stichting mathematisch centrum

MC

AFDELING TOEGEPASTE WISKUNDE TW 154/75 NOVEMBER

J.W. DE ROEVER

ANALYTIC REPRESENTATIONS AND FOURIER TRANSFORMS OF ANALYTIC FUNCTIONALS IN Z' CARRIED BY THE REAL SPACE

Prepublication

2e boerhaavestraat 49 amsterdam

BIBLIOTHEEK MATHEMATISCH CENTRUM
AMSTERDAM

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 42A68, 46F05, 46F15

Analytic representations and fourier transforms of analytic functionals in Z' carried by the real space

by

J.W. de Roever

ABSTRACT

In the space Z', the Fourier transform of the space $\mathcal{D}'$ of Schwartz-distributions, the notion of carrier is introduced. A characterization is given of all distributions in $\mathcal{D}'$ the Fourier transform of which is carried by $\mathbb{R}^n$. Both, such distributions and the analytic functionals in Z' carried by $\mathbb{R}^n$, are represented as sum of boundary values of holomorphic functions. This extends the case of tempered distributions which, regarded as elements of Z', are obviously carried by $\mathbb{R}^n$.

KEY WORDS & PHRASES: *boundary value of a holomorphic function; carrier of an analytic functional; Fourier transforms of distributions and analytic functionals.*

1. INTRODUCTION

In VLADIMIROV [9,26.3 & 26.4 th.2] theorems are derived concerning Fourier transforms of tempered distributions in S' with support in a certain, unbounded, convex set. These Fourier transforms can be represented as boundary values in S' of holomorphic functions in a tubular, radial domain. This yields an analytic representation of distributions in S', i.e., a sum of distributional boundary values of certain holomorphic functions. In this paper these notions are generalized such that the boundary values are no longer attained in S' but in the space Z', which is the Fourier transform of the space $\mathcal{D}'$ of Schwartz-distributions. These boundary values are analytic functionals in Z' carried by $\mathbb{R}^n$ and their inverse Fourier transforms are distributions in $\mathcal{D}'$. Moreover, a characterization is obtained of those distributions in $\mathcal{D}'$ such that their Fourier transforms are all analytic functionals carried by $\mathbb{R}^n$. Then analytic representations of such functionals and distributions are given. This extends the case of tempered distributions which, regarded as elements of Z', are obviously carried by $\mathbb{R}^n$. Finally, the spaces of functions, holomorphic in tubular radial domains of exponential type in Im z and of polynomial growth in Re z, and the spaces of their inverse Fourier transforms are provided with topologies such that Fourier transformation is a topological isomorphism.

2. NOTATIONS AND DEFINITIONS

We will denote vectors in $\mathbb{C}^n$ by $z = (z_1,\dots,z_n) = x + iy$ and by $\zeta = \xi + i\eta$, where x,y,ξ,η are vectors in $\mathbb{R}^n$. The norm in $\mathbb{R}^n$ and $\mathbb{C}^n$ will be denoted as $\|\cdot\|$. For $t,w \in \mathbb{R}^n$ or $\mathbb{C}^n$, $t.w$ will stand for $t_1w_1+\dots+t_nw_n$. Let α be an n-tuple nonnegative integers, then $D_x^\alpha = D_1^{\alpha_1}\dots D_n^{\alpha_n}$, where $D_j = \partial/\partial x_j$, and $\vec{D}_x = (\partial/\partial x_1,\dots,\partial/\partial x_n)$; when no confusion arises the subscript x will be omitted. Similarly, for $t \in \mathbb{R}^n$ or $\mathbb{C}^n$ t^α is defined. Furthermore, $|\alpha| = \alpha_1+\dots+\alpha_n$ and $\alpha! = \alpha_1!\dots\alpha_n!$. For $b \in \mathbb{R}$, $\bar{b}$ will denote the vector $(b,\dots,b) \in \mathbb{R}^n$.

We remind the testfunction spaces $\mathcal{F},\mathcal{D}$ (cf. [7], $\mathcal{K}$ in [4]), S and Z and their duals denoted by ', which also refers to the strong topology. The action of an element $f \in W'$ on functions $\phi \in W$ will be denoted by $\langle f,\phi\rangle$ or

$\langle f,\phi\rangle_{W'}$. Sometimes we will write f_ξ and $\phi(\xi)$, if W consists of functions of ξ. In that case W', W and < , > will be denoted by W'_ξ, W_ξ and $\langle\ ,\ \rangle_\xi$, too. We mention explicitely the action of a function f, regarded as an element of $\mathcal{D}'$, to a testfunction ϕ in $\mathcal{D}$

$$(2.1)\qquad \langle f,\phi\rangle = \int_{\mathbb{R}^n} f(\xi)\phi(\xi)d\xi.$$

The Fourier transform of an L^1-function ϕ is given by

$$F[\phi](x) = F[\phi(\xi)](x) = \int_{\mathbb{R}^n} \phi(\xi)e^{ix\cdot\xi}\, d\xi.$$

Then $\mathcal{D}$ is the Fourier transform of Z and $\mathcal{D}(a)$ of Z(a), where $\mathcal{D}(a)$ is the space of C^∞-functions with support in $\{\xi \mid \|\xi\| \le a\}$. Z(a) consists of entire functions ψ, such that for all m

$$(2.2)\qquad \|\psi\|_m \overset{\text{def}}{=\!=} \sup_{z\in\mathbb{C}^n} (1+\|z\|)^m\, e^{-a\|y\|}\, |\psi(z)| < \infty.$$

The Fourier transform of a distribution $f \in \mathcal{D}'$ is defined as that element $F[f]$ of Z' for which

$$(2.3)\qquad \langle F[f],\psi\rangle_{Z'} = \langle f,F[\psi]\rangle_{\mathcal{D}'}, \qquad\qquad \psi \in Z,$$

cf. SCHWARTZ [7]. [1])

Elements μ of Z'(a) can be written as $\langle\mu,\psi\rangle = \int f(x)\psi(x)dx$ for some entire function f, see GELFAND & SHILOV [4,III §2.3]. Hence μ is a functional on the space of restrictions to $\mathbb{R}^n$ of functions in Z(a). In general, this is no longer true when $\mu \in Z'$. We say that $\mu \in Z'$ is *carried by the closed set* $\Omega \subset \mathbb{C}^n$, if for every ε-neighborhood $\Omega(\varepsilon)$ of Ω μ is already a functional

---

1) Since also Z is the Fourier transform of $\mathcal{D}$, a similar definition gives the Fourier transform of analytic functionals $\mu \in Z'$. Then, if for $\mu \in Z'$ $\check{\mu}$ is defined by $\langle\check{\mu}_\zeta,\psi(\zeta)\rangle = (2\pi)^{-n}\langle\mu_\zeta,\overline{\psi(\bar{\zeta})}\rangle$, $\psi \in Z$, $g = F[\mu]$ implies $F[g] = \check{\mu}$. Therefore, the theorems of section 4 of this paper dealing with inverse Fourier transforms g of elements $\mu \in Z'$ may just as well have been formulated with $g = F[\mu]$, hence with Fourier transforms of analytic functionals instead of inverse Fourier transforms, cf. the title of this paper.

on the space $Z|_{\Omega(\varepsilon)}$ of restrictions to $\Omega(\varepsilon)$ of functions in Z, where $Z|_{\Omega(\varepsilon)}$ carries the topology induced by Z, i.e., in (2.2) the supremum should be taken over all $z \in \Omega(\varepsilon)$. According to EHRENPREIS [2,th.5.13*] a fundamental system of neighborhoods of zero in Z is given by $V(k,\alpha) = \{\psi \in Z \mid |\psi(z)| \leq \alpha k(z)\}$, where $\alpha > 0$ and k is a positive continuous function of the following form: let $\{a_j\}$ be a strictly increasing sequence of integers with $a_0 = a_1 = a_2 = 0$, $a_{j+2} > 2a_j$, and let ℓ be a positive integer; set $k(z) = (1+\|x\|)^{-\ell}(1+\|y\|)^{-\ell}\exp((j-2)\|y\|)$ for $a_j(1+\log(1+\|x\|)) \leq \|y\| \leq \frac{1}{2}a_{j+1}(1+\log(1+\|x\|))$; the definition of k is completed by requiring that k is a function of $\|x\|,\|y\|$ which is continuous and such that, for fixed $\|x\|$, $\log k(\|x\|,\|y\|) + \ell[\log(1+\|x\|) + \log(1+\|y\|)]$ is linear in $\|y\|$ in the regions in which it is not already defined above. Then a fundamental system of neighborhoods of zero in $Z|_{\Omega(\varepsilon)}$ is obtained by $\{\psi \in Z|_{\Omega(\varepsilon)} \mid |\psi(z)| \leq \alpha k(z), z \in \Omega(\varepsilon)\}$. Now the Hahn-Banach theorem and Riesz' representation theorem imply that for every $\varepsilon > 0$ an analytic functional μ carried by Ω can be represented as a measure μ_ε on $\Omega(\varepsilon)$ satisfying

$$\int_{\Omega(\varepsilon)} \frac{|d\mu_\varepsilon(z)|}{k_\varepsilon(z)} \leq K_\varepsilon$$

where k_ε is a function as described above depending on ε.

In this paper we will be concerned with closed sets Ω which are bounded in the imaginary direction, i.e., Ω is contained in a set of the form $\{z \mid \|y\| \leq b\}$, $b \geq 0$. Then, if $Z(a)_m$ denotes the normed space of functions in Z(a) provided with the norm (2.2), the spaces $Z_F = \text{proj}\lim_{m\to\infty} \text{ind}\lim_{a\to\infty} Z(a)_m$ and $Z = \text{ind}\lim_{a\to\infty} \text{proj}\lim_{m\to\infty} Z(a)_m$ induce the same topology on $Z|_{\Omega(\varepsilon)}$. Indeed, according to EHRENPREIS [2,th.5.10] a fundamental system of neighborhoods of zero in Z_F is given by $V(k',\alpha)$, where now $k'(z) = (1+\|x\|)^{-m} k_1'(y)$ with $m \geq 0$ and with k_1' a positive, continuous, function dominating every $\exp a\|y\|$, $a > 0$. Z_F is the Fourier transform of the space $\mathcal{D}_F$, the test space for the finite order distributions. Hence the (inverse) Fourier transforms of all elements μ in Z' carried by Ω are finite order distributions and, moreover, for every $\varepsilon > 0$ these μ satisfy

$$|\langle\mu,\psi\rangle| \leq K_\varepsilon \sup_{z\in\Omega(\varepsilon)} (1+\|x\|)^{m(\varepsilon)} |\psi(z)|, \qquad \psi \in Z,$$

with K_ε and $m(\varepsilon)$ depending on ε and μ. The above representation yields that for every $\varepsilon > 0$ μ can be represented as a measure μ_ε on $\Omega(\varepsilon)$ satisfying

$$(2.4) \qquad \int_{\Omega(\varepsilon)} \frac{|d\mu_\varepsilon(z)|}{(1+\|x\|)^{m(\varepsilon)}} \leq K_\varepsilon .$$

In particular, we will be concerned with analytic functionals in Z' carried by $\mathbb{R}^n$.

The support of a distribution $f \in W'$, where W is a space of C^∞-functions on $\mathbb{R}^n$, is defined as the smallest closed set U such that any $\xi_0 \notin U$ has an open neighborhood V_0 with $\langle f,\phi\rangle = 0$ for every $\phi \in W$ with $\phi(\xi) = 0$ if $\xi \notin V_0$. In section 5 we will show that for certain sets U, in particular for convex sets, f can be represented as sum of weak derivatives of measures on U.

Finally, C will denote an open cone in $\mathbb{R}^n$ (i.e., $t \in C \Rightarrow \lambda t \in C$, $\lambda > 0$), ch(C) its convex hull, $\text{pr } C = \{y \in C \mid \|y\| = 1\}$ and $C' \subset\subset C$ means that C' is a relatively compact subcone of C, i.e., $\overline{\text{pr } C'} \subset \text{pr } C$. The function

$$\mu_C(\xi) = \sup_{y \in \text{pr } C} -y\cdot\xi$$

is the indicatrix of C. C^* denotes the closed cone $\{\xi \mid y\cdot\xi \geq 0,\ y \in C\} = \{\xi \mid \mu_C(\xi) \leq 0\}$ and $C_* = \mathbb{R}^n \setminus C^* = \{\xi \mid \mu_C(\xi) > 0\}$. Furthermore

$$\rho_C = \sup_{\xi \in \mathbb{R}^n} \mu_{ch(C)}(\xi) \Big/ \mu_C(\xi) .$$

We will consider holomorphic functions f in the tubular radial domain $T^C = \mathbb{R}^n + iC$. We say that f(z) has a boundary value f^* in Z' or S' as $y \to 0$, $y \in C$ or $y \in C' \subset\subset C$, respectively, if for all $\phi \in Z$ or S the limit

$$(2.5) \qquad \langle f^*,\phi\rangle = \lim_{\substack{y\to 0\\ y\in C \text{ or } C'}} \int_{\mathbb{R}^n} f(x+iy)\phi(x)dx$$

exists. The boundary value in S' is said to be attained on the distinguished boundary of T^C, i.e., on the set $\{z \mid z \in T^C,\ \text{Im } z = 0\}$. However, if the limit exists in Z', it is less clear a "boundary value on the distinguished boundary", as we will see in section 3.

3. BOUNDARY VALUES IN Z' and S'

We consider functions f holomorphic in the tube domain $T^B = \mathbb{R}^n + iB$, where B is an open set in $\mathbb{R}^n$. For any $y \in B$ $f(z) = f(x+iy)$ is a C^∞-function in x, i.e., $f(z) \in \mathcal{E}_x$. We regard $\mathcal{E}$ as the strong dual of $\mathcal{E}'$, the space of distributions with compact support. By the Paley-Wiener-Schwartz theorem the Fourier transform of $\mathcal{E}'$ is known as the space H of entire functions of exponential type in Im ζ and of polynomial growth in Re ζ, provided with the topology such that the Fourier transformation is a topological isomorphism between $\mathcal{E}'$ and H. Then $\mathcal{E}$ is the Fourier transform of the dual H' of H. With these definitions we have the following lemma.

LEMMA 3.1. *Let* f *be a holomorphic function in* T^B *with* B *an open set in* $\mathbb{R}^n$ *and let* $y_0 \in B$. *Then for all* y *with* $y + y_0 \in B$

$$f(x+iy+iy_0) = \mathcal{F}[e^{-y\cdot\zeta}\,\mu(y_0)_\zeta](x)$$

where

$$\mu(y_0) = \mathcal{F}^{-1}[f(x+iy_0)] \in H'.$$

PROOF. Let $\rho < 1$ and choose ε so small that $y + y_0 \in B$ whenever

$$|y_i| \le \frac{n\varepsilon}{\rho} \overset{\text{not}}{=\!=} \eta, \qquad i = 1,\dots,n.$$

Then also $y + y_0 \in B$ if $|y_i| \le \varepsilon$, $i = 1,\dots,n$. For these y and for $\phi_x \in \mathcal{E}'$ we have

$$\langle\phi_x, f(x+iy+iy_0)\rangle_{\mathcal{E}'} = \langle\phi_x, \sum_{k=0}^{\infty}\frac{(iy\cdot\vec{D}_x)^k}{k!}\, f(x+iy_0)\rangle_{\mathcal{E}'}.$$

In order that the series converges in $\mathcal{E}$ it is sufficient to show that it is bounded in $\mathcal{E}$. Therefore, we first estimate

$$|D^\alpha f(x+iy_0)| \le \frac{\alpha!}{(2\pi)^n}\int_{|z_1|=\eta}\cdots\int_{|z_n|=\eta}\frac{|f(x+iy_0+z)|}{z^{\alpha+\bar{1}}}\,dz \le$$

$$\le \alpha!\,\frac{1}{\eta^{\bar{\alpha}}}\sup_{\substack{|z_i|\le\eta\\ i=1,\dots,n}}|f(x+iy_0+z)|.$$

Hence for any compact set S in $\mathbb{R}^n$ and any nonnegative integer m

$$\sup_{\substack{x\in S\\ |\alpha|\le m}} \left| D^\alpha \sum_{k=0}^{N} \frac{(iy\cdot\vec{D})^k}{k!} f(x+iy_0)\right| \le$$

$$\le \sup_{\substack{x\in S\\ |z_i|\le\eta\\ i=1,\ldots,n}} |f(x+iy_0+z)| \sup_{|\alpha|\le m} \frac{\alpha!}{\bar{\eta}^\alpha} \sum_{k=0}^{N} \left|y\cdot\frac{\bar{1}}{\eta}\right|^k \le$$

$$\le K(S,m) \sum_{k=0}^{N} \rho^k \le \frac{K(S,m)}{1-\rho} ,$$

which is independent of N. With $\psi(\zeta) = F[\phi_x](\zeta) \in H$ we now may write

$$<\phi_x, f(x+iy+iy_0)>_{E'} = \sum_{k=0}^{\infty} <\frac{(-iy\cdot\vec{D})^k}{k!}\phi_x, f(x+iy_0>_{E'} =$$

$$= \sum_{k=0}^{\infty} <\mu(y_0)_\zeta, \frac{(-y\cdot\zeta)^k}{k!}\psi(\zeta)>_{H'} = \sum_{k=0}^{\infty} <\frac{(-y\cdot\zeta)^k}{k!}\mu(y_0)_\zeta, \psi(\zeta)>_{H'} =$$

$$= \lim_{N\to\infty} <\sum_{k=0}^{N} \frac{(-y\cdot\zeta)^k}{k!}\mu(y_0)_\zeta, \psi(\zeta)>_{H'}$$

and this equals $<\mu(y+y_0)_\zeta, \psi(\zeta)>_{H'}$ with $\mu(y+y_0) = F^{-1}[f(x+iy+iy_0)]$. Hence the weak limit for $N\to\infty$ exists in H'. Since H is a Montel space the strong limit exists and equals

$$\sum_{k=0}^{\infty} \frac{(-y\cdot\zeta)^k}{k!}\mu(y_0)_\zeta = e^{-y\cdot\zeta}\mu(y_0)_\zeta .$$

Thus

$$f(x+iy+iy_0) = F[e^{-y\cdot\zeta}\mu(y_0)_\zeta](x)$$

for y with $|y_i| \le \varepsilon$, $i = 1,\ldots,n$. By analytic continuation this formula holds for every y with $y + y_0 \in B$. □

Since $\mathcal{D}$ is dense in E', Z is dense in H. Hence for $f \in E$ the Fourier transform $F[f]$ is also determined by (2.3). The space of restrictions to $\mathbb{R}^n$ of functions in Z is dense in S, so that if, moreover, f belongs to S' then (2.3) implies

$$\langle F[f],\psi\rangle_{S'} = \langle f,F[\psi]\rangle_{S'}, \qquad \psi \in S.$$

Thus we have obtained the following corollary.

<u>COROLLARY 3.2</u>. *If* f(z), *holomorphic in* T^B, *for each* $y \in B$ *belongs to* S'_x, *then with* $y_0 \in B$ *and with* $g(y_0) = F^{-1}[f(x+iy_0)] \in S'$

$$e^{-y\cdot\xi}\, g(y_0)_\xi \in S'_\xi$$

for y *such that* $y + y_0 \in B$ *and*

$$f(z+iy_0) = F[e^{-y\cdot\xi}\, g(y_0)_\xi](x).$$

From this corollary one derives as in VLADIMIROV [9,th.26.1] that f(z) belongs to S'_x for each $y \in B$ if and only if it satisfies

$$(3.1) \qquad |f(z)| \le M(K)(1+\|x\|)^{m(K)}, \qquad y \in K \subset\subset B$$

for all compact sets K in B, where M(K) and m(K) depend on these sets K. Moreover, corollary 3.2 and VLADIMIROV [9, 26.2] yield that, if f(z) belongs to S'_x for each $y \in B$, it necessarily satisfies for every α

$$(3.2) \qquad |D^\alpha f(z)| \le M'(K)(1+\|x\|)^{m'(K)}, \qquad y \in K \subset\subset \mathrm{ch}(B),$$

for all compact sets K in the convex hull ch (B) of B, where M'(K) and m'(K) depend on K and α. Hence f(z) belongs to S'_x for each $y \in B$ means that (2.1) holds for $\phi \in S$. Clearly, in that case for each $y \in B$ f(z) belongs to Z' under definition (2.1), too.

Next we consider a function f(z), holomorphic in T^B, which for each $y \in B$ as a function of x belongs to Z'. This means that f(z) is a continuous linear functional on the space of restrictions to $\mathbb{R}^n$ of functions in Z, where this space carries the topology of S. Its closure is S, hence f(z) belongs to S'_x for each $y \in B$. Thus $f(x+iy) \in S'_x$ is equivalent to

$f(x+iy) \in Z'$ and this should be interpreted in the sense of definition (2.1) for $\phi \in S$ or Z.

Let us now consider the limit in Z' as y tends to zero, if we assume that B is connected. From (3.1) it follows that for any $y \in B$ and y_0 such that $y + y_0 \in B$

$$<f(z),\psi(x)> = \int_{\mathbb{R}^n} f(z+iy_0)\psi(x+iy_0)dx, \qquad \psi \in Z.$$

Therefore, the limit f^* as $y \to 0$ of $f(x+iy)$ exists in Z' and it is given by

$$(3.3) \qquad <f^*,\psi> \overset{\text{def}}{=\!=} \lim_{y\to 0} \int_{\mathbb{R}^n} f(x+iy+iy_0)\psi(x+iy_0)dx = $$

$$= \int_{\mathbb{R}^n} f(x+iy_0)\psi(x+iy_0)dx$$

for all $\psi \in Z$ and $y_0 \in B$. This limit is independent of $y_0 \in B$. Note, that when $0 \notin \bar{B}$ f^* can never exist in S'. If $0 \in \bar{B}$, $0 \notin B$, we may call f^* the boundary value in Z' according to (2.5) and this boundary value is independent of the path $y \to 0$ in B. Still f^* might not belong to S' nor satisfy definition (2.1). For example, take $B = \{y \mid y > 0\}$ in $\mathbb{R}^1$ and $f(z) = $ $= \exp 1/z$. This function satisfies 3.1, but $\int \exp 1/x\ \psi(x)dx$, $\psi \in Z$, does not exist. In general, it follows from (3.1) and (3.3) that f^* is an element of Z' carried by $\mathbb{R}^n$ (see section 2 for the definition of carrier in Z').

The inverse Fourier transform $g = \mathcal{F}^{-1}[f^*]$ is an element of $\mathcal{D}'$. For $\phi \in \mathcal{D}$ and $y_0 \in B$ from (3.3) we derive

$$<g,\phi>_{\mathcal{D}'} = <f^*,\psi>_{Z'} = <f(x+iy_0),\psi(x+iy_0)>_{Z'} =$$

$$= <g(y_0)_\xi, e^{y_0\cdot\xi}\ \phi(\xi)>_{\mathcal{D}'} = <e^{y_0\cdot\xi}\ g(y_0)_\xi, \phi(\xi)>_{\mathcal{D}'} ,$$

where $\psi = \mathcal{F}^{-1}[\phi]$ and $g(y_0) = \mathcal{F}^{-1}[f(x+iy_0)]$. Hence

$$(3.4) \qquad g_\xi = e^{y\cdot\xi}\ g(y)_\xi, \qquad y \in B$$

is independent of y and we have obtained (as in corollary 3.2)

(3.5) $$e^{-y\cdot\xi}\, g_\xi \in S'_\xi, \qquad y \in B$$

and

(3.6) $$f(z) = \mathcal{F}[e^{-y\cdot\xi}\, g_\xi](x).$$

Conversely, for a distribution $g \in \mathcal{D}'$ satisfying (3.5) the function (3.6) satisfies (3.1), hence also (3.2) (see VLADIMIROV [9,th.26.2]). Conditions (3.5) and (3.6) then also hold for $y \in \mathrm{ch}(B)$.

The distribution $g \in \mathcal{D}'$ can be obtained as follows: for $y \in K$

(3.7) $$g_\xi = e^{y\cdot\xi}\, \mathcal{F}^{-1}[f(x+iy)]_\xi = (A(K)-\vec{D}\cdot\vec{D})^{m'(K)} \frac{1}{(2\pi)^n}\int \frac{f(x+iy)e^{-iz\cdot\xi}}{(A(K)+z\cdot z)^{m'(K)}}\,dx,$$

where $m'(K) \geq \frac{1}{2}(m(K) + n+1)$ and where $A(K)$ is so large that for $y \in K$

$$|A(K) + z\cdot z| \geq 1 + x\cdot x .$$

The integral is independent of y. Hence for every $K \subset\subset B$ there are constants $m(K)$, $M_\alpha(K)$ and continuous functions $g_{\alpha,K}$ on $\mathbb{R}^n$, $|\alpha| \leq m(K)$, such that g can be represented as

(3.8) $$\begin{cases} g_\xi = \sum\limits_{|\alpha|\leq m(K)} D^\alpha\, g_{\alpha,K}(\xi) \\ |g_{\alpha,K}(\xi)| \leq M_\alpha(K)\, e^{y\cdot\xi}, \qquad \forall y \in K. \end{cases}$$

In that case g also satisfies (3.8) for every $K \subset\subset \mathrm{ch}(B)$. Conversely, if a distribution $g \in \mathcal{D}'$ satisfies (3.8) then (3.5) holds.

Next we consider the case that the limit f^* exists in S'. Let C be an open cone in $\mathbb{R}^n$ and let for each $C' \subset\subset C$ $R(C')$ be a positive number depending on C'. Let B be an open set in $\mathbb{R}^n$ containing each set $\{y \mid y \in \overline{C'},\ 0 < \|y\| \leq R(C')\}$. Let f be a holomorphic function in T^B which satisfies a stronger condition than (3.1), namely

(3.9) $$|f(z)| \leq M(C')(1+\|x\|)^{m(C')}(1+\|y\|^{-k}), \quad y \in C',\ \|y\| \leq R(C')$$

for each compact subcone C' of C and for some k and m(C') depending on C'. We may let k depend on C', too, but in lemma 3.3 it will be shown that (3.9) is satisfied for a fixed k anyhow. Then the limit

$$f^* = \lim_{\substack{y\to 0 \\ y\in C'}} f(x+iy)$$

exists in S'_x and it is independent of C' and the path $y \to 0$ in C', see VLADIMIROV [9,th.26.3]. Here the most general case arises if R(C') tends to zero as C' approaches C. Now f^* is called the boundary value of f(z) in S' on the distinguished boundary. Clearly f^* is attained in Z', too, but if the limit exists in Z' only, (3.3) shows that the boundary value in Z' may be concentrated on other sets than the distinguished boundary as well.

Let $g = F^{-1}[f^*]$, where f^* is the boundary value in S' of a function f(z) satisfying (3.8), then g belongs to S'. The representation (3.7) now holds for $y \in \overline{C'}$, $0 < \|y\| \le R(C')$ and for $m'(K) \ge \frac{1}{2}(m(C') + n+1)$ and $A(K) \ge R(C')^2 + 1$. Also here the integral is independent of y. Therefore, for any $\xi \in \mathbb{R}^n$ we can choose a suitable $y = y_\xi$. For $\xi \in C'^*$ and $\|\xi\| \ge 1/R(C')$ we choose $y_\xi \in C'$ with $\|y_\xi\| = 1/\|\xi\|$. Then $y_\xi\cdot\xi \le 1$ and $0 \le y\cdot\xi$ for all $y \in C'$, $\|y\| \le R(C')$. For $\xi \in C'_*$ we have $\min_{y\in \mathrm{pr}\, C'} y\cdot\xi < 0$. Let the minimum be attained for y'_ξ, then we take $y_\xi = R(C')y'_\xi$ and we have

$$y_\xi\cdot\xi \le y\cdot\xi, \qquad \forall y \in C',\ \|y\| \le R(C').$$

Now we take $y = y_\xi$ in the integral in (3.7) and we find that for every C' there are a positive integer m(C'), constants $M_\alpha(C')$ and continuous functions $g_{\alpha,C'}$ on $\mathbb{R}^n$, $|\alpha| \le m(C')$, such that g can be represented as

$$(3.10)\qquad \begin{cases} g_\xi = \displaystyle\sum_{|\alpha|\le m(C')} D^\alpha\, g_{\alpha,C'}(\xi) \\[2ex] |g_{\alpha,C'}(\xi)| \le M_\alpha(C')(1+\|\xi\|)^k\, e^{y\cdot\xi}, \quad \forall y \in C',\quad \|y\| \le R(C'). \end{cases}$$

If $R(C') \le R$ for all $C' \subset\subset C$ and if in (3.9) m is independent of C', then in (3.10) m(C') and the functions $g_{\alpha,C'}$ can be chosen independent of C'.

The following lemma shows that f(z) satisfying (3.9) for $C' \subset\subset C$

satisfies (3.9) also for $C' \subset\subset ch(C)$, hence that $g \in S'$ satisfying (3.10) satisfies (3.10) also for $C' \subset\subset ch(C)$.

LEMMA 3.3. *Let* C *be an open cone in* $\mathbb{R}^n$, *let for each compact subcone* C' *of* C $R(C')$ *be a positive number and let* B *be an open set in* $\mathbb{R}^n$ *containing every set* $\{y \mid y \in \overline{C'},\ 0 < \|y\| \le R(C')\}$. *If* $f(z)$ *is a holomorphic function in* T^B *that satisfies* (3.1) *such that the limit* f^* *in* Z' *as* $y \to 0$ *belongs to* S', *then* $f(z)$ *attains this limit already in* S' *as* $y \to 0$, $y \in C' \subset\subset ch(C)$ *and, moreover,* f *satisfies* (3.9) *for each* $C' \subset\subset ch(C)$ *and* $R(C')$ *such that*

$$(3.11) \qquad \{y \mid y \in \overline{C'},\ 0 < \|y\| \le R(C')\} \subset ch(B).$$

If in (3.1) $m(K)$ *does not depend on* K, *then* $m(C')$ *in* (3.9) *does not depend on* C'.

PROOF. Fix $C' \subset\subset ch(C)$ and choose C'' and C''' with $C' \subset\subset C'' \subset\subset C''' \subset\subset ch(C)$. Fix $R(C''')$ such that (3.11) holds for $C' = C'''$. Let U_0 be an open ε-neighborhood of C''^*, then there is a $\delta > 0$ such that for $y \in C'$ $-y\cdot\xi \le -\delta\|y\|\,\|\xi\|$ if $\xi \in U_0$ outside a compact set. Finally choose finitely many vectors $y_j \in pr\ C'''$ and positive numbers $\delta_j < 1$ such that the open sets

$$U_j = \{\xi \mid y_j\cdot\xi < -\delta_j\|\xi\|\} \cap \{\xi \mid -y\cdot\xi < 2\delta_j\|\xi\|,\ y \in pr\ C'\},$$

$j = 1,\dots,p$, cover C''_*. Let $\{\lambda_j\}_{j=0}^p$ be a partition of unity subordinate to the covering $\bigcup_{j=0}^p U_j$ of $\mathbb{R}^n$, such that λ_j is a multiplier in S' for every j. Now for all $y \in C'$, $\|y\| \le 1/4\ R(C''')$ the functions

$$\lambda_j(\xi)\ e^{R(C''')y_j\cdot\xi - y\cdot\xi}, \qquad j = 0,\dots,p, \quad y_0 = 0,$$

belong to S. In lemma 5.2 it will be shown that the Fourier transforms of $\lambda_j(\xi)\ e^{-y\cdot\xi}\ g_\xi$, $j = 0,\dots,p$, where $g \in S'$ satisfies (3.5), are equal to

$$\langle e^{-R(C''')y_j\cdot\xi}\ g_\xi, \lambda_j(\xi)\ e^{R(C''')y_j\cdot\xi + iz\cdot\xi}\rangle_{S'}, \quad j = 0,1,\dots,p,$$

respectively. Hence if $g = F^{-1}[f^*]$, we obtain for $y \in C'$, $\|y\| \le 1/4\ R(C''')$

$$|f(z)| = |\mathcal{F}[e^{-y\cdot\xi}\, g_\xi]| = |\mathcal{F}[\sum_{j=0}^{p} \lambda_j(\xi)\, e^{-y\cdot\xi}\, g_\xi]| \le$$

$$\le |<g_\xi, \lambda_0(\xi)\, e^{iz\cdot\xi}>| + \sum_{j=1}^{p} |<e^{-R(C'')y_j\cdot\xi}\, g_\xi, \lambda_j(\xi) e^{R(C'')y_j\cdot\xi + iz\cdot\xi}>| \le$$

$$\le M_1(1+\|z\|)^m \sup_{t\ge 0}\, (1+t)^k\, e^{-\delta\|y\| t} + M_2(C'')(1+\|z\|)^{m'(C'')} \le$$

$$\le M(C')(1+\|x\|)^{m(C')}(1+\|y\|^{-k})$$

for some M_1, m and k depending on g and some $M_2(C'')$ depending on M(K) in (3.1) and $m'(C'')$ depending on m(K) in (3.1), where $K = \{y \mid y \in C'', \|y\| = R(C'')\}$. Together with (3.1) this yields (3.9) for $C' \subset\subset ch(C)$.

Furthermore, it now follows that g can be represented as in (3.10), hence that the set $\{e^{-y\cdot\xi} g_\xi \mid y \in C', \|y\| \le R(C')\}$ is bounded in S', where R(C') is such that (3.11) holds. Since the limit as $y \to 0$ of $e^{-y\cdot\xi}\, g_\xi$ exists in $\mathcal{D}'$, this limit exists in S' as $y \to 0$, $y \in C'$. Hence $f(x+iy) \to f^*$ in S' as $y \to 0$, $y \in C' \subset\subset ch(C)$. □

Altogether we have obtained the following theorem.

THEOREM 3.4. *Let* f *be a holomorphic function in* T^B, *satisfying* (3.1), *where* B *is an open set in* $\mathbb{R}^n$. *Then it satisfies* (3.2), *too. The limit* f^* *of* f(z) *as* $y \to 0$ *exists in* Z' *and its inverse Fourier transform* g *satisfies* (3.8) *for every* $K \subset\subset ch(B)$. *Conversely, a distribution* $g \in \mathcal{D}'$ *satisfying* (3.8) *for* $K \subset\subset B$ *satisfies* (3.8) *for* $K \subset\subset ch(B)$, *too and then the function* (3.6), *which is defined because* (3.5) *holds, satisfies* (3.2). *Moreover, if* (3.11) *holds, a function* f *holomorphic in* T^B, *has a boundary value* f^* *in* S', *provided that* f *satisfies* (3.9), *which then is satisfied for* $C' \subset\subset ch(C)$, *too. Then the inverse Fourier transform* g *of* f^* *satisfies* (3.10) *for every* $C' \subset\subset ch(C)$. *Conversely, a distribution* $g \in S'$ *satisfying* (3.10) *for* $C' \subset\subset C$ *satisfies* (3.10) *for* $C' \subset\subset ch(C)$, *too and then the function* (3.6), *which is defined because* (3.5) *holds, satisfies* (3.9) *for* $C' \subset\subset ch(C)$. *These is no mixture of these cases, i.e., a holomorphic function* f *in* T^B, B *such that* (3.11) *holds, satisfying* (3.1) *and having a boundary value* f^* *in* Z', *which is an element of* S', *satisfies already* (3.9).

We conclude this section with an example of a function f that satisfies (3.1). Let $B = \{y \mid y > 0\} \subset \mathbb{R}^1$ and let

$$f(z) = z^{\left(\frac{i}{\cos z} + \cos z\right)} .$$

For some positive constants A and B

$$|f(z)| \le M(y)(1+|x|)^{A/\sinh y + B\cosh y} \le$$

$$\le M(r,R)(1+|x|)^{A/r + B\cosh R}, \qquad 0 < r \le y \le R.$$

Let m(r,R) be an integer larger than A/r + B cosh R + 2, then for all $r \le y \le R$

$$g_\xi = e^{y\cdot\xi}\, F^{-1}[f(x+iy)]_\xi =$$

$$= (i\,D_\xi)^{m(r,R)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{z^{i/\cos z + \cos z}\, e^{-iz\xi}}{z^{m(r,R)}}\, dx =$$

$$= (D_\xi)^{m(r,R)}\, g_{r,R}(\xi) \quad \in \mathcal{D}'.$$

The continuous function $g_{r,R}(\xi)$ satisfies for all $r \le y \le R$

$$|g_{r,R}(\xi)| \le M'(r,R)\, e^{y\cdot\xi} ,$$

i.e.,

$$|g_{r,R}(\xi)| \le M'(r,R)\, e^{r\xi} , \qquad \xi > 0$$

$$|g_{r,R}(\xi)| \le M'(r,R)\, e^{-R|\xi|}, \qquad \xi < 0.$$

4. ANALYTIC REPRESENTATIONS OF REAL CARRIED ANALYTIC FUNCTIONALS AND OF THEIR FOURIER TRANSFORMS

In Section 3 we have discussed the behaviour of a function f(z) for $\|y\|$ small and we have seen that it does not change if we deal with a domain T^B or its convex hull $T^{ch(B)}$. In this section we will discuss it for $\|y\|$ large, namely we will consider holomorphic functions $f(z)$ in T^C of exponential type in $\|y\|$. Then $f(z)$ will not satisfy the same estimates for $y \in C$ or for $y \in ch(C)$, but they differ by a factor ρ_C in the exponent. The support of the inverse Fourier transform g of the boundary value f^* of such functions f is contained in a certain, convex set. We will obtain representations of analytic functionals μ in Z', which are carried by $\mathbb{R}^n$, as boundary values of holomorphic functions, and also characterizations and analytic representations of the inverse Fourier transforms $g \in \mathcal{D}'$ of μ. In particular this yields the case that μ and g both are tempered distributions, which case is treated in VLADIMIROV [9, 26.4 th.2].

Let f be a holomorphic function in T^C, where C is an open cone in $\mathbb{R}^n$. It is shown in VLADIMIROV [9, 26.6], that, if f satisfies for every $\sigma > 0$

$$|f(z)| \leq P(C',\sigma)(1+\|z\|)^m(1+\|y\|^{-k})\exp(b+\sigma)\|y\|, \qquad y \in C'$$

for some m, k and $b \geq 0$, then the distribution $g \in S'$ in (3.6) has its support in the set

$$(4.1) \qquad \{\xi \mid -y\cdot\xi \leq b,\ y \in \mathrm{pr}\, C\} = \{\xi \mid \mu_C(\xi) \leq b\}.$$

The proof does not depend on the behaviour of f for $\|y\|$ small, nor on the fact that m is independent of C', cf. (3.9). Hence instead of (3.9) f may satisfy (3.1) for $\|y\|$ small, so that (3.6) holds for some $g \in \mathcal{D}'$. Then the same proof as in VLADIMIROV [9] combined with theorem 3.4 shows that the following theorem is true.

<u>THEOREM 4.1</u>. *Let the holomorphic function* f *in* T^C *satisfy for every* $r > 0$, $\sigma > 0$ *and* $C' \subset\subset C$

$$(4.2) \qquad |f(z)| \leq P(C',r,\sigma)(1+\|z\|)^{m(C',r)} \exp(b+\sigma)\|y\|, \quad y \in C',\ \|y\| \geq r$$

for some $m(C',r)$ *depending on* C' *and* r, *some* $b \geq 0$ *and some constant* $P(C',r,\sigma)$ *depending on* C',r *and* σ. *Then* $f(z) = \mathcal{F}[e^{-y\cdot\xi} g_\xi](x)$ *and* $\lim_{y\to 0} f(x+iy) = \mathcal{F}[g]$ *in* Z' *for some* $g \in \mathcal{D}'$ *with support in the set* (4.1).

The converse of this theorem follows from theorem 3.4 and the proof of [9, 26.4,th.2]. We consider distributions g in $\mathcal{D}'$, which are represented as sum of weak derivatives of measures satisfying a condition like (3.8). That any distribution in $\mathcal{D}'$, satisfying (3.5) with B = C and with support in a convex set, can be represented in this way, will be shown in the next section. The advantage of writing g as sum of derivatives of measures is that it enables us to let the σ vanish in (4.2), if C is convex.

THEOREM 4.2. *Let* g *be a distribution in* $\mathcal{D}'$, *such that for each* $C' \subset\subset C$ *and* $r > 0$ g *can be represented as*

$$g_\xi = \sum_{|\alpha| \le m(C',r)} D^\alpha \mu_{\alpha,C',r}(\xi),$$

where the measures $\mu_{\alpha,C',r}$ *depending on* α, C' *and* r *have their support in the set* (4.1) *and satisfy*

$$(4.3) \qquad \int_{\mathbb{R}^n} e^{-y\cdot\xi} \, |d\mu_{\alpha,C',r}(\xi)| \le M_\alpha(C',r), \qquad \forall y \in C', \quad \|y\| \ge r$$

for some positive integers m(C',r) *depending on* C' *and* r *and for some positive constants* $M_\alpha(C',r)$ *depending on* α, C' *and* r. *Then for* $y \in \mathrm{ch}(C)$ *the function* $f(z) = \mathcal{F}[e^{-y\cdot\xi} g_\xi](x)$ *is holomorphic in* $T^{\mathrm{ch}(C)}$ *and satisfies for each* $C' \subset\subset \mathrm{ch}(C)$ *and* $r > 0$

$$(4.4) \qquad |f(z)| \le P(C',r)(1+\|z\|)^{N(C',r)} \exp \rho_C \, b\|y\|, \quad y \in C', \quad \|y\| \ge r$$

for some positive integers N(C',r) *and constants* P(C',r) *both depending on* C' *and* r. *Moreover,* $\lim_{y\to 0} f(x+iy) = \mathcal{F}[g]$ *in* Z' *and if* m(C',r) *is independent of* C' *and* r, *then* N(C',r), *too.*

Since the distribution g in theorem 4.1 satisfies the conditions of theorem 4.2 (cf. section 3 and the next section) and since $\rho_C = 1$ if C is convex, we obtain the following corollary.

COROLLARY 4.3. *Let the cone* C *be convex, then a function* f *that satisfies* (4.2) *satisfies* (4.4) *with* $\rho_C = 1$, *i.e., in* (4.2) P *is actually independent of* σ.

We now give a characterization of all distributions $g \in \mathcal{D}'$ whose Fourier transforms $\mu \in Z'$ are carried by $\mathbb{R}^n$ (see section 2). We have already seen that, if $\mu \in Z'$ is the boundary value of a holomorphic function, it is carried by $\mathbb{R}^n$. Hence it remains to characterize those $g \in \mathcal{D}'$ whose Fourier transforms admitt an analytic representation and to show that for every $\mu \in Z'$ which is carried by $\mathbb{R}^n$ $F^{-1}[\mu]$ satisfies this characterization. Moreover, this yields an analytic representation of elements in Z' carried by $\mathbb{R}^n$.

If a closed, convex, cone C^* in $\mathbb{R}^n$ is the dual of an open cone C in $\mathbb{R}^n$, it does not contain a straight line. We devide $\mathbb{R}^n$ into such cones so that the following properties hold:

$$(4.5) \qquad \mathbb{R}^n = \bigcup_{j=1}^{p} C_j^*,$$

$$(4.6) \qquad \operatorname{int} C_j^* \cap \operatorname{int} C_k^* = \emptyset, \qquad j \neq k,$$

where each C_j^* is a closed, convex, cone not containing a straight line, while the union of any two cones C_j^* contains a straight line. The last property restricts the number p of cones used, $n+1 \le p \le 2^n$, and furthermore it states that in some sense the cones C_j^* are as large as possible. Let C_j^* be the dual of the open convex cone C_j, then $C = \bigcup_{j=1}^{p} C_j$ is an open cone. Such a cone C can also be obtained directly as follows: let r open half spaces V_k ($V_k = \{y \mid \xi_k \cdot y > 0\}$ for some $\xi_k \in \mathbb{R}^n$, $\|\xi_k\| = 1$) be given such that $\mathbb{R}^n \setminus \{0\} = \bigcup_{k=1}^{r} V_k$, while $\mathbb{R}^n \setminus \{0\}$ is not covered by the union of any r-1 half spaces V_k, hence $n+1 \le r \le 2n$. Then each C_j is the intersection of n half spaces V_k, i.e.,

$$C = \cup\{V_{k_0} \cap \ldots \cap V_{k_{n-1}}\},$$

where the union is taken over all n-tuples $\{k_0,\ldots,k_{n-1}\}$ taken from $\{1,2,\ldots,r\}$. For example, if n=2 we may take

$$C = \{y \mid 0 < \phi < 1/3\pi\} \cup \{y \mid 2/3\pi < \phi < \pi\} \cup \{y \mid 4/3\pi < \phi < 5/3\pi\},$$

where $\phi = \operatorname{tg} y_2/y_1$, or

$$C = \{y \mid 0 < \phi < 1/2\pi\} \cup \{y \mid 1/2\pi < \phi < \pi\} \cup \{y \mid \pi < \phi < 3/2\pi\} \cup$$

$$\cup \{y \mid 3/2\pi < \phi < 2\pi\}.$$

If we write $C' \subset\subset C$, we mean $C' = \bigcup_{j=1}^{p} C'_j$ with $C'_j \subset\subset C_j$. Furthermore, let λ_j be the characteristic function of the set C^*_j.

Let $g \in \mathcal{D}'$ be such that for any $C' \subset\subset C$ and any $\varepsilon > 0$ g can be represented as

$$g_\xi = \sum_{|\alpha| \le m(C',\varepsilon)} D^\alpha g_{\alpha,C',\varepsilon}(\xi)$$

where the continuous functions $g_{\alpha,C',\varepsilon}$ satisfy

$$|\lambda_j(\xi)\ g_{\alpha,C',\varepsilon}(\xi)| \le M_{\alpha,j}(C',\varepsilon)\ e^{y\cdot\xi}, \qquad \forall y \in C'_j,\ \|y\| \ge \varepsilon, \tag{4.7}$$

cf. (4.3). Then by (4.5) and (4.6) for $\phi \in \mathcal{D}$ we have

$$<g,\phi> = \sum_{|\alpha| \le m(C',\varepsilon)} (-1)^{|\alpha|} \int g_{\alpha,C',\varepsilon}(\xi)\ D^\alpha\ \phi(\xi)d\xi =$$

$$= \sum_{|\alpha| \le m(C',\varepsilon)} \sum_{j=1}^{p} (-1)^{|\alpha|} \int \lambda_j(\xi)\ g_{\alpha,C',\varepsilon}(\xi)\ D^\alpha\ \phi(\xi)d\xi,$$

hence

$$g_\xi = \sum_{j=1}^{p} \sum_{|\alpha| \le m(C',\varepsilon)} D^\alpha\ \lambda_j(\xi) g_{\alpha,C',\varepsilon}(\xi). \tag{4.8}$$

It follows from theorem 4.2 that $F[g]$ is the sum of boundary values in Z' of functions f_j, holomorphic in T^{C_j}, satisfying for all $C'_j \subset\subset C_j$ and $\varepsilon > 0$

$$|f_j(z)| \le P(C'_j,\varepsilon)(1+\|z\|)^{N(C'_j,\varepsilon)}, \qquad y \in C'_j,\ \|y\| \ge \varepsilon,$$

for $j = 1,\ldots,p$, respectively. Hence $F[g]$ is an element of Z' carried by $\mathbb{R}^n$. Since for any $C' \subset\subset C$ there is a positive number $\delta(C') < 1$ such that for $y \in C'_j$ and $\xi \in C^*_j$, $j = 1,\ldots,p$,

$$(4.9) \qquad \delta(C')\|y\|\|\xi\| \le y\cdot\xi \le \|y\|\|\xi\|,$$

condition (4.7) is equivalent to: for every $\varepsilon > 0$ g can be written as

$$g_\xi = \sum_{|\alpha|\le m(\varepsilon)} D^\alpha g_{\alpha,\varepsilon}(\xi),$$

where the continuous functions $g_{\alpha,\varepsilon}$ on $\mathbb{R}^n$ depend on ε and satisfy

$$(4.10) \qquad |g_{\alpha,\varepsilon}(\xi)| \le M_\alpha(\varepsilon)\, e^{\varepsilon\|\xi\|}.$$

Next consider an element μ of Z' carried by $\mathbb{R}^n$. As in section 2, for any $\varepsilon > 0$ μ can be represented as a measure μ_ε on an ε-neighborhood $\Omega(\varepsilon)$ of $\mathbb{R}^n$ in $\mathbb{C}^n$ satisfying (2.4). Let μ be the Fourier transform of a distribution $g \in \mathcal{D}'$. Then for any positive $\varepsilon < 1$ and $\phi \in \mathcal{D}$

$$<g,\phi> = \int_{\Omega(\varepsilon)} \frac{1}{(2\pi)^n} \int e^{-iz\cdot\xi}\, \phi(\xi)d\xi\, d\mu_\varepsilon(z) =$$

$$= \int_{\Omega(\varepsilon)} \frac{1}{(2\pi)^n} \int \phi(\xi)\, (1-\vec{D}_\xi\cdot\vec{D}_\xi)^{m(\varepsilon)}\, e^{-iz\cdot\xi}\, d\xi\, \frac{d\mu_\varepsilon(z)}{(1+z\cdot z)^{m(\varepsilon)}} =$$

$$= \int \{(1-\vec{D}_\xi\cdot\vec{D}_\xi)^{m(\varepsilon)}\phi(\xi)\} \int_{\Omega(\varepsilon)} \frac{e^{-iz\cdot\xi}}{(2\pi)^n(1+z\cdot z)^{m(\varepsilon)}}\, d\mu_\varepsilon(z)\, d\xi,$$

hence $g_\xi = (1-\vec{D}\cdot\vec{D})^{m(\varepsilon)}\, g_\varepsilon(\xi)$, where g_ε is a continuous function on $\mathbb{R}^n$ which satisfies

$$|g_\varepsilon(\xi)| \le \frac{1}{(2\pi)^n} \frac{1}{(1-\varepsilon^2)^{m(\varepsilon)}} K_\varepsilon\, e^{\varepsilon\|\xi\|},$$

i.e., g satisfies (4.10).

In the following theorem not only $F[g]$ but also g is represented as sum of boundary values of holomorphic functions.

THEOREM 4.4. *The following four conditions for a distribution* $g \in \mathcal{D}'$ *are equivalent:*

(1) $F[g] \in Z'$ *is carried by* $\mathbb{R}^n$.

(2) *For any* $\varepsilon > 0$ g *can be represented as* $g_\xi = \sum_{|\alpha| \le m(\varepsilon)} D^\alpha g_{\alpha,\varepsilon}(\xi)$, *where* $g_{\alpha,\varepsilon}$ *are continuous functions on* $\mathbb{R}^n$ *satisfying*

$$|g_{\alpha,\varepsilon}(\xi)| \le M_\alpha(\varepsilon)\, e^{\varepsilon\|\xi\|}.$$

(3) $F[g]$ *is the sum of boundary values in* Z' *of functions* f_j *holomorphic in* T^{C_j} *satisfying for any* $C_j' \subset\subset C_j$ *and any* $\varepsilon > 0$

$$|f_j(z)| \le P(C_j',\varepsilon)(1+\|z\|)^{N(C_j',\varepsilon)}, \qquad y \in C_j',\ \|y\| \ge \varepsilon$$

for $j = 1,\dots,p$, *respectively, where* C_j, $j = 1,\dots,p$, *are any cones satisfying* (4.5) *and* (4.6) *and where* $P(C_j',\varepsilon)$ *and* $N(C_j',\varepsilon)$ *are constants depending on* C_j' *and* ε.

(4) g *is the sum of boundary values in* $\mathcal{D}'$ *of holomorphic functions* h_k *in* $T^{\tilde{C}_k}$ *satisfying for any* $\tilde{C}_k' \subset\subset \tilde{C}_k$ *and any* $\varepsilon > 0$

$$|h_k(\zeta)| \le M(\tilde{C}_k',\varepsilon)(1+\|\eta\|^{-m(\varepsilon)})\, e^{\varepsilon\|\xi\|}, \qquad \eta \in \tilde{C}_k'$$

for $k = 1,\dots,\tilde{p}$, *respectively, where* $\tilde{C}_k$, $k = 1,\dots,\tilde{p}$, *are any cones satisfying* (4.5) *and* (4.6), *where* $M(\tilde{C}_k',\varepsilon)$ *depends on* $\tilde{C}_k'$ *and* ε, *and where* $m(\varepsilon)$ *depends on* ε *only.*

PROOF. We only have to prove that (4) is equivalent to one of the equivalent conditions (1), (2) or (3). As in VLADIMIROV [9,th.26.3] it follows that (4) implies (2). Now let $g \in \mathcal{D}'$ satisfy (3) and let $\tilde{C} = \cup_{k=1}^{\tilde{p}} \tilde{C}_k$ be a cone satisfying (4.5) and (4.6). In what follows $C' \subset\subset C$ in (3) is fixed, but ε is variable. Let us write $z_j = x + iy_j$ if $y_j \in C_j$ and $\zeta_k = \xi + i\eta_k$ if $\eta_k \in \tilde{C}_k$. Since all the integrals exist and Lebesgue's dominated convergence theorem may be applied, for $\phi \in \mathcal{D}$, $N \ge \frac{1}{2}\{N(C',\varepsilon) + n+1\}$ and $y_j \in C_j'$, $\|y_j\| \ge \varepsilon$, $j = 1,\dots,p$, we get

$$\langle g,\phi\rangle = \sum_{j=1}^{p} \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{f_j(z_j)e^{-iz_j\cdot\xi}}{(z_j\cdot z_j)^N}\, dx\,(-\vec{D}\cdot\vec{D})^N\, \phi(\xi)d\xi =$$

$$= \sum_{j=1}^{p} \int_{\mathbb{R}^n} \sum_{k=1}^{\tilde{p}} \frac{1}{(2\pi)^n} \int_{-\tilde{C}_k^*} \frac{f_j(z_j)e^{-iz_j\cdot\xi}}{(z_j\cdot z_j)^N} dx\ (-\vec{D}\cdot\vec{D})^N\ \phi(\xi)d\xi =$$

$$= \sum_{k=1}^{\tilde{p}} \int_{\mathbb{R}^n} \lim_{\substack{\eta_k\to 0\\ \eta_k\in\tilde{C}_k'}} \sum_{j=1}^{p} \frac{1}{(2\pi)^n} \int_{-\tilde{C}_k^*} \frac{f_j(z_j)e^{-iz_j\cdot\zeta_k}}{(z_j\cdot z_j)^N} dx\ (-\vec{D}\cdot\vec{D})^N\ \phi(\xi)d\xi =$$

$$= \sum_{k=1}^{\tilde{p}} \lim_{\substack{\eta_k\to 0\\ \eta_k\in\tilde{C}_k'}} \int_{\mathbb{R}^n} \sum_{j=1}^{p} \frac{1}{(2\pi)^n} \int_{-\tilde{C}_k^*} \frac{f_j(z_j)e^{-iz_j\cdot\zeta_k}}{(z_j\cdot z_j)^N} dx\ (-\vec{D}\cdot\vec{D})^N\ \phi(\xi)d\xi =$$

$$= \sum_{k=1}^{\tilde{p}} \lim_{\substack{\eta_k\to 0\\ \eta_k\in\tilde{C}_k'}} \int_{\mathbb{R}^n} \sum_{j=1}^{p} \frac{1}{(2\pi)^n} \int_{-\tilde{C}_k^*} f_j(z_j)e^{-iz_j\cdot\zeta_k} dx\ \phi(\xi)\ d\xi,$$

where $\tilde{C}_k' \subset\subset \tilde{C}_k$ for $k = 1,\ldots,\tilde{p}$. The function

$$h_k(\zeta) = \sum_{j=1}^{p} \frac{1}{(2\pi)^n} \int_{-\tilde{C}_k^*} f_j(z_j)e^{-iz_j\cdot\zeta}\ dx \tag{4.11}$$

is holomorphic in $\mathbb{R}^n + i\tilde{C}_k$ and the terms in the sum are independent of y_j, respectively. Furthermore, by taking $y_j \in C_j'$ with $\|y_j\| = \varepsilon$ in (4.11), we see that for any $\varepsilon > 0$ and any $\tilde{C}_k' \subset\subset \tilde{C}_k$ h_k satisfies

$$|h_k(\zeta)| \le$$

$$\le \frac{p}{(2\pi)^n} P(C',\varepsilon)(1+\varepsilon)^{N(C',\varepsilon)} \int_{-\tilde{C}_k^*} \frac{(1+\|x\|)^{N(C',\varepsilon)+n+1}\ e^{\varepsilon\|\xi\| - \delta(\tilde{C}_k')\|x\|\,\|\eta\|}}{(1+\|x\|)^{n+1}} dx \le$$

$$\le M(\tilde{C}_k',\varepsilon)(1+\|\eta\|^{-m(\varepsilon)})e^{\varepsilon\|\xi\|}, \qquad \eta\in\tilde{C}_k',$$

where $m(\varepsilon) = N(C',\varepsilon) + n+1$. Hence g satisfies condition (4). □

COROLLARY 4.5. *Any analytic functional* μ in Z' *carried by* $\mathbb{R}^n$ *is the sum of boundary values in* Z' *of functions* f_j *satisfying* (3) *in theorem* 4.4.

REMARK 4.6. The above mentioned analytic representations of μ and g are in two ways not unique. Firstly, there is an arbitrariness in the number (between n+1 and 2^n) and the choice of the cones C_j, $j = 1,\dots,p$, or $\tilde{C}_k$, $k = 1,\dots,\tilde{p}$, as long as they satisfy (4.5) and (4.6). Secondly, if the cones $C = \cup_{j=1}^p C_j$ and $\tilde{C}$ are fixed, the functions f in T^C (i.e., $f = f_j$ in T^{C_j}) and h in $T^{\tilde{C}}$ are not unique. For example, instead of (4.8) g may just as well be written as

$$g_\xi = \sum_{j=1}^{p} \left\{ \sum_{|\alpha|\leq m(\varepsilon)} D^\alpha \lambda_j(\xi)\, g_{\alpha,\varepsilon}(\xi) + \sum_{k=1}^{p} h^{jk}{}_\xi \right\}$$

with $g_{\alpha,\varepsilon}$ satisfying (4.10), where h^{jk} are arbitrary distributions with support in $C_j^* \cap C_k^*$ with the restrictions that $h^{jk} = -h^{kj}$ and that for each $\varepsilon > 0$ h^{jk} can be represented as sum of weak derivatives up to order $m(\varepsilon)$ of measures $\mu^{jk}_{\alpha,\varepsilon}(\xi)$ on $C_j^* \cap C_k^*$ satisfying $\int e^{-\varepsilon\|\xi\|} |d\mu^{jk}_{\alpha,\varepsilon}(\xi)| \leq M_\alpha(\varepsilon)$. Now $\mathcal{F}[h^{jk}]$ is the boundary value of a function holomorphic in $\mathbb{R}^n + i\,\mathrm{ch}\,(C_j \cup C_k)$. Hence f' is another representation of μ if its difference with f satisfies

$$f(z) - f'(z) = \sum_{k=1}^{p} F_{jk}(z), \qquad y \in C_j$$

for $j = 1,\dots,p$, where F_{jk} are arbitrary functions, holomorphic in $\mathbb{R}^n + i\,\mathrm{ch}\,(C_j \cup C_k)$ satisfying

$$|F_{jk}(z)| \leq P(C_j',C_k',\varepsilon)(1+\|z\|)^{N(C_j',C_k',\varepsilon)}, \quad y \in \mathrm{ch}\,(C_j' \cup C_k'),\ \|y\| \geq \varepsilon,$$

such that $F_{jk} = -F_{jk}$. Or, when $C = \cup\{V_{k_0} \cap \dots \cap V_{k_{n-1}}\}$, $\{f_{k_0 \dots k_{n-1}}\}$ and $\{f'_{k_0 \dots k_{n-1}}\}$ represent the same μ if

$$f_{k_0\dots k_{n-1}} - f'_{k_0\dots k_{n-1}} = \sum_{j=0}^{n-1} (-1)^j F_{k_0\dots\hat{k}_j\dots k_{n-1}},$$

where $\hat{k}_j$ denotes that the index k_j is omitted, for arbitrary functions $F_{i_0\dots i_{n-2}}$ holomorphic in $\mathbb{R}^n + i\,\{V_{i_0} \cap \dots \cap V_{i_{n-2}}\}$ satisfying estimates as above, provided that f, f' and F are antisymmetric in their indices. Here μ is represented by the $\binom{r}{n}$ functions

$$\{f_{k_0\ldots k_{n-1}}\}_{\{k_0,\ldots,k_{n-1}\} \in \{1,\ldots,r\}}$$

if the order $k_0\ldots k_{n-1}$ in the n-tuples $\{k_0,\ldots,k_{n-1}\}$ is such that

$$\sum_{\{k_0,\ldots,k_{n-1}\} \in \{1,\ldots,r\}} \sum_{j=0}^{n-1} (-1)^j \, [k_0\ldots\hat{k}_j\ldots k_{n-1}] = 0$$

where $[i_0\ldots i_{n-2}]$ denotes the rearrangement $i'_0\ldots i'_{n-2}$ of $i_0\ldots i_{n-2}$, such that $i'_0 < \ldots < i'_{n-2}$, preceded by the sign of the permutation $i_0\ldots i_{n-2} \to i'_0\ldots i'_{n-2}$. For example, if $r = n+1$

$$\mu = \sum_{j=1}^{n+1} (-1)^{j+1} f^*_{1\ldots\hat{j}\ldots n+1}$$

and if $r = 2^n$, $C_{\varepsilon_1\ldots\varepsilon_n} = \bigcap_{j=1}^{n} \{y \mid \varepsilon_j y_j > 0\}$ where $\varepsilon_j = \pm 1$, then

$$\mu = \sum_{\varepsilon_j \in \{-1,1\}, j=1,\ldots,n} \varepsilon_1\ldots\varepsilon_n \, f^*_{\varepsilon_1\ldots\varepsilon_n} ,$$

cf. MARTINEAU [5]. The representations equivalent to $h \in T^{\tilde{C}}$ are obtained similarly.

REMARK 4.7. At a first glance theorems 4.1, 4.2 and 4.4 (2) and (3) (with (4.7) instead of (4.10)) resemble theorems 10, 11 and 12 in section 6 of CARMICHAEL [1]. There too, on the one hand holomorphic functions in T^C are considered satisfying a condition similar to (4.2), namely with N constant and with P independent of r and σ in [1, formula (34)] as well as dependent on r in [1, last formula on p.753 and formula (54), hence (49)] and on the other hand distributions U in $\mathcal{D}'$ satisfying conditions similar to ours (4.3) and (4.7), but with M_α independent of C' and r in [1, formula (48) and the last formula on p.756] as well as dependent on C' in [1, formula (47)]. Therefore, the functions g_α in [1] are bounded or even identically zero (the function g_k in [1, last formula on p.758]). In [1, theorems 10,11 and 12] finite Fourier transforms of elements $U \in \mathcal{D}'(A)$

with support in a certain, unbounded, convex set are represented as boundary values in $Z'(\overline{2\pi})$ of certain holomorphic functions. Furthermore, the distributions U are represented as weak derivatives of continuous functions on $\mathbb{R}^n$, so that also $U \in \mathcal{D}'$. However, the support of U as element of $\mathcal{D}'(A)$ is not the same as the support of U as element of $\mathcal{D}'$. According to the definition of support (see section 2) any element $U \in \mathcal{D}'(A)$ has compact support, hence its (finite) Fourier transform is the boundary value of an entire function, which result indeed is obtained in [1, theorem 1], cf. [4,III §2.3]. It turns out that the proofs of theorems 10, 11 and 12 in [1] yield a stronger result than the statements, namely they give the analytic representation in Z' of the ordinary Fourier transform of U, where U is regarded as an element of $\mathcal{D}'$. In this form [1, theorems 10,11 and 12] resemble our theorems 4.1, 4.2 and 4.4 (2) and (3), but although nowhere mentioned explicitely, the boundary values in [1] are always attained in S', too, and actually the theorems in [1, section 6] are particular cases of the theorems in VLADIMIROV [9]. Only the one dimensional "corollary" to theorem 10 on [1, p.753-754] shows that boundary values in Z' are really intended. Furthermore, there is one more difference between [1, section 6] and this paper, namely, before taking the Fourier transform in [1, theorems 11 and 12] U is reflected to $\check{U}$. This is due to the fact that the definition of Fourier transformation in [1], the one of [4], has not been "motivated by a Parseval relation". For, defining $\mathcal{F}[U]$ by requiring $\langle \mathcal{F}[U], \mathcal{F}[\phi] \rangle_{Z'} =$ $= (2\pi)^n \langle U, \phi \rangle_{\mathcal{D}'}$, $\phi \in \mathcal{D}$, one should take the complex conjugate of f in (2.1) in order to get a Parseval relation.

EXAMPLE 1

Any $g \in S'$ satisfies the conditions of theorem 4.4. Then the functions f_j in (3) of theorem 4.4 satisfy

$$|f_j(z)| \leq P(C')(1+\|z\|)^N(1+\|y\|^{-k}), \qquad y \in C'_j$$

for some N and k, cf. corollary 5.4, and a similar estimate holds for the functions h_k in (4).

EXAMPLE 2

μ given by

$$\langle\mu,\psi\rangle = \oint \psi(z) \exp \frac{1}{z}\, dz$$

is an element of Z' carried by the origin. Its analytic representation is

$$\mu = \lim_{y\downarrow 0} \{\exp \frac{1}{x-iy} - \exp \frac{1}{x+iy}\} ,$$

which is unique modulo polynomials. The inverse Fourier transform $\frac{1}{2\pi} \oint \exp(1/z-iz\xi)dz$ is a function in $\mathbb{R}^n$, which can be continued to an entire function g with for every $\varepsilon > 0$

$$|g(\zeta)| \leq M(\varepsilon)\, e^{\varepsilon\|\zeta\|} .$$

Here $F^{-1}[\mu]$ does not belong to S'.

EXAMPLE 3

μ represented as

$$\mu = \lim_{y\downarrow 0} \{z^{i/\cos z} - \bar{z}^{i/\cos \bar{z}}\}$$

is an element of Z' carried by $(-\infty,0] \cup \{\frac{1}{2}\pi\} \cup \{\frac{3}{2}\pi\} \cup \ldots$ and its inverse Fourier transform is a finite order distribution g in $\mathcal{D}'$, which does not belong to S'; for any $\varepsilon > 0$ g can be represented as in theorem 4.4 (2) (cf. the example at the end of section 3) or as sum of boundary values in $\mathcal{D}'$ as $\eta \to 0$, $\eta \in \tilde{C}_k' \subset\subset \tilde{C}_k$, of holomorphic functions h_k given by (4.11).

5. FOURIER TRANSFORMATION AS A TOPOLOGICAL ISOMORPHISM

In this section we topologize the space of distributions $g \in \mathcal{D}'$ satisfying (4.3) and the space of analytic functions f satisfying (4.2), such that the Fourier transformation F in theorems 4.1 and 4.2 is a topological isomorphism. We also prove a representation theorem of such functions f and for these functions lemma 3.3 can be improved. In DE ROEVER [6] the theorems of this section are used to derive the Fourier transformation between functions f of exponential type in both $\|x\|$ and $\|y\|$, holomorphic in tubular radial domains, and certain spaces of analytic functionals with unbounded, convex, carrier. In general, these functions f do not have distributional boundary values on the distinguished boundary, but as a particular case the analytic representation is obtained of distributions in $\mathcal{D}'$ being the Fourier transform of elements in Z' carried by certain, unbounded, convex sets in $\mathbb{C}^n$. In this form the theorems of [6] are opposite to the theorems of this section.

First we remark that the correspondence between the exponential type $b\|y\|$ of f in theorem 4.1 and the set (4.1) $\{\xi \mid -y\cdot\xi \le b,\ y \in \text{pr}\, C\}$ can be generalized to exponential types varying with the direction of y and arbitrary convex sets. Let b be a convex function of $y \in C$ homogeneous of degree one, where C is an open convex cone in $\mathbb{R}^n$. This means that b(y) is determined by its value on pr C

$$b(y) = \|y\|\ b\left(\frac{y}{\|y\|}\right).$$

The convex open cone C and the convex homogeneous function b on C determine a closed convex set U in $\mathbb{R}^n$ by

$$(5.1)\qquad U \overset{\text{not}}{=\!=} U(b,C) = \{\xi \mid -y\cdot\xi \le b(y),\ y \in C\}.$$

If b can be continuously continued to pr $\overline{C}$, then C and $\overline{C}$ determine the same convex set U(b,C).

Conversely, each closed convex set U in $\mathbb{R}^n$ determines an open (possibly in some linear subspace of $\mathbb{R}^n$), convex cone C in $\mathbb{R}^n$ and a convex, homogeneous function b on C by: let for $y \in \mathbb{R}^n$ and for some real number α $H(y,\alpha)$ be the affine halfspace in $\mathbb{R}^n$

$$H(y,\alpha) = \{\xi \mid -y\cdot\xi \le \alpha\};$$

then C is the interior (possibly in some linear subspace of $\mathbb{R}^n$) of the set of all $y \in \mathbb{R}^n$ such that $U \subset H(y,\alpha)$ for a real number α depending on y and

$$(5.2) \qquad b(y) = \sup_{\xi \in U} -y \cdot \xi .$$

C is open in $\mathbb{R}^n$ (hence, C is not contained in a proper linear subspace) if and only if U does not contain a straight line. Note that b(y) might not be positive for all $y \in \mathrm{pr}\, C$, in which case U determined by (5.1) does not contain the origin.

Secondly we discuss representations of distributions g with support in a closed set U as sum of weak derivatives of measures. For arbitrary sets U such a representation is not always possible, because U has to satisfy certain properties, see WHITNEY [11] and more generally SCHWARTZ [7] or VLADIMIROV [9] and [10]. Here we only need that it is sufficient if U is the closure of a convex, open set. It is shown in WHITNEY [12] that a C^∞-function ϕ on the closed convex set U (see WHITNEY [11]), whose derivatives are uniformly continuous and bounded on U, can be extended to a C^∞-function on an ε-neighborhood of U, which is bounded there. Hence ϕ can be extended to a C^∞-function $\tilde{\phi}$ on $\mathbb{R}^n$ which, together with its derivatives, is bounded. Moreover, it follows from the construction of $\tilde{\phi}$ in [12] that, if $D^\alpha \phi(\xi) \to 0$ as $\xi \to \infty$ in U (hence then $D^\alpha \phi$ is uniformly continuous in U), this also holds for $\tilde{\phi}$ as $\xi \to \infty$ in $\mathbb{R}^n$.

Let $\alpha(m)$ and $\beta(m)$ be two non-decreasing sequences, where for every m at least one of the inequalities $\alpha(m+1) > \alpha(m)$ or $\beta(m+1) > \beta(m)$ holds and let $M_m(\xi)$ be a positive C^∞-function on $\mathbb{R}^n$ which outside the unit ball equals

$$(1+\|\xi\|)^{\alpha(m)} \, e^{\beta(m)\|\xi\|} .$$

For any closed set V the norms

$$\sup_{\substack{\xi \in V \\ |\alpha| \le m}} |D^\alpha M_m(\xi)\phi(\xi)|$$

and

$$\sup_{\substack{\xi \in V \\ |\alpha| \le m}} M_m(\xi) \, |D^\alpha \phi(\xi)|$$

are equivalent and also it is equivalent to assert

$$\lim_{\substack{\xi\to\infty \\ \xi\in V}} D^{\alpha} M_m(\xi)\phi(\xi) = 0, \qquad |\alpha| \le m$$

or

$$\lim_{\substack{\xi\to\infty \\ \xi\in V}} M_m(\xi)\, D^{\alpha}\, \phi(\xi) = 0, \quad |\alpha| \le m.$$

Let us denote by $W^m_{\infty,0}\,(M_m;V)$ the Banach space of C^m-functions ϕ on the closure V of an open, convex set (in the sense of WHITNEY [11]) with the norm

$$\|\phi\|_m = \sup_{\substack{\xi\in V \\ |\alpha|\le m}} M_m(\xi)\ |D^{\alpha}\ \phi(\xi)|$$

and with

$$\lim_{\substack{\xi\to\infty \\ \xi\in V}} M_m(\xi)\, D^{\alpha}\, \phi(\xi) = 0, \qquad |\alpha| \le m,$$

cf. WLOKA [13] and by W(V) the Fréchet space

$$W(V) = \operatorname{proj}\lim_{m\to\infty} W^m_{\infty,0}\ (M_m;V).$$

Then the restriction map I from $W(\mathbb{R}^n)$ into W(U) is surjective, cf. VLADIMIROV [10] in case $\beta(m) \equiv 0$. According to TREVES [8, th.37.2] the transposed map I' from the dual W(U)' of W(U) into the dual $W(\mathbb{R}^n)'$ of $W(\mathbb{R}^n)$ is injective and has weakly closed range. Therefore, W(U)' can be identified (by means of I') with the subspace W'_U of $W(\mathbb{R}^n)'$ consisting of the elements with support in U. Indeed, W'_U, by the definition of support (see section 2) vanishing on the space of all $\phi \in W(\mathbb{R}^n)$ with support in U^c, also vanishes on the closure of this space, which is just Ker I. Then according to TREVES [8, prop. 35.4] W'_U is the weak closure of Im I' and since this is already weakly closed $I'(W(U)') = W'_U$.

Furthermore, we may conclude that W(U)' is a closed linear subspace of $W(\mathbb{R}^n)'$. For, it follows from WLOKA [13] that the identity map from $W^{m+1}_{\infty,0}(M_{m+1};\mathbb{R}^n)$ into $W^m_{\infty,0}(M_m;\mathbb{R}^n)$ is compact, hence that $W(\mathbb{R}^n)$ is an $F\bar{S}$-space, see FLORET & WLOKA [3] ($W(\mathbb{R}^n)$ is even a nuclear $F\bar{S}$-space). Hence $W(\mathbb{R}^n)'$ can be written as inductive limit (LS-space) and it is reflexive. Therefore, Im I' is even strongly closed. Now the following natural embedding maps are bijective and continuous

$$\operatorname{ind}\lim_{m\to\infty} W^m_{\infty,0}(M_m;U)' \longrightarrow W(U)' \xrightarrow{I'} W'_U \subset W(\mathbb{R}^n)'.$$

By a property of LS-spaces (FLORET & WLOKA [3,25.1]) the closed sets of the first space are closed in W'_U, where W'_U = Im I' is regarded as a closed subspace of $W(\mathbb{R}^n)'$. Therefore, the three spaces are equal also as topological spaces. Thus W(U)' is an LS-space, hence W(U) is an F$\bar{S}$-space, and W(U)' is a closed linear subspace of $W(\mathbb{R}^n)'$.

Finally, by Riesz' representation theorem distributions $g \in W(U)'$ can be represented as sum of weak derivatives of measures μ_α on U such that

$$g = \sum_{|\alpha|\le m} D^\alpha \mu_\alpha,$$

$$\int_U \frac{|d\mu_\alpha(\xi)|}{M_k(\xi)} < \infty, \qquad |\alpha| \le m,$$

where m and k depend on g. For this reason we required that $M_m(\xi)\, D^\alpha\, \phi(\xi) \to 0$ as $\xi \to \infty$ in U, for, in that case the measures μ_α, according to Riesz' theorem being defined on a compactification of U, yet are concentrated on U.

Now we discribe the space of the distributions g of theorem 4.2. In the remaining C will be an open, convex, cone in $\mathbb{R}^n$. Theorem 4.2 holds for convex homogeneous functions b(y) instead of $b\|y\|$ as well. Let

$$(5.3) \qquad S^k(b,C) \stackrel{\text{def}}{=\!=} \operatorname{proj}\lim_{m\to\infty} W^m_{\infty,0}((1+\|\xi\|)^m \exp(1/k\|\xi\|);U(b,C)),$$

where U(b,C) is given by (5.1). For p > k the identity map maps $S^k(b,C)$ continuously into $S^p(b,C)$, hence the strong dual $S^p(b,C)'$ of $S^p(b,C)$ into $S^k(b,C)'$. Now define the space

$$S^*(b,C)' \stackrel{\text{def}}{=\!=} \operatorname{proj}\lim_{k\to\infty} S^k(b,C)'.$$

Deleting $(1+\|\xi\|)^m$ from the weight functions in (5.3) would yield the same space $S^*(b,C)'$, but $S^k(b,C)$ in the form (5.3) is an $F\bar{S}$-space. We choose an increasing sequence $\{C_k\}_{k=1}^{\infty}$ of convex subcones of C exhausting C, such that, when $\delta_k > 0$ is a number with for $y \in C_k$ and $\xi \in C^*_{k+1}$

$$(5.4) \qquad y\cdot\xi \geq \delta_k \|y\|\|\xi\|, \qquad \text{cf. (4.9)},$$

then $\{1/k\ \delta_k\}_{k=1}^{\infty}$ is a decreasing sequence of positive numbers. In view of the fact that for any k the set $(C^*_{k+1})^c \cap U(b,C)$ is compact (see the proof of lemma 5.1) the distributions g in $S^*(b,C)'$ are just the $g \in \mathcal{D}'$ of theorem 4.2.

LEMMA 5.1. *For all* k *there is a* $p > k$ *such that for any* $z = x+iy$ *with* $y \in C_k$ *and* $\|y\| > 1/k$

$$e^{iz\cdot\xi} \in S^p(b,C)_\xi.$$

PROOF. For every $\xi \in (C^*_{k+1})^c$ there is an $y_\xi \in \mathrm{pr}\,\overline{C_{k+2}}$ with

$$\cos(y_\xi, \tfrac{\xi}{\|\xi\|}) \leq -\delta_{k+1},$$

thus $-y_\xi\cdot\xi \geq \delta_{k+1}\|\xi\|$. Then for every $\xi \in (C^*_{k+1})^c \cap U(b,C)$ it follows from (5.2) that

$$\delta_{k+1}\|\xi\| \leq \sup_{y\in \mathrm{pr}\,\overline{C_{k+2}}} -y\cdot\xi \leq \sup_{y\in \mathrm{pr}\,\overline{C_{k+2}}} b(y) \stackrel{\mathrm{def}}{=\!=} b_k.$$

Thus with $d_k = \dfrac{b_k}{\delta_{k+1}}$ we get for these ξ

$$\|\xi\| \leq d_k.$$

Now let $p > k/\delta_k$, then using (5.4) and (5.2) we find for $y \in C_k$, $\|y\| > 1/k$ and every m

$$\sup_{\substack{\xi\in U(b,C)\\ |\alpha|\leq m}} (1+\|\xi\|)^m \exp(\tfrac{1}{p}\|\xi\|)\ |D^\alpha e^{iz\cdot\xi}| \leq$$

$$\leq \sup_{\substack{\xi\in C^*_{k+1}\\ |\alpha|\leq m}} (1+\|\xi\|)^m\ |z^\alpha|\ \exp(\tfrac{1}{p}-\delta_k\tfrac{1}{k})\|\xi\| +$$

$$+ \sup_{|\alpha|\le m} (1+d_k)^m \, |z^\alpha| \, \exp(\tfrac{1}{p}\, d_k + b(y)) \le$$

$$\le M_k(1+\|z\|)^m \, e^{b(y)} \sup_{t\ge 0} (1+t)^m \exp-(\tfrac{1}{k}\,\delta_k - \tfrac{1}{p})\, t \, .$$

The lemma follows from the fact that for all $t \ge 0$

$$(1+t)^m \exp - \delta t \le K_m \, 1/\delta^m$$

for some constant K_m depending on m. □

As an element of S' the Fourier transform of $e^{-y\cdot\xi}\, g_\xi$ with $g \in S^*(b,C)'$ is known. We can now formulate a simple representation of this Fourier transform.

<u>LEMMA 5.2</u>. *For any* $y \in C$ *and* $g \in S^*(b,C)'$

(5.5) $\qquad F[e^{-y\cdot\xi}\, g_\xi](x) = \langle g\, , e^{iz\cdot\xi}\rangle.$

<u>PROOF</u>. Let us first assume that moreover $g \in S'$. Let α be a C^∞-function with support in $U(b+1,C)$, equal to 1 in $U(b,C)$, such that α is a multiplier in S' (here $(b+1)(y) = b(y)+\|y\|$). Then $\alpha(\xi)\, e^{-y\cdot\xi} \in S_\xi$ for every $y \in C$. If $\phi \in S$, we obtain

$$\langle F[e^{-y\cdot\xi} g_\xi], \phi\rangle = \langle e^{-y\cdot\xi} g_\xi, \int_{\mathbb{R}^n} e^{ix\cdot\xi} \phi(x)dx\rangle =$$

$$= \langle \alpha(\xi) e^{-y\cdot\xi} g_\xi, \int_{\mathbb{R}^n} e^{ix\cdot\xi} \phi(x)dx\rangle =$$

$$= \langle g_\xi, \int_{\mathbb{R}^n} \alpha(\xi) e^{iz\cdot\xi} \phi(x)dx\rangle_\xi \; .$$

Furthermore, $\alpha(\xi) e^{iz\cdot\xi}\phi(x) \in S_{\xi,x}$ and considering g_ξ as a distribution in $S'_{\xi,x}$, by TREVES [8,(51.7)] we get

$$<F[e^{-y\cdot\xi}g_\xi],\phi>_x = <g_\xi,\alpha(\xi)e^{iz\cdot\xi}\phi(x)>_{\xi,x} =$$

$$= \int_{\mathbb{R}^n} <g_\xi,\alpha(\xi)e^{iz\cdot\xi}>_\xi \phi(x)dx =$$

$$= <<g_\xi,\alpha(\xi)e^{iz\cdot\xi}>_\xi,\phi>_x ,$$

hence

$$F[e^{-y\cdot\xi}g_\xi](x) = <g_\xi,\alpha(\xi)e^{iz\cdot\xi}>_{S'}.$$

Now we take $g \in S^*(b,C)'$ and $y \in C$. Let $y_0 \in C$ be such that $y-y_0 \in C$ and let p be such that for ξ in U(b,C) outside a compact set $-y_0\cdot\xi \leq \leq -1/p\|\xi\|$. Then multiplication by exp $-y_0\cdot\xi$ and restriction to U(b,C) is a continuous map from S into $S^p(b,C)$, so its transpose is continuous from $S^p(b,C)'$ into S'. Hence $e^{-y_0\cdot\xi}\, g_\xi \in S'$ and according to above

$$F[e^{-(y-y_0)\cdot\xi}\, e^{-y_0\cdot\xi} g_\xi](x) = <e^{-y_0\cdot\xi} g_\xi,\alpha(\xi)e^{i(z-iy_0)\cdot\xi}>_{S'} =$$

$$= <g_\xi, e^{-y_0\cdot\xi}\alpha(\xi)e^{i(z-iy_0)\cdot\xi}\Big|_{U(b,C)}>_{S^p(b,C)'} = <g_\xi,e^{iz\cdot\xi}>. \qquad \square$$

Theorem 4.1 also holds for convex homogeneous functions b(y) instead of $b\|y\|$, if C is convex. Therefore, we get a representation of the function f of theorem 4.1 (cf. CARMICHAEL [1,th.13]). Let f be holomorphic in T^C and satisfy

$$(5.6) \qquad |f(z)| \leq P(C',r,\sigma)(1+\|z\|)^{N(C',r)} \exp\{b(y)+\sigma\|y\|\}, \qquad z \in T^{C'},\ \|y\| \geq r$$

for all $C' \subset\subset C$, $r > 0$ and $\sigma > 0$.

<u>COROLLARY 5.3</u>. *For any function* f *that satisfies* (5.6) *there is a distribution* $g \in S^*(b,C)'$ *such that*

$$(5.7) \qquad f(z) = <g_\xi,e^{iz\cdot\xi}>.$$

BIBLIOTHEEK MATHEMATISCH CENTRUM
AMSTERDAM

With this representation lemma 3.3 can be improved so that $m(C')$ in (3.8) no longer depends on C'.

COROLLARY 5.4. *Let the boundary value* f^* *in* Z' *of a function* f *satisfying* (5.6) *belong to* S'. *Then* f *attains this boundary value already in* S' *as* $y \to 0$, $y \in C'$ *and* f *satisfies the stronger condition*

$$(5.8) \qquad |f(z)| \le P(C')(1+\|z\|)^m(1+\|y\|^{-k}) \exp b(y), \qquad y \in C',$$

for every $C' \subset\subset C$ *and some* m *and* k.

PROOF. In view of (5.7) it is sufficient to represent $g = F^{-1}[f^*]$ as sum of weak derivatives of measures in $U(b,C)$ and to estimate

$$\sup_{\substack{\xi \in U(b,C) \\ |\alpha| \le m}} (1+\|\xi\|)^k \; |D^\alpha e^{iz\cdot\xi}|$$

as in the proof of lemma 5.1. □

We now define a topology on the space $H^*(b,C)$ of functions f satisfying (5.6) by

$$(5.9) \qquad H^*(b,C) \overset{\text{def}}{=\!=} \operatorname{proj} \lim_{k\to\infty} \operatorname{ind} \lim_{m\to\infty} A_\infty\left(\frac{\exp -b(y)}{(1+\|z\|)^m}; \mathbb{R}^n + iC(k)\right),$$

where $A_\infty(M(z);\Omega)$ denotes the Banach space of holomorphic functions in Ω with finite sup norm $\sup_{z\in\Omega} M(z)|f(z)|$ and where $C(k) = C_k \cap \{y \mid y > 1/k\}$. Here the continuous maps from

$$H_m^p \overset{\text{def}}{=\!=} A_\infty\left(\frac{\exp -b(y)}{(1+\|z\|)^m}; \mathbb{R}^n + iC(p)\right)$$

into H_ℓ^k, $\ell \ge m$, $p \ge k$, are the natural injections. By changing this representation of the space $H^*(b,C)$ somewhat, one can see that $H^*(b,C)$, just as $S^*(b,C)'$, is the projective limit of nuclear LS-spaces: let for each k $\{C_{k+1/m}\}_{m=1}^\infty$ be a decreasing sequence of convex, relatively compact subcones of C_{k+1} with intersection $\overline{C_k}\backslash\{0\}$ and let $C(k+1/m) =$ $= C_{k+1/m} \cap \{y \mid \|y\| > k+1/m\}$. Then also

$$H^*(b,C) = \text{proj}\lim_{k\to\infty} \ \text{ind}\lim_{m\to\infty} H_m^{p+1/m}$$

and from the compact embedding theorems between A-spaces in WLOKA [13, theorem 2, §4.2, where the condition $d(S_n, CS_{n+1}) > 0$ may be replaced by $\overline{S_n} \subset G_1$] follows the above mentioned property.

The following theorem gives the Fourier transformation in theorems 4.1 and 4.2 as an isomorphism.

THEOREM 5.5. *The Fourier transformation* F: $S^*(b,C)' \longrightarrow H^*(b,C)$ *given by* $F(g)(z) = \langle g_\xi, e^{iz\cdot\xi}\rangle$ *for* $g \in S^*(b,C)'$ *is a topological isomorphism.*

PROOF. In the proof of lemma 5.2 it is shown that for $g \in S^*(b,C)'$ and $y_0 \in C$ $e^{-y_0\cdot\xi} g_\xi$ indeed belongs to S', so that the Fourier transformation (5.5) is 1-1. Hence F is an injective map from $S^*(b,C)'$ into $H^*(b,C)$ according to theorem 4.2. Theorem 4.1 says that this map is moreover surjective. In order to prove the continuity of F it is sufficient to show that for each k there is a p such that F is a bounded map from $S^p(b,C)'$ into $H^k(b,C) \overset{\text{def}}{=\!=} \text{ind}\lim_{m\to\infty} H_m^k$, because as an LS-space $S^p(b,C)'$ is bornologic, see FLORET & WLOKA [3]. So, let B be a bounded set in $S^p(b,C)'$, where p is still to be chosen. This means that for some m B is bounded in the strong dual of $W^m_{\infty,0}(M_m^p; U(b,C))$, where $M_m^p(\xi) = (1+\|\xi\|)^m \exp(1/p\|\xi\|)$. Hence there is a K such that for all $\phi \in W^m_{\infty,0}(M_m^p; U(b,C))$ and $g \in B$

$$|\langle g,\phi\rangle| \le K \sup_{\substack{\xi\in U(b,C)\\ |\alpha|\le m}} M_m^p(\xi)\ |D^\alpha \phi(\xi)|.$$

Now we choose $p > k$ as in lemma 5.1 and replacing $\phi(\xi)$ by $\exp iz\cdot\xi$ in the above estimate yields for the images $f = F(g)$

$$|f(z)| \le K\, M_{k,m}(1+\|z\|)^m\, e^{b(y)}, \qquad y \in C_k, \quad \|y\| > 1/k.$$

Hence $F(B)$ is bounded in H_m^k, thus bounded in $H^k(b,C)$.

Next we prove the continuity of F^{-1}. Again it would be sufficient to show that for each k there is a p such that F^{-1} is a bounded map from the LS-space $\text{ind}\lim_{m\to\infty} H_m^{p+1/m}$ into $S^k(b,C)'$. So, let us start with a bounded set in $H_m^{p+1/m}$ for some m. This set is certainly bounded in H_m^p. However, the elements of a bounded set A in H^p_m are holomorphic in $\mathbb{R}^n+iC(p)$, so that we cannot expect that $F^{-1}(A) \subset S^k(b,C)'$.

Let $p > k$ and let $y_0 \in C_p$ be such that

$$\frac{1}{p} \leq \|y_0\| \leq \frac{1}{k}.$$

Then $f(z+iy_0)$ is holomorphic in $\mathbb{R}^n+iC_p$ if $f \in A$ and it satisfies there

$$|f(z+iy_0)| \leq M(1+\|z\|)^m \exp b(y+y_0) \leq M'(1+\|z\|)^m \exp b(y),$$

because $b(y)$ is homogeneous and convex: $\frac{1}{2}b(y+y_0) = b(\frac{1}{2}y+\frac{1}{2}y_0) \leq \frac{1}{2}b(y) + \frac{1}{2}b(y_0)$. Hence the set $B' = \{g' \mid g' = F^{-1}[f(x+iy_0)],\ f \in A\}$ is a bounded set in S' and every $g' \in B'$ has its support in the set $U(b,C_p)$. Since $y_0\cdot\xi \leq 1/k\|\xi\|$, multiplication by $\exp y_0\cdot\xi$ maps B' into a bounded set B in $S^k(b,C_p)'$. According to corollary 3.2 and (5.5) we have for $f \in A$ and all y such that $\exp i(z+iy_0)\cdot\xi \in S^k(b,C_p)$

$$f(z+iy_0) = F[e^{-y\cdot\xi} g'_\xi] = F[e^{-(y+y_0)\cdot\xi}\, e^{y_0\cdot\xi}\, g'_\xi] = \langle g_\xi, e^{i(z+iy_0)\cdot\xi}\rangle, \tag{5.10}$$

for some $g \in B$. By (3.4) g is independent of y_0, so that $F^{-1}(f) = g$. Hence $F^{-1}(A)$ is bounded in $S^k(b,C_p)'$. If f also belongs to H^p_ℓ for a larger p, $\ell \geq m$, then still we would have found the same g. Therefore, F^{-1} is a continuous map from $H^*(b,C)$ into $S^k(b,C_p)'$ for any p with the same image in every space $S^k(b,C_p)'$, $p = 1,2,\ldots$. Thus F^{-1} is continuous from $H^*(b,C)$ into $\text{proj}\lim_{p\to\infty} S^k(b,C_p)$, which equals $S^k(b,C)'$ because $S^k(b,C_{p+1})'$ is a closed linear subspace of $S^k(b,C_p)'$. Hence F^{-1} is continuous from $H^*(b,C)$ into $S^k(b,C)'$ for all k and since in (5.10) g is also independent of k, it follows that F^{-1} is a continuous map from $H^*(b,C)$ into $S^*(b,C)'$. □

Similarly, when the boundary values of the functions f exist in S', we can topologize the spaces of these functions and of their inverse Fourier transforms, so that the Fourier transformation is a topological isomorphism. For that purpose, let

$$S(b,C) \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{m\to\infty} W^m_{\infty,0}((1+\|\xi\|)^m;U(b,C))$$

and let

$$H(b,C) \overset{\text{def}}{=\!=} \operatorname*{ind\,lim}_{m\to\infty} \; \operatorname*{proj\,lim}_{k\to\infty} A_\infty\left(\frac{\exp - b(y)}{(1+\|z\|)^m(1+\|y\|^{-m})};\mathbb{R}^n+iC_k\right).$$

$S(b,C)$ is an $F\overline{S}$-space and as a consequence of the following theorem $H(b,C)$ is an LS-space. Also here for any $z \in T^C$ $\exp iz\cdot\xi \in S(b,C)_\xi$ and lemma 5.2 holds for $g \in S(b,C)'$. Similarly to theorem 5.5 with the aid of VLADIMIROV [9, th.26.3 and 26.4, th.2] one can prove the following theorem, which gives the Fourier transformation of [9, 26.4 th.2] as a topological isomorphism.

THEOREM 5.6. *The Fourier transformation* F: $S(b,C)' \longrightarrow H(b,C)$ *given by* $F(g)(z) = \langle g_\xi, e^{iz\cdot\xi}\rangle$ *for* $g \in S(b,C)'$ *is a topological isomorphism.*

REMARK 5.7. At the beginning of this section it is shown that $S(b,C)'$ is a closed linear subspace of S'. However, $S^*(b,C)'$ as a subset of $\mathcal{D}'$ carries a finer topology than the one induced by $\mathcal{D}'$. For, let $n=1$, $C = \{y \mid y > 0\}$ and $b=0$, then the function $\theta(\xi)e^\xi$ belongs to $\mathcal{D}'$, but not to $S^*(b,C)'$, because $\theta(\xi)e^{(1-y)\xi} \notin S'$ if $0 < y < 1$ (here $\theta(\xi) = 1$ if $\xi > 0$ and $\theta(\xi) = 0$ if $\xi < 0$). Furthermore $\theta(\xi)\ \Sigma_{k=0}^N 1/k!\ \xi^k$ belongs to $S^*(b,C)'$ for every N and the limit for $N \to \infty$ converges in $\mathcal{D}'$ to $\theta(\xi)e^\xi$, hence it does not converge in $S^*(b,C)'$, which as projective limit of complete spaces is itself complete.

We end this paper with a last striking property of functions in $H^*(b,C)$ or $H(b,C)$, when the function b is moreover uniformly continuous in C. Let $w \in \text{pr}\ C$, then $w \in \text{pr}\ C'$ for some $C' \subset\subset C$. Since for $C' \subset\subset C'' \subset\subset C$ $U(b,C) \cap C''_*$ is bounded and since for $\xi \in C''^*$

$$\delta\|\xi\| \le w\cdot\xi \le \|\xi\|$$

for some $\delta > 0$, in (5.3) the weight functions $\exp(1/k\|\xi\|)$ in the definition of $S^*(b,C)'$ may be replaced by $\exp(1/k\ w\cdot\xi)$. Then a function $f \in H^*(b,C)$ satisfies for each $\varepsilon > 0$ and $z \in \mathbb{R}^n+i\{\varepsilon w + C\}$

$$|f(z)| \le |<g_\xi, e^{iz\cdot\xi}>| \le$$

$$\le K''_\varepsilon (1+\|z\|)^{m(\varepsilon)} \sup_{\xi\in U(b,C)} (1+\|\xi\|)^{m(\varepsilon)} e^{\frac{1}{2}\varepsilon w\cdot\xi - y\cdot\xi} \le$$

$$\le K'_\varepsilon (1+\|z\|)^{m(\varepsilon)} \sup_{\xi\in U(b,C)} \exp(\varepsilon w\cdot\xi - y\cdot\xi) \le K'_\varepsilon (1+\|z\|)^{m(\varepsilon)} \exp b(y-\varepsilon w) \le$$

$$\le K_\varepsilon (1+\|z\|)^{m(\varepsilon)} e^{b(y)},$$

because b is uniformly continuous in C. Since this property for functions $f \in H^*(b,C)$ is not true for general convex homogeneous functions b, we see that it was right to divide C into $\bigcup_{k=1}^{\infty} C(k)$ instead of $\bigcup_{\varepsilon>0}\{\varepsilon w+C\}$ in the definition (5.9) of $H^*(b,C)$. A similar property holds for functions f in H(b,C).

For example, we may take b constant on pr C, i.e., $b(y) = b\|y\|$ where now b is a number. Then we have obtained the following corollary.

COROLLARY 5.8. *If a holomorphic function* f *in* T^C, C *open and convex, satisfies* (4.2), *then it also satisfies for every* $\varepsilon > 0$

$$|f(z)| \le P(\varepsilon)(1+\|z\|)^{N(\varepsilon)} \exp b\|y\|, \qquad y \in \varepsilon w + C$$

for some fixed $w \in$ pr C. *If* f *satisfies* (5.8), *where* $b(y) = b\|y\|$ *for some number* b, *then* f *also satisfies for some* $m' \ge m$ *and every* $\varepsilon > 0$

$$|f(z)| \le P(\varepsilon)(1+\|z\|)^{m'} \exp b\|y\|, \qquad y \in \varepsilon w + C.$$

REFERENCES

[1] CARMICHAEL, R.D., *Analytic representation of the distributional finite Fourier transform*, SIAM J. Math. Anal. 5(1974) p.737-761.

[2] EHRENPREIS, L., *Fourier analysis in several complex variables*, Wiley, New York, 1970.

[3] FLORET, K. & J. WLOKA, *Einführung in die Theorie der lokalkonvexen Räume*, Lecture Notes in Mathematics no. 56, Springer-Verlag, Berlin, 1968.

[4] GEL'FAND, I.M. & G.E. SHILOV, *Generalized functions*, Vol.2, Academic Press, New York, 1968.

[5] MARTINEAU, A., *Distributions et valeurs au bord des fonctions holomorphes*, Proc. of the Intern. Summer Inst. Lisbon, 1964, p.193-326.

[6] ROEVER, J.W. de, *Ehrenpreis' fundamental principle for non-entire functions*, to appear.

[7] SCHWARTZ, L., *Théorie des distributions*, Hermann, Paris, 1966.

[8] TREVES, F., *Topological vector spaces, distributions and kernels*, Pure and applied mathematics, series no. 25, Academic Press, N.Y., 1967.

[9] VLADIMIROV, V.S., *Methods of the theory of functions of many complex variables*, MIT Press, Cambridge, Mass., 1966.

[10] VLADIMIROV, V.S., *Functions which are holomorphic in tubular cones*, (Russian), Izv. Akad. Nauk. SSSR, Ser. Mat. 27(1963) p.75-100.

[11] WHITNEY, H., *Functions differentiable on the boundary of regions*, Ann. of Math. 35(1934) 482-485.

[12] WHITNEY, H., *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc., 36(1934) 63-89.

[13] WLOKA, J., *Grundräume und verallgemeinerte Funktionen*, Lecture Notes in Mathematics no. 82, Springer-Verlag, Berlin, 1969.

ONTVANGEN 18 NOV. 1975