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POSITIVITY PROOFS FOR LINEARIZATION AND CONNECTION COEFFICIENTS OF ORTHOGONAL POLYNOMIALS SATISFYING AN ADDITION FORMULA

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Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition formula *)

by

T.H. Koornwinder

ABSTRACT

For orthogonal polynomials in one or several variables which satisfy an addition formula of certain type it is proved that the linearization coefficients are nonnegative. For two classes of orthogonal polynomials which satisfy related addition formulas a sufficient condition for nonnegativity of the connection coefficients is given. The results are applied to Jacobi, Laguerre and disk polynomials. In particular, Dunkl's recent expression of a certain Jacobi polynomial times a simple polynomial as a sum of Gegenbauer polynomials with nonnegative coefficients is generalized to all real $\alpha \geq 0$.

KEY WORDS & PHRASES : *positivity of linearization coefficients; positivity of connection coefficients; addition formula; Jacobi polynomials; Laguerre polynomials; disk polynomials; spherical functions on compact homogeneous spaces.*

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*) This paper is not for review; it is meant for publication elsewhere.

CONTENTS

1. INTRODUCTION

Consider orthogonal polynomials $p_n(x)$ such that

$$\int p_m(x)p_n(x)d\alpha(x) = (\pi_n)^{-1}\delta_{m,n}.$$

The coefficients in the formula

$$p_m(x)p_n(x) = \sum_{\ell} a(\ell,m,n)\pi_\ell \; p_\ell(x) \tag{1.1}$$

are called linearization coefficients. If $\{q_n(x)\}$ is another orthogonal system of polynomials then the coefficients in the formula

$$p_n(x) = \sum_m b_{n,m} \, q_m(x) \tag{1.2}$$

are called connection coefficients. For the harmonic analysis with respect to such polynomials it is important to know whether the coefficients $a(\ell,m,n)$ and $b_{n,m}$ are nonnegative, cf. the survey papers by ASKEY [2] and GASPER [10]. In the case of Jacobi polynomials the two problems have been solved by GASPER [7], [8] and by ASKEY & GASPER [3], respectively.

The first main result in the present paper is a positivity proof for the coefficients $a(\ell,m,n)$ in the case that the polynomials $p_n(x)$ (which may be polynomials in several variables) satisfy an addition formula with certain qualitative features. In particular, the addition formula for Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$, $\alpha > \beta > -\frac{1}{2}$, has the required form. Thus we obtain a new proof for part of GASPER's [7] results. This proof is much less computational than Gasper's original proof. The same method also applies to disk polynomials (a class of orthogonal polynomials in two variables on the unit disk). Here the results are probably new.

Our second main result is the derivation of a sufficient condition for the positivity of the coefficients in (1.2), whenever $p_n(x)$ and $q_m(x)$ satisfy related addition formulas. In the case of Jacobi polynomials this approach is no improvement compared with the proofs in ASKEY & GASPER [3]. However, if the polynomials $p_n(x)$ are disk polynomials restricted to the real axis and the polynomials $q_m(x)$ are Gegenbauer polynomials then it can be shown by our methods that

$$x^k R_n^{(\alpha,k)}(2x^2-1) = \sum_i b_i \, R_{2n+k-2i}^{(\frac{1}{2}\alpha-\frac{1}{2},\frac{1}{2}\alpha-\frac{1}{2})}(x) \tag{1.3}$$

with $b_i \geq 0$ if $\alpha \geq 0$, thus extending DUNKL's [5] result in the cases $\alpha = 0,1,2,\ldots$.

If the polynomials $p_n(x)$ and $q_n(x)$ can be interpreted as spherical functions on compact homogeneous spaces then the positivity of $a(\ell,m,n)$ and, sometimes, the positivity of $b_{n,m}$ can be obtained from this group theoretic interpretation. For Jacobi polynomials and disk polynomials such an interpretation is possible for certain discrete values of the parameters. Addition formulas can be considered as formulas containing much analytic information of group theoretic nature, even if the polynomials have such parameter values that no group theoretic interpretation exists. The proofs of our main results imitate the proofs for the corresponding results on spherical functions.

2. PRELIMINARIES

Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$ $(\alpha,\beta>-1)$ are orthogonal polynomials on the interval $(-1,1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. They are normalized such that $R_n^{(\alpha,\beta)}(1) = 1$. We have

$$\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} R_m^{(\alpha,\beta)}(x)\, R_n^{(\alpha,\beta)}(x) \cdot (1-x)^\alpha(1+x)^\beta\, dx = (\pi_n^{(\alpha,\beta)})^{-1}\, \delta_{m,n}, \tag{2.1}$$

where

$$\pi_n^{(\alpha,\beta)} := \frac{(2n+\alpha+\beta+1)(\alpha+1)_n(\alpha+\beta+2)_n}{(n+\alpha+\beta+1)\ \ (\beta+1)_n\ \ n!} .$$

Laguerre polynomials $L_n^\alpha(x)$ $(\alpha>-1)$ are orthogonal polynomials on the interval $(0,\infty)$ with respect to the weight function $e^{-x}x^\alpha$. They are normalized such that $L_n^\alpha(0) = (\alpha+1)_n/n!$. Laguerre polynomials can be obtained from Jacobi polynomials by the limit formula

(2.2) $$L_n^\alpha(x)/L_n^\alpha(0) = \lim_{\beta\to\infty} R_n^{(\alpha,\beta)}(1-2\beta^{-1}x).$$

Disk polynomials $R_{m,n}^\alpha(z)$ ($\alpha>-1$, $m,n=0,1,2,\ldots$, $z \in \mathbb{C}$) are defined in terms of Jacobi polynomials by

(2.3) $$R_{m,n}^\alpha(r\,e^{i\phi}) := R_{m\wedge n}^{(\alpha,|m-n|)}(2r^2-1)\,r^{|m-n|}\,e^{i(m-n)\phi},$$

where

$$m \wedge n := \min\{m,n\}.$$

We have

(2.4) $$\frac{\alpha+1}{\pi} \iint_{x^2+y^2<1} R_{m,n}^\alpha(x+iy)\,\overline{R_{k,\ell}^\alpha(x+iy)}\cdot$$

$$\cdot\,(1-x^2-y^2)^\alpha\,dx\,dy = (\pi_{m,n}^\alpha)^{-1}\,\delta_{m,k}\delta_{n,\ell},$$

where

$$\pi_{m,n}^\alpha := \frac{(m+n+\alpha+1)(\alpha+1)_m(\alpha+1)_n}{(\alpha+1)\;m!\;n!}.$$

Orthogonal polynomials $R_{n,k}^{\alpha,\beta}(x,y)$ ($\alpha,\beta>-1$, n,k integers, $n\ge k\ge 0$) on a region bounded by a straight line and a parabola can be defined in terms of Jacobi polynomials by

(2.5) $$R_{n,k}^{\alpha,\beta}(x,y) := R_k^{(\alpha,\beta+n-k+\frac{1}{2})}(2y-1)\;y^{\frac{1}{2}(n-k)}\cdot R_{n-k}^{(\beta,\beta)}(y^{-\frac{1}{2}}x).$$

We have

(2.6) $$\frac{\Gamma(\alpha+\beta+\frac{5}{2})}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(\frac{1}{2})} \int_{y=0}^{1} \int_{x=-y^{\frac{1}{2}}}^{y^{\frac{1}{2}}} R_{n,k}^{\alpha,\beta}(x,y)R_{m,\ell}^{\alpha,\beta}(x,y)\;\cdot$$

$$\cdot\,(1-y)^\alpha(y-x^2)^\beta\,dx\,dy = (\pi_{n,k}^{\alpha,\beta})^{-1}\delta_{n,m}\delta_{k,\ell},$$

where

$$\pi_{n,k}^{\alpha,\beta} := \frac{(2n-2k+2\beta+1)(n+k+\alpha+\beta+\frac{3}{2})}{(n-k+2\beta+1)(n+\alpha+\beta+\frac{3}{2})}\cdot$$

$$\cdot\,\frac{(\alpha+1)_k(2\beta+2)_{n-k}(\alpha+\beta+\frac{5}{2})_n}{k!\;(n-k)!\;(\beta+\frac{3}{2})_n}.$$

The following addition formulas are well-known. For Gegenbauer polynomials $R_n^{(\alpha,\alpha)}(x)$ ($\alpha>-\frac{1}{2}$) we have

$$(2.7)\qquad R_n^{(\alpha,\alpha)}(xy+(1-x^2)^{\frac12}(1-y^2)^{\frac12}t) =$$

$$= \sum_{k=0}^{n} \frac{(-1)^k(-n)_k(n+2\alpha+1)_k}{2^{2k}(\alpha+1)_k(\alpha+1)_k}(1-x^2)^{\frac12 k}\, R_{n-k}^{(\alpha+k,\alpha+k)}(x)\cdot$$

$$\cdot\ (1-y^2)^{\frac12 k}\, R_{n-k}^{(\alpha+k,\alpha+k)}(y)\,\pi_k^{(\alpha-\frac12,\alpha-\frac12)}R_k^{(\alpha-\frac12),\alpha-\frac12)}(t),$$

cf. ERDéLYI [6,3.15 (20)].

For Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$, $\alpha > \beta > -\frac12$, we have

$$(2.8)\qquad R_n^{(\alpha,\beta)}(\tfrac12(1+x)(1+y) + \tfrac12(1-x)(1-y)t + (1-x^2)^{\frac12}(1-y^2)^{\frac12} s - 1) =$$

$$= \sum_{k=0}^{n}\sum_{\ell=0}^{k} \frac{(-1)^{k+\ell}(-n)_k(-n-\beta)_\ell(n+\alpha+\beta+1)_k(n+\alpha+1)_\ell}{2^{2k}(\alpha+1)_{k+\ell}\ (\alpha+1)_{k+\ell}}\ \cdot$$

$$\cdot\ (1-x)^{\frac12(k+\ell)}(1+x)^{\frac12(k-\ell)}R_{n-k}^{(\alpha+k+\ell,\beta+k-\ell)}(x)\ \cdot$$

$$\cdot\ (1-y)^{\frac12(k+\ell)}(1+y)^{\frac12(k-\ell)}R_{n-k}^{(\alpha+k+\ell,\beta+k-\ell)}(y)\ \cdot$$

$$\cdot\ \pi_{k,\ell}^{\alpha-\beta-1,\beta-\frac12}\ R_{k,\ell}^{\alpha-\beta-1,\beta-\frac12}(s,t),$$

cf. KOORNWINDER [10],[11],[12],[13],[14].

Finally, for disk polynomials $R_{m,n}^{\alpha}(z)$, $\alpha > 0$, we have

$$(2.9)\qquad R_{m,n}^{\alpha}(z_1\bar z_2 + (1-z_1\bar z_1)^{\frac12}\ (1 - z_2\bar z_2)^{\frac12}\ w) =$$

$$= \sum_{k=0}^{m}\sum_{\ell=0}^{n} \frac{(-1)^{k+\ell}(-m)_k(-n)_\ell(n+\alpha+1)_k(m+\alpha+1)_\ell}{(\alpha+1)_{k+\ell}\ (\alpha+1)_{k+\ell}}\ \cdot$$

$$\cdot\ (1-z_1\bar z_1)^{\frac12(k+\ell)}R_{m-k,n-\ell}^{\alpha+k+\ell}(z_1)\ \cdot$$

$$\cdot\ (1-z_2\bar z_2)^{\frac12(k+\ell)}\ \overline{R_{m-k,n-\ell}^{\alpha+k+\ell}(z_2)}\ \pi_{k,\ell}^{\alpha-1}\ R_{k,\ell}^{\alpha-1}\ (w),$$

cf. ŠAPIRO [15] and KOORNWINDER [11].

We conclude this section with the following lemma.

LEMMA 2.1. *Let* $\alpha,\ \beta > -1$, $n = 0,1,2,\ldots$. *Then*

$$\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,\Gamma(\beta+1)} \int_{-1}^{1} R_n^{(2\alpha+1,2\beta+1)}(x)(1-x)^{\alpha}(1+x)^{\beta}\,dx =$$

$$= \begin{cases} 0 & \textit{if } n \textit{ is odd,} \\[2ex] \dfrac{(\frac{1}{2})_m\,(\beta+1)_m}{(\alpha+\frac{3}{2})_m\,(\alpha+\beta+2)_m} > 0 & \textit{if } n = 2m. \end{cases}$$

PROOF. By expanding the Jacobi polynomial as a hypergeometric power series in $\frac{1}{2}(1-x)$ we obtain that the left-hand side equals

$${}_3F_2\left(\begin{matrix} -n,\ n+2\alpha+2\beta+3,\ \alpha+1; \\ \alpha+\beta+2,\ 2\alpha+2 \qquad ; \end{matrix}\ 1\right) =$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{3}{2})\Gamma(\alpha+\beta+2)\Gamma(-\beta)}{\Gamma(-\frac{1}{2}n+\frac{1}{2})\Gamma(\frac{1}{2}n+\alpha+\beta+2)\Gamma(\frac{1}{2}n+\alpha+\frac{3}{2})\Gamma(-\frac{1}{2}n-\beta)}$$

$$= \frac{\sin\pi(\frac{1}{2}-\frac{1}{2}n)\sin\pi(-\frac{1}{2}n-\beta)}{\sin\pi(-\beta)} \cdot \frac{\Gamma(\frac{1}{2}+\frac{1}{2}n)}{\Gamma(\frac{1}{2})} \cdot \frac{\Gamma(1+\beta+\frac{1}{2}n)}{\Gamma(1+\beta)} \cdot$$

$$\cdot \frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+\frac{3}{2}+\frac{1}{2}n)} \cdot \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+\beta+2+\frac{1}{2}n)}\ ,$$

where the first equality follows from Watson's theorem (cf. ERDéLYI [6, 4.4 (6)]). □

3. THE CASE OF SPHERICAL FUNCTIONS ON COMPACT HOMOGENEOUS SPACES

Let G be a compact group with closed subgroup K such that each irreducible representation of G contains the representation 1 of K at most once. Let $\Omega := G/K$, let $\xi_0 \in \Omega$ be stabilized by K and let $d\omega$ be the G-invariant normalized measure on Ω. Let

(3.1) $$L^2(\Omega) = \sum_n{}^{\oplus} \mathcal{H}_n$$

be the unique orthogonal decomposition of $L^2(\Omega)$ into irreducible subspaces with respect to G. Let ϕ_n be the (zonal) spherical function in H_n, i.e. the unique zonal (K-invariant) function in H_n which has value 1 in ξ_0.
If ϕ is a zonal and continuous function on Ω then let Φ be the unique function on $\Omega \times \Omega$ satisfying (i) $\Phi(g\xi, g\eta) = \Phi(\xi,\eta)$ for all $\xi,\eta \in \Omega$, $g \in G$, and (ii) $\Phi(\xi,\xi_0) = \phi(\xi)$ for all $\xi \in \Omega$. Let π_n denote the dimension of H_n. Then

(3.2) $$\int_\Omega \phi_m(\xi)\overline{\phi_n(\xi)}d\omega(\xi) = (\pi_n)^{-1}\delta_{m,n}.$$

If $\{f_k^n \mid k = 1,\ldots,\pi_n\}$ is an orthonormal basis of H_n then we have the addition formula

(3.3) $$\Phi_n(\xi,\eta) = (\pi_n)^{-1} \sum_{k=1}^{\pi_n} f_k^n(\xi)\overline{f_k^n(\eta)} .$$

In the formula

(3.4) $$\phi_m(\xi)\phi_n(\xi) = \sum_\ell a(\ell,m,n)\ \pi_\ell\overline{\phi_\ell(\xi)}$$

only finitely many terms are nonzero.

For Theorem 3.2 below we need the following additional assumptions. Let G_1 be a closed subgroup of G, let $K_1 := K \cap G_1$ and suppose that each irreducible representation of G_1 contains the representation 1 of K_1 at most once. Let $\Omega_1 := \{g\xi_0 \mid g \in G_1\}$. Then $\Omega_1 = G_1/K_1$. Let $d\omega_1$ be the G_1-invariant normalized measure on Ω_1. Let the irreducible subspaces of $L^2(\Omega_1)$ with respect to G_1 be denoted by K_n, let K_n have dimension ρ_n and let ψ_n be the unique K_1-invariant function in K_n which has value 1 in ξ_0. If ψ is a K_1-invariant and continuous function on Ω_1 then let Ψ be the function on $\Omega_1 \times \Omega_1$ satisfying (i) $\Psi(g\xi,g\eta) = \Psi(\xi,\eta)$ for all $\xi\ \eta \in \Omega_1$, $g \in G_1$ and (ii) $\Psi(\xi,\xi_0) = \psi(\xi)$ for all $\xi \in \Omega_1$. Note that if ψ is the restriction to Ω_1 of a K-invariant continuous function ϕ on Ω then Ψ is the restriction of Φ to $\Omega_1 \times \Omega_1$. In the formula

(3.5) $$\phi_n(\xi) = \sum_m b_{n,m}\ \psi_m(\xi) \quad , \xi \in \Omega_1 ,$$

only finitely many terms are nonzero.

It is well-known that the coefficient $a(\ell,m,n)$ in (3.4) and $b_{n,m}$ in (3.5) are nonnegative. Usually this is proved by using the positive definiteness of spherical functions. For didactic reasons we now give the proofs in a somewhat different form adapted to the proofs of the main theorems in section 4.

THEOREM 3.1. *The coefficients* $a(\ell,m,n)$ *in* (3.4) *are nonnegative.*

PROOF. Choose an orthonormal basis $\{f_k^n\}$ of H_n and let $f_1^n := (\pi_n)^{\frac{1}{2}}\phi_n$. We have

$$\Phi_m(\xi,\eta)\;\Phi_n(\xi,\eta) = \sum_\ell a(\ell,m,n)\pi_\ell\;\overline{\Phi_\ell(\xi,\eta)} =$$

$$= \sum_\ell \sum_{k=1}^{\pi_\ell} a(\ell,m,n)\overline{f_k^\ell(\xi)}\;f_k^\ell(\eta).$$

Hence

$$a(\ell,m,n) = \pi_\ell \int_\Omega\int_\Omega \Phi_m(\xi,\eta)\Phi_n(\xi,\eta)\;\overline{\phi_\ell(\xi)\phi_\ell(\eta)}\;d\omega(\xi)\;d\omega(\eta) =$$

$$= \pi_\ell(\pi_m\pi_n)^{-1}\sum_{i=1}^{\pi_m}\sum_{j=1}^{\pi_n}\int_\Omega\int_\Omega f_i^m(\xi)\;\overline{f_i^m(\eta)}\;\cdot$$

$$f_j^n(\xi)\;\overline{f_j^n(\eta)}\;\phi_\ell(\xi)\;\overline{\phi_\ell(\eta)}\;d\omega(\xi)\;d\omega(\eta) =$$

$$= \pi_\ell(\pi_m\pi_n)^{-1}\sum_{i=1}^{\pi_m}\sum_{j=1}^{\pi_n}\left|\int_\Omega f_i^m(\xi)\;f_j^n(\xi)\;\phi_\ell(\xi)\;d\omega(\xi)\right|^2 \geq 0. \quad \square$$

Essentially the same positivity proof can be used in the case of linearization coefficients of Q-functions on an association scheme (cf. DELSARTE [4, Lemma 2.4]).

THEOREM 3.2. *The coefficients* $b_{n,m}$ *in* (3.5) *are nonnegative.*

PROOF. Choose an orthonormal basis $\{f_k^n\}$ of H_n and $\{g_k^m\}$ of K_m. Let $g_1^m := (\rho_m)^{\frac{1}{2}}\,\psi_m$. For $\xi,\,\eta \in \Omega_1$ we have

$$\Phi_n(\xi,\eta) = \sum_m b_{n,m} \Psi_m(\xi,\eta) = \sum_m \sum_{k=1}^{\rho_m} b_{n,m}(\rho_m)^{-1} g_k^m(\xi) \overline{g_k^m(\eta)}.$$

Hence

$$b_{n,m} = (\rho_m)^2 \int_{\Omega_1}\int_{\Omega_1} \Phi_n(\xi,\eta) \overline{\psi_m(\xi)} \psi_m(\eta) d\omega_1(\xi) d\omega_1(\eta) =$$

$$= (\rho_m)^2(\pi_n)^{-1} \sum_{k=1}^{\pi_n} \int_{\Omega_1}\int_{\Omega_1} f_k^n(\xi) \overline{f_k^n(\eta)} \overline{\psi_m(\xi)} \psi_m(\eta) d\omega_1(\xi) d\omega_1(\eta) =$$

$$= (\rho_m)^2(\pi_n)^{-1} \sum_{k=1}^{\pi_n} \left| \int_{\Omega_1} f_k^n(\xi) \overline{\psi_m(\xi)} d\omega_1(\xi) \right|^2 \geq 0. \quad \square$$

4. THE CASE OF GENERAL ORTHOGONAL SYSTEMS OF FUNCTIONS SATISFYING AN ADDITION FORMULA

Let A and B be compact Hausdorff spaces with Borel measures α respectively β such that $0 < \alpha(E_1) < \infty$ and $0 < \beta(E_2) < \infty$ on nonempty open subsets E_1 of A and E_2 of B. Let $\{p_n\}$ and $\{r_n\}$ be families of continuous functions on A respectively B such that $r_0(t) \equiv 1$ and

(4.1) $$\int_A p_m(x) \overline{p_n(x)} d\alpha(x) = (\pi_n)^{-1} \delta_{m,n},$$

(4.2) $$\int_B r_m(t) \overline{r_n(t)} d\beta(t) = (\rho_n)^{-1} \delta_{m,n},$$

where $0 < \pi_n < \infty$, $0 < \rho_n < \infty$. Suppose that

(4.3) $$p_m(x) p_n(x) = \sum_\ell a(\ell,m,n)\pi_\ell \overline{p_\ell(x)}$$

with only finitely many nonzero terms. Suppose that Λ is a continuous mapping from A × A × B to A such that for each n there is an addition formula of the form

(4.4) $$p_n(\Lambda(x,y,t)) = \sum_k c_{n,k} p_n^k(x) \overline{p_n^k(y)} \rho_k r_k(t),$$

where p_n^k is continuous on A, $p_n^0 = p_n$, $c_{n,k} \geq 0$, $c_{n,0} > 0$ and where for each n only finitely many coefficients $c_{n,k}$ are nonzero.

Observe that the addition formulas (2.7), (2.8) and (2.9) have the form just described.

THEOREM 4.1. *Under the above assumptions the coefficients* $a(\ell,m,n)$ *in* (4.3) *are nonnegative.*

PROOF. We have

$$p_m(\Lambda(x,y,t))\ p_n(\Lambda(x,y,t)) = \sum_\ell a(\ell,m,n)\pi_\ell\ \overline{p_\ell(\Lambda(x,y,t))} =$$

$$= \sum_{\ell,k} a(\ell,m,n)\pi_\ell c_{\ell,k}\ p_\ell^k(x)\ \overline{p_\ell^k(y)}\rho_k\ r_k(t).$$

Hence

$$a(\ell,m,n) = (c_{\ell,0})^{-1}\ \pi_\ell \int_A \int_A \int_B p_m(\Lambda(x,y,t)) \cdot$$

$$\cdot\ p_n(\Lambda(x,y,t))\ p_\ell(x)\ \overline{p_\ell(y)}\ d\alpha(x)\ d\alpha(y)\ d\beta(t) =$$

$$= (c_{\ell,0})^{-1}\ \pi_\ell \sum_{i,j} c_{m,i}\ \rho_i\ c_{n,j}\ \rho_j \cdot$$

$$\cdot \int_A \int_A \int_B p_m^i(x)\ \overline{p_m^i(y)}\ r_i(t)\ p_n^j(x)\ \overline{p_n^j(y)}\ r_j(t)\ \cdot$$

$$\cdot\ p_\ell(x)\ \overline{p_\ell(y)}\ d\alpha(x)\ d\alpha(y)\ d\beta(t) =$$

$$= (c_{\ell,0})^{-1}\ \pi_\ell \sum_i c_{m,i}\ c_{n,i}\ \rho_i \int_A \int_A p_m^i(x)\ \overline{p_m^i(y)}\ \cdot$$

$$\cdot\ p_n^i(x)\ \overline{p_n^i(y)}\ p_\ell(x)\ \overline{p_\ell(y)}\ d\alpha(x)\ d\alpha(y) =$$

$$= (c_{\ell,0})^{-1}\ \pi_\ell \sum_i c_{m,i}\ c_{n,i}\ \rho_i\ \left| \int_A p_m^i(x)\ p_n^i(x)\ p_\ell(x)\ d\alpha(x) \right|^2 \geq$$

$$\geq 0. \quad \square$$

For the next theorem we make some additional assumptions. Let A_1 and B_1 be closed subspaces of A and B, respectively, with Borel measures α_1 on

A_1 and β_1 on B_1, which are finite and positive on nonempty open subsets. Let $\{q_m\}$ and $\{s_m\}$ be orthogonal systems of continuous functions on A_1 respectively B_1 with respect to these measures and denote the inverse quadratic norms by κ_m and σ_m, respectively.
Let $s_0(t) \equiv 1$. Suppose that

$$p_n(x) = \sum_m b_{n,m}\, q_m(x), \qquad x \in A_1, \tag{4.5}$$

with only finitely many nonzero terms. Suppose that for each m there is an addition formula of the form

$$q_m(\Lambda(x,y,t)) = \sum_k d_{m,k}\, q_m^k(x)\, \overline{q_m^k(y)}\, \sigma_k s_k(t), \tag{4.6}$$

$x,y \in A_1$, $t \in B_1$, where q_m^k is continuous on A_1, $q_m^0 = q_m$, $d_{m,k} \geq 0$, $d_{m,0} > 0$ and where for each m only finitely many coefficients $d_{m,k}$ are nonzero.

The above assumptions are satisfied, for instance, by two classes of Jacobi polynomials of different orders or by a class of disk polynomials together with a class of Gegenbauer polynomials.

THEOREM 4.2. *If the above assumptions are satisfied and if for all* k

$$\int_{B_1} r_k(t)\, d\beta_1(t) \geq 0 \tag{4.7}$$

then the coefficients $b_{n,m}$ *in* (4.5) *are nonnegative.*

PROOF. For $x,y \in A_1$, $t \in B_1$ we have

$$p_n(\Lambda(x,y,t)) = \sum_m b_{n,m}\, q_m(\Lambda(x,y,t)) =$$

$$= \sum_{m,k} b_{n,m}\, d_{m,k}\, q_m^k(x)\, \overline{q_m^k(y)}\, \sigma_k\, s_k(t).$$

Hence

$$b_{n,m} = (d_{m,0})^{-1}(\kappa_m)^2 \int_{A_1}\int_{A_1}\int_{B_1} p_n(\Lambda(x,y,t)) \cdot$$

$$\cdot\, \overline{q_m(x)}\, q_m(y)\, d\alpha_1(x)\, d\alpha_1(y)\, d\beta_1(t) =$$

$$= (d_{m,0})^{-1}(\kappa_m)^2 \sum_k c_{n,k}\,\rho_k \int_{A_1}\int_{A_1}\int_{B_1} p_n^k(x)\,\overline{p_n^k(y)}\,\cdot$$

$$\cdot\, r_k(t)\,\overline{q_m(x)}\,q_m(y)\,d\alpha_1(x)\,d\alpha_1(y)\,d\beta_1(t) =$$

$$= (d_{m,0})^{-1}\,(\kappa_m)^2 \sum_k c_{n,k}\rho_k\Big(\int_{B_1} r_k(t)\,d\beta_1(t)\Big)\,\cdot$$

$$\cdot\,\Big|\int_{A_1} p_n^k(x)\,\overline{q_m(x)}\,d\alpha_1(x)\Big|^2. \quad \square$$

5. APPLICATIONS TO THE LINEARIZATION OF PRODUCTS OF JACOBI, LAGUERRE AND DISK POLYNOMIALS

In this section Theorem 4.1 will be applied to Jacobi and disk polynomials. Some further results for Jacobi and Laguerre polynomials will follow.

COROLLARY 5.1. *For Jacobi polynomials* $R_n^{(\alpha,\beta)}(x)$, $\alpha \geq \beta \geq -\frac{1}{2}$, *we have*

$$R_m^{(\alpha,\beta)}(x)\,R_n^{(\alpha,\beta)}(x) = \sum_{\ell=|m-n|}^{m+n} a(\ell,m,n)\,\pi_\ell^{(\alpha,\beta)}R_\ell^{(\alpha,\beta)}(x)$$

with $a(\ell,m,n) \geq 0$.

This result was first obtained by GASPER [7], [8]. In fact he proved the nonnegativity of $a(\ell,m,n)$ for a larger region in the (α,β)-plane including $\{(\alpha,\beta) \mid \alpha \geq \beta,\ \alpha+\beta \geq -1\}$.

COROLLARY 5.2. *For disk polynomials* $R_{m,n}^{\alpha}(z)$, $\alpha \geq 0$, *we have*

$$R^{\alpha}_{m_1,n_1}(z)R^{\alpha}_{m_2,n_2}(z) = \sum_{m_3,n_3} a(m_1,n_1;m_2,n_2;m_3,n_3)\pi^{\alpha}_{m_3,n_3}\overline{R^{\alpha}_{m_3,n_3}(z)}$$

with $a(m_1,n_1;m_2,n_2;m_3,n_3) \geq 0$. *In the above sum the pair* (m_3,n_3) *takes such values that* $m_1+m_2+m_3 = n_1+n_2+n_3$ *and* $|m_1+n_1-m_2-n_2| \leq m_3+n_3 \leq$ $\leq m_1+n_1+m_2+n_2$.

Except for the cases $\alpha = 0,1,2,\ldots$, where the nonnegativity follows from the group theoretic interpretation, this result is probably new.

COROLLARY 5.3. *If* $\alpha \geq 0$ *and* k,ℓ,m,n *are nonnegative integers then*

$$R_m^{(\alpha,k)}(x)R_n^{(\alpha,\ell)}(x) = \sum_i c_i\, R_{m+n-i}^{(\alpha,k+\ell)}(x)$$

with $c_i \geq 0$. *In the above sum* i *takes such integer values that* $0 \leq i \leq (m+n) \wedge (2m+k) \wedge (2n+\ell)$.

PROOF. Use (2.2) and Corollary 5.2. □

COROLLARY 5.4. *If* k,ℓ,m,n *are nonnegative integers then*

$$L_m^k(x)\, L_n^\ell(x) = \sum_i (-1)^i\, c_i\, L_{m+n-i}^{k+\ell}(x)$$

with $c_i \geq 0$.

PROOF. Use Corollary 5.3, formula (2.2) and the identity

$$R_n^{(\alpha,\beta)}(-x) = \frac{(-1)^n(\beta+1)_n}{(\alpha+1)_n}\, R_n^{(\beta,\alpha)}(x). \quad \square$$

COROLLARY 5.5. *If* $\alpha \geq 0$, $0 \leq \lambda \leq 1$ *and* m,n *are nonnegative integers then*

$$L_m^\alpha(\lambda x)\, L_n^\alpha((1-\lambda)x) = \sum_i c_i(\lambda)\, L_{m+n-i}^\alpha(x)$$

with $c_i(\lambda) \geq 0$.

PROOF. It follows from Corollary 5.3 that for nonnegative integers m,n,p,k,ℓ,t we have

$$\int_0^t R_m^{(\alpha,kt)}(1-2t^{-1}x)\, R_n^{(\alpha,\ell t)}(1-2t^{-1}x)\, \cdot$$
$$\cdot\, R_p^{(\alpha(k+\ell)t)}(1-2t^{-1}x)\, x^\alpha (1-t^{-1}x)^{(k+\ell)t}\, dx \geq 0.$$

If $t \to \infty$ then it follows by (2.2) that

$$\int_0^\infty L_m^\alpha(kx)\, L_n^\alpha(\ell x)\, L_p^\alpha((k+\ell)x)\, x^\alpha e^{-(k+\ell)x}\, dx \geq 0.$$

Hence

$$\int_0^\infty L_m^\alpha(\lambda x)\ L_n^\alpha((1-\lambda)x)\ L_p^\alpha(x)\ x^\alpha e^{-x}\ dx \ge 0$$

for all rational λ, $0 \le \lambda \le 1$. By continuity this is also true for real λ, $0 \le \lambda \le 1$. □

6. APPLICATIONS TO CONNECTION COEFFICIENTS FOR JACOBI POLYNOMIALS AND DISK POLYNOMIALS

In this section we consider applications of Theorem 4.2.

COROLLARY 6.1. *Let* $a > b > -\frac{1}{2}$, $\alpha > \beta > -\frac{1}{2}$,

$$R_n^{(a,b)}(x) = \sum_{m=0}^{n} b_{n,m} R_m^{(\alpha,\beta)}(x). \tag{6.1}$$

If for all nonnegative integers k,j

$$(b-\beta)_j \int_{-1}^{1} R_k^{(a-b-1,b+2j)}(x)(1-x)^{\alpha-\beta-1}(1+x)^{\beta+j}\ dx \ge 0 \tag{6.2}$$

then the coefficients $b_{n,m}$ *in* (6.1) *are nonnegative.*

PROOF. It follows from Theorem 4.2 and the addition formula (2.8) that all $b_{n,m} \ge 0$ if

$$\int_{y=0}^{1} \int_{x=-y^{\frac{1}{2}}}^{y^{\frac{1}{2}}} R_{k,\ell}^{a-b-1,b-\frac{1}{2}}(x,y)(1-y)^{\alpha-\beta-1}\ \cdot$$

$$\cdot\ (y-x^2)^{\beta-\frac{1}{2}}dx\ dy \ge 0 \quad \text{for all } k,\ell.$$

By (2.5) this is equivalent to

$$\int_{-1}^{1} R_\ell^{(a-b-1,b+k-\ell)}(y)(1-y)^{\alpha-\beta-1}(1+y)^{\beta+\frac{1}{2}(k-\ell)}dy\ \cdot$$

$$\cdot \int_{-1}^{1} R_{k-\ell}^{(b-\frac{1}{2},b-\frac{1}{2})}(x)(1-x^2)^{\beta-\frac{1}{2}}\ dx \ge 0.$$

The second integral is zero if $k-\ell$ is odd and it has the same sign as $(b-\beta)_{\frac{1}{2}(k-\ell)}$ if $k-\ell$ is even (cf. ASKEY [2, (7.34)]). □

To a large extent the problem on the positivity of the coefficients $b_{n,m}$ in (6.1) has been solved by ASKEY & GASPER [3]. It does not seem that our reduction of this problem to inequality (6.2) is a big improvement compared with the methods of proof used in [3].
However, in the particular case that $a = 2\alpha + 1$, $b = 2\beta + 1$, $\alpha > \beta > -\frac{1}{2}$, the inequality (6.2) immediately follows from Lemma 2.1. Note that, in a certain sense, the point $(2\alpha+1, 2\beta+1)$ is extreme in the set of all (a,b) such that all $b_{n,m} \geq 0$, cf. [3, Theorem 2].

COROLLARY 6.2. *If* $\alpha \geq 0$, $-1 < \beta \leq \frac{1}{2}\alpha - \frac{1}{2}$ *and if* n,k *are nonnegative then*

$$(6.3) \qquad x^k R_n^{(\alpha,k)}(2x^2-1) = \sum_{i=0}^{[n+\frac{1}{2}k]} b_i \, R_{2n+k-2i}^{(\beta,\beta)}(x)$$

with $b_i \geq 0$.

PROOF. If $-1 < \beta < \frac{1}{2}\alpha - \frac{1}{2}$ then $R_n^{(\frac{1}{2}\alpha-\frac{1}{2},\frac{1}{2}\alpha-\frac{1}{2})}(x)$ is a linear combination with positive coefficients of polynomials $R_k^{(\beta,\beta)}(x)$ (cf. ASKEY [2, (7.34)]). Hence it is sufficient to give a proof in the case $\beta = \frac{1}{2}\alpha - \frac{1}{2}$. By (2.3) the left-hand side of (6.3) equals $R^{\alpha}_{n+k,n}(x)$, where x is real. It follows from Theorem 4.2 and the addition formulas (2.9) and (2.7) that the coefficients b_i in (6.3) are nonnegative if

$$\int_{-1}^{1} R_{i,j}^{\alpha-1}(x)(1-x^2)^{\frac{1}{2}\alpha-1}\,dx \geq 0 \quad \text{for all } i,j,$$

or, equivalently, if

$$\int_{-1}^{1} R_m^{(\alpha-1,2\ell)}(x)(1-x)^{\frac{1}{2}\alpha-1}(1+x)^{\ell-\frac{1}{2}}\,dx \geq 0$$

for all m,ℓ. Lemma 2.1 shows that this last equality is valid. □

The above result was proved by DUNKL [5] in the cases $\alpha = 0,1,2,\ldots$ by using the group theoretic interpretation. As a corollary ASKEY [1] derived that the function

$$\{(1-r)(1-s)(1-t)\,[(1-r)(1-s)(1+t) + (1-r)(1+s)(1-t) + (1+r)(1-s)(1-t)]\}^{-\frac{1}{2}\alpha-\frac{1}{2}}$$

has nonnegative power series coefficients for $\alpha = 0,1,2,\ldots$. Because of our Corollary 6.2 Askey's result is valid for all real $\alpha \geq 0$.

REFERENCES

[1] ASKEY, R., *Certain rational functions whose power series have positive coefficients, II*, SIAM J. Math. Anal. 5 (1974), 53-57.

[2] ASKEY, R., *Orthogonal polynomials and special functions*, Regconal Conference Series in Applied Mathematics, No. 21, Society for Industrial and Applied Mathematics, Philadelphia, 1975.

[3] ASKEY, R. & G. GASPER, *Jacobi polynomial expansions of Jacobi polynomials with nonnegative coefficients*, Proc. Camb. Phil. Soc. 70 (1971), 243-255.

[4] DELSARTE, P., *An algebraic approach to the association schemes of coding theory*, Philips Res. Repts Suppl. 1973, No. 10.

[5] DUNKL, C.F., *An expansion in ultraspherical polynomials with nonnegative coefficients*, SIAM J. Math. Anal. 5 (1974), 51-52.

[6] ERDéLYI, A., W. MAGNUS, F. OBERHETTINGER & F.G. TRICOMI, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.

[7] GASPER, G., *Linearization of the product of Jacobi polynomials, I*, Canad. J. Math. 22 (1970), 171-175.

[8] GASPER, G., *Linearization of the product of Jacobi polynomials, II*, Canad. J. Math. 22 (1970), 582-593.

[9] GASPER, G., *Positivity and special functions*, in "Theory and application of special functions" (R. Askey, ed.), pp. 375-433, Academic Press, New York, 1975.

[10] KOORNWINDER, T.H., *The addition formula for Jacobi polynomials, I, Summary of results*, Indag. Math. 34 (1972), 188-191.

[11] KOORNWINDER, T.H., *The addition formula for Jacobi polynomials, III. Completion of the proof*, Math. Centrum Amsterdam, Report TW 135 (1972).

[12] KOORNWINDER, T.H., *The addition formula for Jacobi polynomials and spherical harmonics*, SIAM J. Appl. Math. 25 (1973), 236-246.

[13] KOORNWINDER, T.H., *Jacobi polynomials, III. An analytic proof of the addition formula*, SIAM J. Math. Anal. 6 (1975), 533-543.

[14] KOORNWINDER, T.H., *Yet another proof of the addition formula for Jacobi polynomials*, Math. Centrum Amsterdam, Report TW 150 (1975), part I, also to appear in J. Math. Anal. Appl.

[15] ŠAPIRO, R.L., *Special functions related to representations of the group group* SU(n), *of class I with respect to* SU(n-1) ($n \geq 3$) (Russian), Izv. Vysš. Učeb. Zaved. Matematika (1968), no. 4 (71), pp. 97-107.