if x < 3 then
    begin b:= x/2; d:= -ln(b); e:= a × d;
        c:= if abs(a) < ₁₀-15 then 1/pi else a/sin(a × pi);
        s:= if abs(e) < ₁₀-15 then 1    else sinh(e)/e;
        e:= exp(e); g:= recip gamma(a,p,q) × e; e:= (e + 1/e)/2;
        f:= 2 × c × (p × e + q × s × d); e:= a × a;
        p:= g × c; q:= 1/g/pi; c:= a × pi/2;
        r:= if abs(c) < ₁₀-15 then 1 else sin(c)/c; r:= pi × c × r × r;
        c:= 1; d:= -b × b; ya:= f + r × q; ya1:= p;
        for n:= 1, n + 1 while
        abs(g/(1 + abs(ya))) + abs(h/(1 + abs(ya1))) > eps do
        begin f:= (f × n + p + q)/(n × n - e); c:= c × d/n;
            p:= p/(n- a); q:= q/(n + a);
            g:= c × (f + r × q); h:= c × p - n × g;
            ya:= ya + g; ya1:= ya1 + h
        end;
        f:= -ya; g:= -ya1/b
    end else
    begin b:= x - pi × (a + .5)/2; c:= cos(b); s:= sin(b);
        d:= sqrt(2/x/pi);
        besspqa(a,x,eps,p,q,b,h);
        f:= d × (p × s + q × c); g:= d × (h × s - b × c)
    end;
    if rev then
    begin x:= 2/x; na:= -na - 1;
        for n:= 0 step 1 until na do
        begin h:= x × (a - n) × f - g; g:= f; f:= h end
    end else if rec then
    begin x:= 2/x;
        for n:= 1 step 1 until na do
        begin h:= x × (a + n) × g - f; f:= g; g:= h end
    end;
    ya:= f; ya1:= g
end bessya;

procedure besspqa(a,x,eps,pa,qa,pa1,qa1); value a,x,eps;
          real a,x,eps,pa,qa,pa1,qa1;
begin real b,c,d,e,f,g,h,p,p0,q,q0,r,s; integer n,na; boolean rec,rev;
    rev:= a < -.5; if rev then a:= -a-1;
    rec:= a ≥  .5; if rec then
    begin na:= entier(a+.5); a:= a - na end;
    if a = -.5 then
    begin pa:= pa1:= 1; qa:= qa1:= 0 end else
    begin c:= .25 - a × a; b:= x + x; p:= 4 × arctan(1);
        e:= (x × cos(a × p)/p/eps) ↑ 2; p:= 1; q:= -x; r:= s:= 1 + x × x;
```

```
        for n:= 2, n + 1 while r × n × n < e do
        begin d:= (n - 1 + c/n)/s; p:= (2 × n - p × d)/(n + 1);
            q:= (-b + q × d)/(n + 1); s:= p × p + q × q; r:= r × s
        end;
        f:= p:= p/s; g:= q:= -q/s;
        for n:= n, n - 1 while n > 0 do
        begin r:= (n+1) × (2-p) - 2; s:= b + (n+1) × q; d:= (n - 1 + c/n)/
            (r × r + s × s); p:= d × r; q:= d × s; e:= f;
            f:= p × (e + 1) - g × q; g:= q × (e + 1) + p × g
        end;
        f:= 1 + f; d:= f × f + g × g;
        pa:= f/d; qa:= -g/d; d:= a + .5 - p; q:= q + x;
        pa1:= (pa × q - qa × d)/x;
        qa1:= (qa × q + pa × d)/x
    end;
    if rec then
    begin x:= 2/x; b:= (a + 1) × x;
        for n:= 1 step 1 until na do
        begin p0:= pa - qa1 × b; q0:= qa + pa1 × b;
            pa:= pa1; pa1:= p0; qa:= qa1, qa1:= q0; b:= b + x
        end
    end;
    if rev then
    begin p0:= pa1; pa1:= pa; pa:= p0;
        q0:= qa1; qa1:= qa; qa:= q0;
    end
end besspqa;
```

TABLE I

x \ a	0.0	0.2	0.4
0.5	$1.4_{10}-14$	$7.1_{10}-15$	$0.0_{10}+00$
1.0	$0.0_{10}+00$	$7.1_{10}-15$	$7.1_{10}-15$
2.0	$7.1_{10}-15$	$2.8_{10}-14$	$7.1_{10}-15$
3.0	$7.1_{10}-15$	$0.0_{10}+00$	$0.0_{10}+00$
5.0	$7.1_{10}-15$	$1.4_{10}-14$	$0.0_{10}+00$
7.0	$7.1_{10}-15$	$7.1_{10}-15$	$1.4_{10}-14$
10.0	$7.1_{10}-15$	$7.1_{10}-15$	$7.1_{10}-15$
20.0	$0.0_{10}+00$	$7.1_{10}-15$	$0.0_{10}+00$
50.0	$2.1_{10}-14$	$1.4_{10}-14$	$0.0_{10}+00$
100.0	$2.1_{10}-14$	$7.1_{10}-15$	$7.1_{10}-15$

TABLE II

a	eps	5.0_{10}-06	5.0_{10}-09	5.0_{10}-12	5.0_{10}-14
0.0	d0	5.2_{10}-08	$4,3_{10}$-11	3.4_{10}-14	5.3_{10}-15
	d1	6.4_{10}-08	1.8_{10}-11	3.6_{10}-14	5.3_{10}-15
	(n,N)	(9,17)	(11,37)	(13,64)	(14,87)
0.2	d0	4.8_{10}-08	5.3_{10}-11	5.0_{10}-14	1.8_{10}-15
	d1	9.4_{10}-08	4.9_{10}-11	2.2_{10}-14	1.3_{10}-14
	(n,N)	(9,17)	(11,36)	(13,63)	(14,86)
0.4	d0	6.8_{10}-09	2.2_{10}-11	2.1_{10}-14	8.9_{10}-15
	d1	2.3_{10}-08	1.1_{10}-10	2.5_{10}-14	2.3_{10}-14
	(n,N)	(10,15)	(11,33)	(13,59)	(14,81)
0.6	d0	2.0_{10}-07	8.2_{10}-12	3.4_{10}-14	1.6_{10}-14
	d1	9.9_{10}-08	4.8_{10}-11	1.6_{10}-14	2.4_{10}-14
	(n,N)	(8,15)	(11,33)	(13,59)	(14,81)
0.8	d0	3.5_{10}-08	4.7_{10}-12	4.1_{10}-14	1.1_{10}-14
	d1	5.7_{10}-08	4.7_{10}-11	0.0_{10}+00	2.1_{10}-14
	(n,N)	(9,17)	(11,36)	(13,63)	(14,86)
1.0	d0	6.4_{10}-08	1.8_{10}-11	3.2_{10}-14	3.6_{10}-15
	d1	9.5_{10}-08	5.5_{10}-11	7.1_{10}-15	1.4_{10}-14
	(n,N)	(9,17)	(11,37)	(13,64)	(14,87)

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