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UNIFORM ASYMPTOTIC EXPANSIONS OF CONFLUENT HYPERGEOMETRIC FUNCTIONS

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Uniform asymptotic expansions of confluent hypergeometric functions *)

by

N.M. Temme

ABSTRACT

New asymptotic expansions are derived for the confluent hypergeometric functions M(a,b,x) and U(a,b,x) for large b. The results are uniformly valid with respect to x in a neighbourhood containing x = b; a is a fixed parameter. The expansions contain parabolic cylinder functions and asymptotic series.

KEY WORDS & PHRASES: *confluent hypergeometric functions, asymptotic expansion, parabolic cylinder functions.*

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*) This paper is not for review; it is meant for publication elsewhere.

1. INTRODUCTION

In a recent paper [7], we derived new asymptotic expansions for the incomplete gamma functions

$$\gamma(a,x) = \int_0^x t^{a-1} e^{-t} dt, \qquad \Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t} dt$$

and for the incomplete beta function

$$I_x(p,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^x t^{p-1} (1-t)^{q-1} dt.$$

In each case, the expansion contains the complementary error function defined by

$$\text{(1.1)} \qquad \operatorname{erfc}(x) = 2\pi^{-\frac{1}{2}} \int_x^\infty e^{-t^2} dt$$

and an asymptotic series. The expansions are uniformly valid with respect to certain domains of the parameters.

The incomplete gamma functions may be considered as special cases of the confluent hypergeometric functions, which, in the notation of ABRAMOWITZ & STEGUN [1], are denoted as $M(a,b,x)$ and $U(a,b,x)$. Explicitly we have

$$\text{(1.2)} \qquad \begin{aligned} \gamma(a,x) &= a^{-1} x^a M(a,a+1,-x) = a^{-1} x^a e^{-x} M(1,a+1,x), \\ \Gamma(a,x) &= x^a e^{-x} U(1,a+1,x) = e^{-x} U(1-a,1-a,x). \end{aligned}$$

For large values of a and x with $x \sim a$, the functions $\gamma(a,x)$ and $\Gamma(a,x)$ exhibit a nonuniform behaviour. The expansions given in TEMME [7] describe this behaviour adequately. The same phenomena are expected for $M(a,b,x)$ and $U(a,b,x)$ for large values of x and b with $x \sim b$.

In this paper, we are concerned with the asymptotic expansions of the confluent hypergeometric functions for large positive values of b and/or x, which are uniformly valid with respect to $\lambda = x/b$ in a λ-interval containing $\lambda = 1$; a is considered as a fixed parameter.

The Whittaker functions are closely connected with the confluent hypergeometric functions. The relations are

$$M_{\kappa,\mu}(x) = e^{-x/2}\, x^{\mu+\frac{1}{2}}\, M(\tfrac{1}{2}+\mu-\kappa, 1+2\mu, x)$$
$$W_{\kappa,\mu}(x) = e^{-x/2}\, x^{\mu+\frac{1}{2}}\, U(\tfrac{1}{2}+\mu-\kappa, 1+2\mu, x).$$

There is a vast literature on confluent hypergeometric functions and Whittaker functions and on asymptotic expansions of these functions. Two recent books with many references are DINGLE [3] and OLVER [5]. Apart from the well-known expansions in inverse powers of the large argument x, expansions may be found which are uniformly valid with respect to certain parameters. The theory for large x and b, however, is still incomplete.

The results in the present paper can be considered as an extension of some of the results of Dingle, who gives expansions of M(a,b,x) and U(a,b,x) for $b < x$, $b > x$ and also in a neighbourhood of the transition point $b = x$. From Dingle's expansions, we learn that the qualitative behaviour of M and U in this neighbourhood can be described by parabolic cylinder functions of which the error function in (1.1) is a special case. The parabolic cylinder functions, which are also important in our paper, are special cases of the confluent hypergeometric functions. Explicitly, we have

$$D_\nu(x) = 2^{\nu/2}\, e^{-x^2/4}\, U(-\tfrac{1}{2}\nu, \tfrac{1}{2}, \tfrac{1}{2}x^2).$$

As in our previous paper, the starting point of the investigations will be an integral, which can be considered as an Laplace-type inversion formula. This representation turns out to be very suitable for obtaining uniform asymptotic expansions. For $b \sim x$, the saddle point of the integrand lies near by a singularity. By expanding the integrand or by integrating by parts as suggested by BLEISTEIN [2], we obtain two types of uniform asymptotic expansions in terms of functions allied to parabolic cylinder functions.

WONG [8] gives an asymptotic expansion of the Whittaker function $W_{\kappa,\mu}(z)$ for large values of the three parameters. In his expansion, parabolic cylinder functions also occur. For $z \to \infty$, $|\arg z| < \pi - \delta$, $\kappa = o(z)$, $\mu = o(z^{\frac{1}{2}})$ the expansion is

$$W_{\kappa,\mu}(z^2) \sim 2^{\frac{1}{4}-\kappa} z^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2}+2\mu+n)}{n!\,\Gamma(\frac{1}{2}+2\mu-n)}\; \frac{D_{2\kappa-n-\frac{1}{2}}(z2^{\frac{1}{2}})}{(z2^{3/2})^n}.$$

In our expansions we can take both κ and μ of order $O(z^2)$, but we have the condition $|\mu-\kappa| \leq M$ for some positive constant M.

In a recent paper, OLVER [6] announces a publication with asymptotic expansions for Whittaker functions with both κ and μ large. Olver's results are derived by using the differential equation for the Whittaker functions. His expansions will also be given in terms of parabolic cylinder functions.

2. CONTOUR INTEGRALS

U(a,b,x) and M(a,b,x) are solutions of Kummer's differential equation

$$x\,y'' + (b-x)\,y' - ay = 0. \tag{2.1}$$

If $a \neq 0,-1,-2,\ldots$, M and U are linearly independent. In general, U is singular at $x = 0$, whereas M is an entire function with the expansion

$$M(a,b,x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+n)} \frac{x^n}{n!}. \tag{2.2}$$

For fixed values of a,b and as $x \to \infty$ we have

$$\begin{aligned} M(a,b,x) &= \Gamma(b)\, e^x\, x^{a-b}/\Gamma(a)\ (1+O(x^{-1})),\\ U(a,b,x) &= x^{-a}\ (1+O(x^{-1})). \end{aligned} \tag{2.3}$$

In this section, we consider integrals of the type

$$\frac{1}{2\pi i} \int_L e^s\, s^c\, (s-x)^{-a}\, ds, \tag{2.4}$$

where a,c and x are real numbers. Throughout this paper we take $x \geq 0$ and

$$c = a - b;$$

a and b are the parameters of the confluent hypergeometric functions. L is a contour either so that it is a closed circuit such that the integrand of (2.4) returns to its initial value after s has described the circuit, or so that the integrand vanishes at each limit. Of course the integral is supposed to converge on L.

LEMMA 2.1. *Let* L *be specified as above. Then the integral in* (2.4), *considered as a function of* x, *satisfies Kummer's equation* (2.1).

PROOF. Denoting the function in (2.4) by y(x), we obtain by standard methods (cf. HOCHSTADT [4, p. 100]).

$$xy'' + (b-x)\,y' - ay = \frac{-a}{2\pi i}\int_L \frac{d}{ds}\left[e^s\, s^{c+1}\,(s-x)^{-a-1}\right] ds,$$

from which the lemma follows. □

After a further specification of L, we wish to write the integral (2.4) as a linear combination of the M- and U-function. In the following three lemmas, the many-valued functions are supposed to be real for positive values of their arguments.

LEMMA 2.2. *Let* L *be given as in figure* 2.1. *On* L *the phase of* s *increases from* $-\pi$ *to* π *as* s *describes the contour.* L *encircles the point* x *in positive direction. Let the branch-cuts of* s^c *and* $(s-x)^{-a}$ *be chosen from* 0, *respectively* x, *to* $-\infty$, *such that they are both enclosed by* L. *Then*

$$(2.5)\qquad \frac{1}{2\pi i}\int_{-\infty}^{(x+)} e^s\, s^c\,(s-x)^{-a}\, ds = \frac{1}{\Gamma(b)}\, M\,(a,b,x).$$

PROOF. By considering the behaviour of U and M near x = 0, and using Hankel's integral

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+)} e^s\, s^{-z}\, ds = 1/\Gamma(z),$$

the lemma is easily verified. □

LEMMA 2.3. *Let* L *and the branch-cuts of* s^c *and* $(x-s)^{-a}$ *be as indicated in figure* 2.2. *Then*

(2.6) $$\frac{1}{2\pi i}\int_{-\infty}^{(0^+)} e^s\, s^c\, (x-s)^{-a}\, ds = \frac{1}{\Gamma(-c)}\; U(a,b,x).$$

PROOF. In this case we consider (2.3) and we use Watson's lemma for loop integrals (see OLVER [6, p. 120]). □

LEMMA 2.4. *For* $\varepsilon = \pm 1$, *let* L_ε *be as indicated in figures* 2.3 *and* 2.4. L_ε *encloses the branch-out of* $(s-x)^{-a}$ *and it passes above (beneath) the origin for* $\varepsilon = +1(-1)$. *Then*

(2.7) $$\frac{1}{2\pi i}\int_{L_\varepsilon} e^s\, s^c\, (s-x)^{-a} = \frac{e^{\varepsilon i\pi c+x}}{\Gamma(a)}\; U\,(-c,b,e^{-\varepsilon i\pi}x).$$

PROOF. After a shift $s \to s + x$ in (2.6), we obtain an integrand resembling that of (2.6). Next, the value of x is changed into $e^{\varepsilon i\pi}x$, which gives (2.7). □

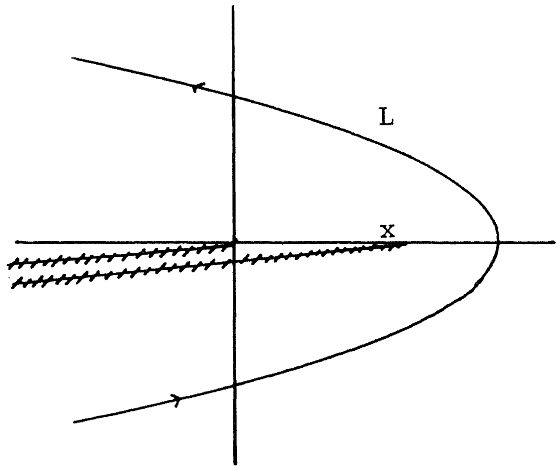

Figure 2.1

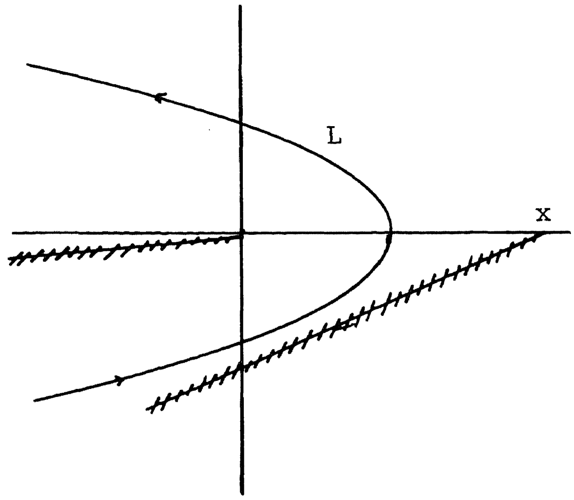

Figure 2.2

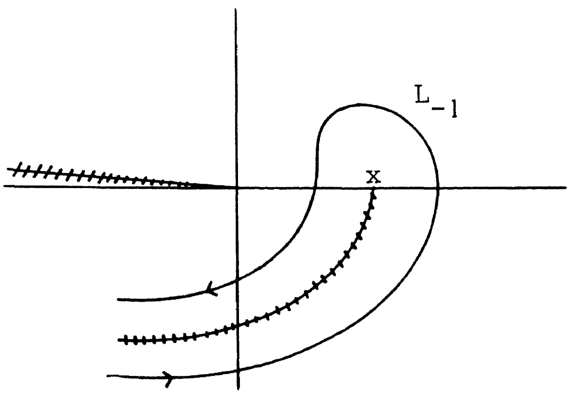

Figure 2.3

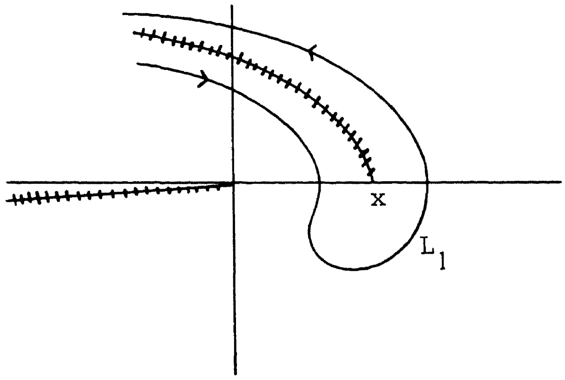

Figure 2.4

REMARK 2.5. The confluent hypergeometric functions in (2.5), (2.6) and (2.7) are related to each other, as follows from the connection formula

$$M(a,b,x) = \frac{\Gamma(b)}{\Gamma(b-a)} e^{i\varepsilon\pi a} U(a,b,x) + \frac{\Gamma(b)}{\Gamma(a)} e^{i\pi\varepsilon c+x} U(b-a,b,e^{-i\varepsilon\pi}x) \tag{2.8}$$

where $\varepsilon = \pm 1$. This formula follows from our results by deforming the contour in figure 2.1 into the contours of figures 2.2 and 2.3 (for $\varepsilon = -1$) or into those of figures 2.2 and 2.4 (for $\varepsilon = +1$).

More integral representations can be derived from (2.1), but in this paper we only use the above ones. Of course, the results (2.5) through (2.7) are valid for wider ranges of the parameters.

The following function is important in the asymptotic expansions of this paper

$$W_a(v) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} (u-v)^{-a} \, du, \tag{2.9}$$

with $a \in \mathbb{R}$, $v \in \mathbb{C}$. The contour of integration passes the singularity at $u = v$ as in the following picture.

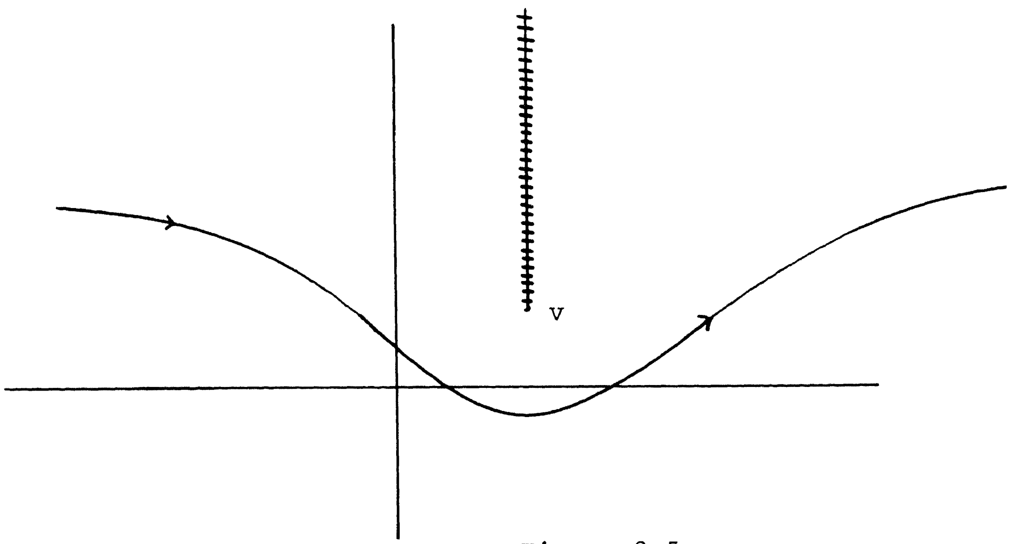

Figure 2.5

As follows from ABRAMOWITZ & STEGUN [1, p. 688], $W_a(v)$ is related to the parabolic cylinder function, this relation being given by

$$W_a(v) = (2\pi)^{\frac{1}{2}} \exp(-\tfrac{1}{4}v^2+\tfrac{1}{2}ia\pi)\, D_{-a}(-iv). \tag{2.10}$$

Clearly, we have

$$\frac{d}{dv} W_a(v) = a\, W_{a+1}(v). \tag{2.11}$$

Furthermore, we use the functions

$$(2.12)\qquad F_k(v) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, u^k (u-v)^{k-a} du,\qquad G_k(v) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\, u^{k+1} (u-v)^{k-a} du$$

for k = 0,1,..., where, again, the integration is as in figure 2.5. By expanding $u^k = [(u-v)+v]^k$ in a finite binomial series, $F_k(v)$ and $G_k(v)$ can be expressed as finite linear combinations of $W_{a-n}(v)$. By integration by parts, recurrence relations can be derived, for instance

$$(2.13)\qquad \begin{aligned} G_k(v) &= (2k-a)G_{k-1}(v) - vk\, F_{k-1}(v), \\ F_k(v) &= (2k-a-1)F_{k-1}(v) - v(k-1)[G_{k-2}(v) - vF_{k-2}(v)]. \end{aligned}$$

3. UNIFORM EXPANSIONS

3.1. *Saddle point contours*. Let us start with M(a,b,x). We derive an asymptotic expansion of this function for $x \to \infty$ and/or $b \to \infty$, uniformly valid with respect to λ, where

$$\lambda = x/b. \tag{3.1}$$

From (2.5) we obtain

$$M(a,b,x) = \frac{\Gamma(b+1)e^b b^{-b}}{2\pi i} \int_{-\infty}^{(\lambda^+)} e^{b\phi(t)}\, (1-\lambda/t)^{-a} dt, \tag{3.2}$$

where

(3.3) $\phi(t) = t - 1 - \ln t.$

Let us suppose temporarily $\lambda < 1$. As in [7], we choose the contour in (3.2) through the saddle point of ϕ at $t = 1$ along the steepest descent curve L given by $\operatorname{Im} \phi(t) = 0$, or explicitly

(3.4) $\sigma = \tau \operatorname{cotg} \tau, \quad -\pi < \tau < \pi,$

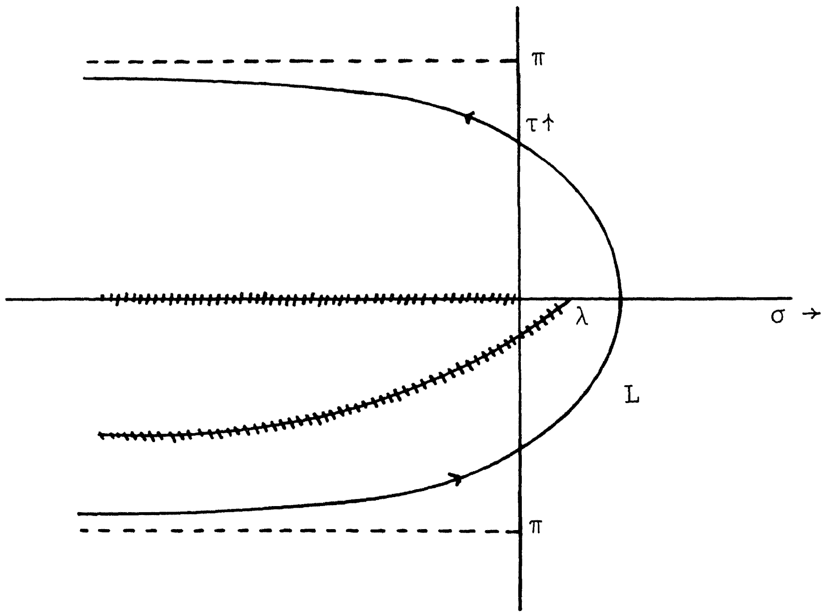

Figure 3.1

where $t = \sigma + i\tau$ $(\sigma,\tau \in \mathbb{R})$, see figure 3.1. On L the function ϕ is real and non-positive. Next, we define a mapping of the t-plane into the u-plane by the equation

(3.5) $-\frac{1}{2}u^2 = \phi(t)$

with the condition that $t \in L$ corresponds with $u \in \mathbb{R}$, and $u < 0$ if $\tau < 0$, $u > 0$ if $\tau > 0$. From these conditions it follows that

(3.6) $u = i(1-t)\,[2(t-1-\ln t)/(1-t)^2]^{\frac{1}{2}},$

where the square root is positive for positive values of its argument.

Integration with respect to u gives for (3.2)

$$(3.7) \qquad M(a,b,x) = \frac{\Gamma(b+1)e^b b^{-b}}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2} \frac{dt}{du} \frac{du}{(1-\lambda/t)^a} .$$

The singular points of the integrand in (3.7) are of two different types. First, we have the singularity due to the factor $(1-\lambda/t)^{-a}$ (of course, a singularity will only occur if $a \neq 0,-1,-2,...$). The singular point $t = \lambda$ in the t-plane corresponds with a point $u = u(\lambda) = u_1$, say, in the u-plane explicitly given by (cf. (3.6))

$$(3.8) \qquad u_1 = i(1-\lambda) \left[2(\lambda-1-\ln \lambda)/(1-\lambda)^2\right]^{\frac{1}{2}},$$

and if $\lambda \to 1$, then $u_1 \to 0$. The contour in (3.7) is as in figure 2.5, with $v = u_1$. If $\mathrm{Im}\, u_1 > 0$, an ideal contour of integration is the steepest descent path $\mathrm{Im}\, u = 0$. If $\mathrm{Im}\, u_1 \leq 0$, the contour in (3.7) will be deformed around the branch-cut of $(1-\lambda/t)^{-a}$. Hence, we may dispose of the condition $0 \leq \lambda < 1$ and we suppose henceforth $\lambda \geq 0$.

The second type of singularities of the integrand are due to the factor dt/du, which, by using (3.3) and (3.5) can be written as

$$(3.9) \qquad \frac{dt}{du} = \frac{ut}{1-t}$$

The point $t = 1$, corresponding to $u = 0$, gives a regular point. But, on account of the many-valuedness of the logarithm in (3.3), we also must consider the points $\exp(2\pi i n)$, for integer values of n, giving a sequence of singular points

$$(3.10) \qquad 2(\pi i n)^{\frac{1}{2}}, \qquad n = \pm 1, \pm 2,...$$

in the u-plane. When distorting the contour in (3.7) in order to allow non-positive values of $\mathrm{Im}\, u_1$, the singularities (3.10) must be avoided.

It is important to note that the singularities of the second type, given in (3.10), are fixed points in the u-plane, whereas u_1 given in (3.8) may be close to the origin (the saddle point). The point u_1 causes a non-uniform behaviour in (3.7) while the points in (3.10) are of a secundary interest.

The standard method for obtaining an asymptotic expansion via (3.7) is based on the substitution of the expansion

$$\frac{dt}{du}(1 - \lambda/t)^{-a} = \sum c_k(\lambda)\, u^k$$

in (3.7) followed by termwise integration. Owing to the singularity at u_1, a non-uniform expansion is obtained in this way. In fact, all $c_k(\lambda)$ are singular for $\lambda = 1$. In the following subsections, we give two types of uniform asymptotic expansions.

3.2. *Bleistein's method.* In the first place, we use an integration by parts procedure suggested by BLEISTEIN [2]. The integral in (3.7) is written as

$$J(a,b,u_1) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2}\, G(u)\frac{du}{(u-u_1)^a}, \tag{3.11}$$

where

$$G(u) = [(u-u_1)/(\lambda/t-1)]^a \frac{dt}{du}. \tag{3.12}$$

Except for the points given in (3.10), G is a holomorphic function of u. Especially, it is regular at $u = u_1$. Let us write

$$G(u) = \gamma_0 + (u-u_1)\,\gamma_1 + u(u-u_1)G_1(u), \tag{3.13}$$

where γ_0, γ_1 and G_1 must be determined. Substituting $u = u_1$, $u = 0$ respectively, we obtain

$$\gamma_0 = G(u_1), \qquad \gamma_1 = [G(u_1) - G(0)]/u_1, \tag{3.14}$$

and with γ_0 and γ_1, G_1 follows from (3.13). As G, it is regular except for the points in (3.10).

Upon inserting (3.13) into (3.11), we can rewrite $J(a,b,u_1)$ in the form

$$(3.15)\qquad J(a,b,u_1) = b^{\frac{1}{2}(a-1)}[\gamma_0\, W_a(u_1 b^{\frac{1}{2}}) + b^{-\frac{1}{2}}\gamma_1\, W_{a-1}(u_1 b^{\frac{1}{2}})] + J_1(a,b,u_1),$$

where W_a is defined in (2.7) and

$$(3.16)\qquad J_1(a,b,u_1) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2}\, u(u-u_1)^{-a+1}\, G_1(u)\, du.$$

We integrate by parts in (3.15) and obtain

$$b\, J_1(a,b,u_1) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2}(u-u_1)^{-a}\, [(1-a)G_1(u) + (u-u_1)G_1'(u)]\, du.$$

The procedure of (3.15) and (3.16) can now be applied to $b\, J_1(a,b,u_1)$ if we set

$$(1-a)G_1(u) + (u-u_1)G_1'(u) = \gamma_2 + (u-u_1)\gamma_3 + u(u-u_1)G_2(u).$$

It then follows that

$$J(a,b,u_1) = b^{\frac{1}{2}(a-1)}[(\gamma_0 + \gamma_2\, b^{-1})W_a(u_1 b^{\frac{1}{2}}) + b^{-\frac{1}{2}}\, (\gamma_1 + \gamma_3 b^{-1})W_{a-1}(u_1 b^{\frac{1}{2}})]$$

$$+\, b^{-1}\, J_2(a,b,u_1),$$

$$J_2(a,b,u_1) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2}\, u(u-u_1)^{-a+1} G_2(u)\, du.$$

This process can be continued to obtain an arbitrary number of terms. The final result is the asymptotic expansion

$$(3.17)\qquad M(a,b,x) \sim \frac{\Gamma(b+1)e^b b^{-b+\frac{1}{2}(a-1)}}{2\pi i}\left[W_a(u_1 b^{\frac{1}{2}}) \sum_{n=0}^{\infty} \gamma_{2n} b^{-n} + \right.$$

$$\left. +\, b^{-\frac{1}{2}}\, W_{a-1}\, (u_1 b^{\frac{1}{2}}) \sum_{n=0}^{\infty} \gamma_{2n+1} b^{-n} \right].$$

From (3.14), (3.12) and (3.8) it follows that

$$\gamma_0 = \lambda\,[(1-\lambda)/u_1]^{a-1},$$

(3.18)

$$\gamma_1 = \{\lambda\,[(1-\lambda)/u_1]^{a-1} - i\,[u_1/(\lambda-1)]^{a}\}/u_1 .$$

In general,

$$\gamma_{2n} = (1-a)\; G_n(u_1),$$

$$\gamma_{2n+1} = (1-a)\;[G_n(u_1) - G_n(0)]/u_1 + G_n'(0),$$

the functions G_n are determined recursively from the equations

(3.19) $$(1-a)G_n(u) + (u-u_1)G_n'(u) = \gamma_{2n} + (u-u_1)\;\gamma_{2n+1} + u(u-u_1)G_{n+1}(u),$$

$n = 0,1,\ldots,$ with $G_0(u) = G(u)$ given in (3.12). By inspection $\gamma_k = O(1)$ in λ if $\lambda \to 1$.

By using the more familiar parabolic cylinder functions we obtain, by considering (2.8) and the recurrence relation

$$D_{\nu+1}(x) = (x/2)\;D_\nu(x) - D_\nu'(x),$$

the asymptotic expansion in which all variables are real

(3.20) $$M(a,b,x) \sim (2\pi)^{-\frac{1}{2}}\Gamma(b+1)b^{-b+\frac{1}{2}(a-1)}e^{b+\frac{1}{4}\zeta^2}\Big[D_{-a}(\zeta)\sum_{n=0}^{\infty}\delta_{2n}b^{-n} + b^{-\frac{1}{2}}\;D_{-a}'(\zeta)\sum_{n=0}^{\infty}\delta_{2n+1}b^{-n-\frac{1}{2}}\Big]$$

where

(3.21) $$\zeta = -iu_1b^{\frac{1}{2}},\quad \delta_{2n} = -ie^{\frac{1}{2}ai\pi}(\gamma_{2n} - \tfrac{1}{2}u_1\gamma_{2n+1}),\quad \delta_{2n+1} = e^{\frac{1}{2}ai\pi}\gamma_{2n+1}.$$

REMARK 3.1. For $a = 1$, the confluent hypergeometric functions can be expressed as incomplete gamma functions, see (1.2). For this case

$W_0(u_1 b^{\frac{1}{2}}) = (2\pi)^{\frac{1}{2}}$, $W_1(u_1 b^{\frac{1}{2}}) = i\pi \exp(-\frac{1}{2}bu_1^{\ 2})\operatorname{erfc}(\zeta/\sqrt{2})$, $\gamma_0 = \lambda$, $\gamma_{2n} = 0$ $(n > 0)$. As can be verified, expansion (3.17) reduces indeed to the result of our previous paper.

Bleisten showed that an expansion like (3.17) is uniformly valid with respect to λ in a neighbourhood of $\lambda = 1$. In the case of the incomplete gamma functions, the expansions turned out to be uniformly valid for $\lambda \geq 0$, and we might expect the expansion (3.17) to hold in the same λ-domain. In terms of ζ given in (3.21), the expansions then are expected to hold uniformly for all real ζ. For that purpose, the following properties have to be verified.

(i) The sequences $\{\gamma_{2n} b^{-n}\}$ and $\{\gamma_{2n+1} b^{-n}\}$ are uniform asymptotic sequences. That is to say, the elements of the sequences have to satisfy

$$\gamma_{2n+2}\, b^{-n-1} = o(\gamma_{2n} b^{-n}),\quad \gamma_{2n+3}\, b^{-n-1} = o(\gamma_{2n+1} b^{-n}),$$

$n = 0,1,2,\ldots$, for $b \to \infty$, uniform in $\zeta \in \mathbb{R}$.

(ii) There are sequences $\{\alpha_{2n}\}$, $\{\alpha_{2n+1}\}$, which are uniform asymptotic sequences for $b \to \infty$, uniformly in $\zeta \in \mathbb{R}$, such that for $n, m = 0,1,2,\ldots$

$$M(a,b,x) = \frac{\Gamma(b+1)e^b b^{-b+\frac{1}{2}(a-1)}}{2\pi i}\,[W_a(u_1 b^{\frac{1}{2}})\{\sum_{i=0}^{n-1} \gamma_{2i} b^{-i} + O(\alpha_{2n})\} +$$

$$+\, b^{-\frac{1}{2}} W_{a-1}(u_1 b^{\frac{1}{2}})\{\sum_{i=0}^{m-1} \gamma_{2i+1} b^{-i} + O(\alpha_{2m+1})\}]$$

for $b \to \infty$, uniform in $\zeta \in \mathbb{R}$.

A drawback of Bleinstein's method is a lack of an explicit expression for the coefficients γ_n and the functions G_n, which are only given recursively. As a consequence, it is difficult to verify the properties in (i) and (ii), even in our case where the function G is given explicitly. However, inspection of the first ratio γ_2/γ_0 for $\zeta \to \pm\infty$ indicates that the uniformity with respect to ζ cannot be given for the whole domain $\mathbb{R}$. The most we can expect is uniformity with respect to an interval $[-A,B]$, where A,B depend on a,b, such that $A,B \to \infty$ for $b \to \infty$ (a fixed).

A pleasant feature of Bleistein's method is the form of the asymptotic expansion, in which only two parabolic cylinder functions occur. In the following section, we give an alternative expansion, but first we give an alternative expansion, but first we give the results for U(a,b,x) corresponding to (3.17).

The starting point is (2.6). After some transformations, we obtain

$$U(a,b,x) = \frac{b^{1-b}e^{b}\Gamma(b-a)}{2\pi i}\int_{-\infty}^{\infty} e^{-\frac{1}{2}bu^2} G(u) \frac{du}{(u_1-u)^a},$$

where G and u_1 are given in (3.12) and (3.8). The contour passes the singularity at $u = u_1$ as in figure 3.2.

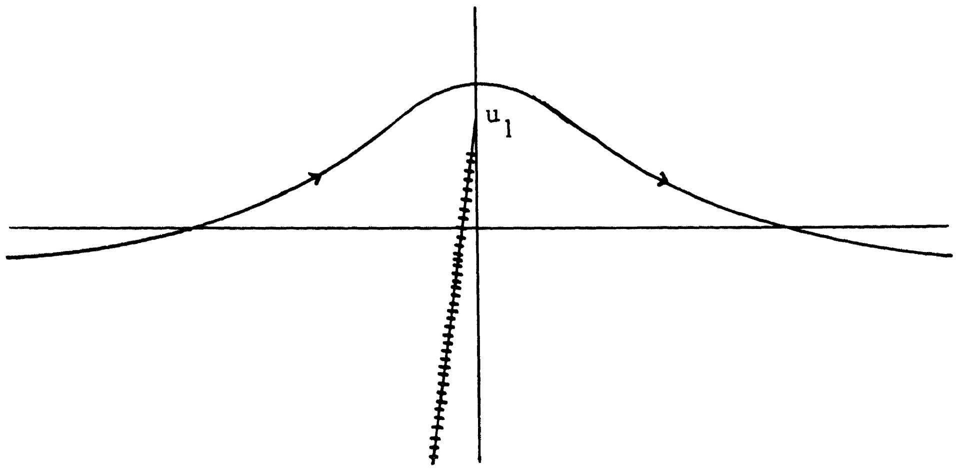

Figure 3.2

The asymptotic expansion now contains functions of the type

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}(v-u)^{-a}du.$$

If we make the change of variable $u \to -u$, it turns out that this integral equals $W_a(-v)$.

Proceeding as before, we arrive at the expansion

$$(3.22)\quad U(a,b,x) \sim \frac{b^{\frac{1}{2}(1+a)-b}e^{b}\Gamma(b-a)}{2\pi i}\left[W_a(-u_1 b^{\frac{1}{2}})\sum_{n=0}^{\infty}\gamma_{2n}b^{-n}\right.$$

$$\left.-b^{-\frac{1}{2}}W_{a-1}(-u_1 b^{\frac{1}{2}})\sum_{n=0}^{\infty}\gamma_{2n+1}b^{-n}\right].$$

The coefficients γ_n are the same as those for M(a,b,x) in (3.17).

The expansions for the integrals along the contours in figures 2.3 and 2.4 follow now from (2.8). By using (3.17), (3.20) and a connection formula for the parabolic cylinder functions, viz.

$$D_\nu(z) = e^{-\varepsilon i\nu\pi}D_\nu(-z) + (2\pi)^{\frac{1}{2}}\Gamma(-\nu)^{-1}D_{-\nu-1}(\varepsilon iz)e^{-\varepsilon i\pi(\nu+1)/2},$$

with $\varepsilon = \pm 1$, we obtain

$$(3.23)\quad U(b-a,b,e^{-\varepsilon i\pi}x) \sim -(2\pi)^{-\frac{1}{2}}b^{\frac{1}{2}(a+1)-b}e^{-\frac{1}{2}v^2+\frac{1}{2}i\pi[2a+\varepsilon(1-a+2b)]+b-x}$$

$$\left\{W_{1-a}(-i\varepsilon u_1 b^{\frac{1}{2}})\sum_{n=0}^{\infty}\gamma_{2n}b^{-n} - (1-a)b^{-\frac{1}{2}}e^{\frac{1}{2}\pi i\varepsilon}W_{2-a}(-i\varepsilon u_1 b^{-\frac{1}{2}})\sum_{n=0}^{\infty}\gamma_{2n+1}b^{-n}\right\}.$$

<u>REMARK 3.2.</u> The expansions in (3.17), (3.22) and (3.23) are given as series of inverse powers of b. The results are valid for $b \to \infty$, uniformly valid with respect to λ in a neighbourhood of $\lambda = 1$. By considering in (2.5) and (2.6) x as a large parameter, we can derive asymptotic expansions for U(a,b,x) and M(a,b,x) with series in inverse powers of x for $x \to \infty$, uniformly valid with respect to λ, again in a neighbourhood of $\lambda = 1$.

3.3. *Alternative expansions.* If we expand the function G in (3.12) in a two-points Maclaurin expansion

$$(3.23)\quad G(u) = \sum_{k=0}^{\infty} c_k u^k (u-u_1)^k + u\sum_{k=0}^{\infty} d_k u^k(u-u_1)^k,$$

were the c_k and d_k are to be determined, we obtain by termwise integration in (3.11)

$$(3.24)\quad J(a,b,u_1) \sim b^{\frac{1}{2}(a-1)}\left[\sum_{k=0}^{\infty} c_k F_k(u_1 b^{\frac{1}{2}})b^{-k} + \sum_{k=0}^{\infty} d_k G_k(u_1 b^{\frac{1}{2}})b^{-k-\frac{1}{2}}\right].$$

The functions F_k and G_k are given in (2.12) and recursion relations between them in (2.13). The coefficients c_k and d_k may be obtained by substituting the values $u = 0$ and $u = u_1$ and differentiating the series. The first few are

$$c_0 = G(0), \qquad d_0 = [G(u_1) - G(0)]/u_1,$$

$$c_1 = G'(0) + d_0/u_1, d_1 = [G'(u_1) - d_0 - c_1u_1]/u_1^2.$$

The following lemma gives an explicit formula for c_k and d_k for general k.

<u>LEMMA 3.3.</u> *Let* $w(v) = (v + \frac{1}{4}u_1^2)^{\frac{1}{2}}$ *and let the functions* H_1 *and* H_2 *be given by*

$$H_1(v) = [G(\tfrac{1}{2}u_1 + w(v)) - G(\tfrac{1}{2}u_1 - w(v))]/w(v),$$

$$H_2(v) = [(w(v) - \tfrac{1}{2}u_1)\ G(\tfrac{1}{2}u_1 + w(v)) + (w(v) + \tfrac{1}{2}u_1)\ G(\tfrac{1}{2}u_1 - w(v))]/w(v),$$

where the square root in w *is real for positive arguments, then the coefficients* c_k *and* d_k *in* (3.23) *are given by*

$$(3.25) \qquad \begin{aligned} 2c_k &= \frac{1}{2\pi i} \int_{C_2} \frac{H_2(v)}{v^{k+1}}\, dv, \\ 2d_k &= \frac{1}{2\pi i} \int_{C_1} \frac{H_1(v)}{v^{k+1}}\, dv, \end{aligned}$$

where C_i *are simple closed contours encircling* $v = 0$ *but not encircling any singularity of* H_i, $i = 1,2$.

<u>PROOF.</u> If we substitute $u = \frac{1}{2}u_1 + v$ in (3.23), we obtain

$$G(\tfrac{1}{2}u_1+v) = \sum c_k(v^2-\tfrac{1}{4}u_1^2)^k + (v+\tfrac{1}{2}u_1) \sum d_k(v^2-\tfrac{1}{4}u_1^2)^k.$$

Splitting up the right-hand side in odd and even parts (with respect to v) and using Cauchy's integral formula for the coefficients of the Maclaurin expansion of holomorphic functions we obtain the representations for c_k and d_k. Since G is holomorphic except at the points (3.10), H_1 and H_2 are holomorphic in a neighbourhood of $v = 0$. □

It is not difficult to prove that the sequences $\{c_k F_k b^{-k}\}$ and $\{d_k G_k b^{-k}\}$ are uniform asymptotic sequences for $b \to \infty$, uniform in a neighbourhood of $\lambda = 1$, and that (3.24) gives a uniform asymptotic expansion. An optimal interval of uniformity has not been obtained, but we expect that the investigations on this subject are carried out easier with (3.24) than by using the expansion of the previous subsection.

4. CONCLUDING REMARKS

4.1. In our previous paper [7], we obtained the following expansions for the incomplete gamma functions (in the notation of the present paper)

$$(4.1)\quad \begin{aligned} \gamma(b,x)/\Gamma(b) &= \tfrac{1}{2}\mathrm{erfc}(\zeta 2^{-\frac{1}{2}}) - R(b,x) \\ \Gamma(b,x)/\Gamma(b) &= \tfrac{1}{2}\mathrm{erfc}(-\zeta 2^{-\frac{1}{2}}) + R(b,x), \end{aligned}$$

where ζ is given in (3.21). For the function R(b,x) we gave an asymptotic expansion in inverse powers of b. As a first approximation we have

$$R(b,x) = e^{-b\phi(\lambda)}\, b^{-\frac{1}{2}} O(1), \qquad b \to \infty,$$

where ϕ is given in (3.3) and the O-symbol in uniformly valid in $x \geq 0$. From these formulas, it is easy to see how the incomplete gamma functions behave qualitatively. See figure 4.1.

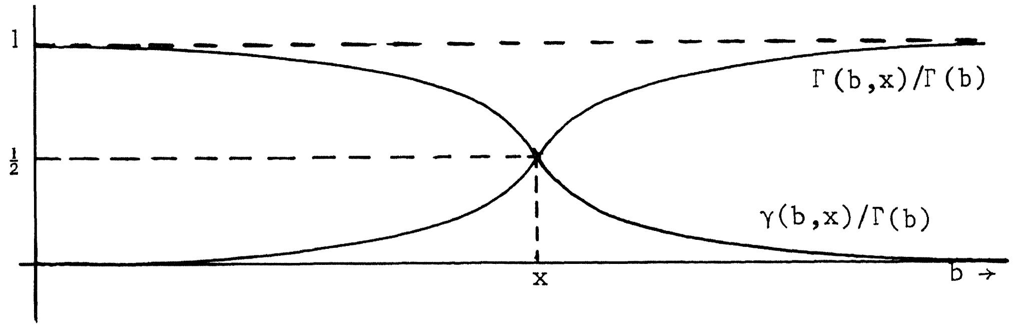

Figure 4.1

At $\zeta = 0$, i.e., $b = x$, $\gamma(b,x)$ and $\Gamma(b,x)$ take over each others asymptotic behaviour. If $\zeta > 0$ (i.e., $b > x$), $\gamma(b,x)/\Gamma(b)$ is exponentially small.

The same phenomenon occurs for $M(a,b,x)$ and $U(a,b,x)$. Formulas (3.17) and (3.22) are not as transparent as (4.1), but we can consider the asymptotic behaviour of $W_a(u_1 b^{\frac{1}{2}})$ for $b \to \infty$ and fixed λ. If $\lambda < 1$, $b \to \infty$ we have

(4.2) $$W_a(u_1 b^{\frac{1}{2}}) = O(b^{-\frac{1}{2}a}),$$

and if $\lambda > 1$, $b \to \infty$ we have

(4.3) $$W_a(u_1 b^{\frac{1}{2}}) = O\left(b^{\frac{1}{2}(a-1)} e^{-\frac{1}{2}bu_1^2}\right)$$

for $a \neq 0, -1, -2, \ldots$, while for $a = 0, 1, \ldots$ (4.2) applies. Hence if λ passes the value 1, the qualitative behaviour of $W_a(u_1 b^{\frac{1}{2}})$ and $W_{a-1}(u_1 b^{\frac{1}{2}})$ changes abruptly.

4.2. If we compare the methods for uniform asymptotic expansions with those for the non-uniform expansions, we observe the following. When applying the non-uniform saddle point method to the integral (3.2), we can first suppose $\lambda < 1$. The saddle point contour is drawn in figure 3.1 and is described in (3.4). It cuts the real t-axis at the saddle point in $t = 1$. If a is a positive integer and if we take $\lambda > 1$, the pole at $t = \lambda$ passes the contour L. If we do not modify L we have to take into account the contribution of the residue of the pole. Hence, the treatment of the case $\lambda < 1$ is equivalent to that of the case $\lambda > 1$. However, in the latter case the asymptotic expansion contains the residue of the pole. If $a = 1$ (the incomplete gamma function case) the pole is simple and the residue is easily calculated.

In the above description, we notice some non-uniformity with respect to λ, whereas the case $\lambda = 1$ is not discussed. In the uniform saddle point method, the error function in (4.1) takes over the role of the residue, but it has a smoothing effect. In fact, the expansions (4.1) hold uniformly in $\lambda \geq 0$, which is due to the error function.

If a is not a positive integer and λ passes the value 1, the contour can be split up and the contour integral along the contours in figures 2.3

and 2.4 is the analogue of the residue in the foregoing case. Beginning for $\lambda < 1$ with (2.5) or (3.2) for M(a,b,x), we can take (2.6) for U(a,b,x) if $\lambda > 1$. Then the transition from the first case to the second one is established by using (2.7). In the uniform method, the parabolic cylinder functions, however, again smooth the effect of the integral (2.7), and the result is a uniform expansion with respect to λ.

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