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AFDELING TOEGEPASTE WISKUNDE  
(DEPARTMENT OF APPLIED MATHEMATICS)

TW 163/76

NOVEMBER

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A JACOBI SERIES EXPANSION WITH NONNEGATIVE COEFFICIENTS  
RELATED TO A SPECIAL CLASS OF ORTHOGONAL POLYNOMIALS IN  
TWO VARIABLES

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*Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.*

*The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.*

A Jacobi series expansion with nonnegative coefficients related to a special class of orthogonal polynomials in two variables

by

I.G. Sprinkhuizen-Kuyper

ABSTRACT

A special class of orthogonal polynomials in two variables is considered for which the region of orthogonality is bounded by two straight lines and a parabola. It is proved that these polynomials when restricted to the parabolic boundary line of its orthogonality region, have a certain Jacobi series expansion with nonnegative coefficients. As a special case Gasper's positivity result for the linearization coefficients of the product of two Jacobi polynomials is obtained.

KEY WORDS & PHRASES: *orthogonal polynomials in two variables, linearization coefficients for the product of two Jacobi polynomials, Appell's hypergeometric function  $F_4$ .*



## 1. INTRODUCTION

Let  $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$  denote the orthogonal polynomial in two variables of degree  $(n,k)$  and order  $(\alpha,\beta,\gamma)$  as defined in KOORNWINDER & SPRINKHUIZEN [7]. These polynomials are orthogonal over a region bounded by two straight lines and a parabola. On each of the three parts of the boundary of the region of orthogonality there exists a natural expansion in Jacobi polynomials of certain order. On the straight lines the coefficients in these expansions are explicitly known and, in particular, they are nonnegative if the parameters  $\alpha,\beta$  and  $\gamma$  satisfy some inequalities (cf. [7, section 6]). This nonnegativity and the property of Jacobi polynomials  $R_n^{(\alpha,\beta)}(x)$  that

$$|R_n^{(\alpha,\beta)}(x)| \leq R_n^{(\alpha,\beta)}(1),$$

with certain conditions on  $\alpha$  and  $\beta$ , results in the property that

$$|R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)| \leq R_{n,k}^{\alpha,\beta,\gamma}(0,0),$$

for  $(\xi,\eta)$  on the straight boundary lines of the region of orthogonality if  $\alpha,\beta,\gamma$  satisfy the inequalities  $\alpha \geq \beta$ ,  $\gamma \geq -\frac{1}{2}$  and  $\max(\alpha,\beta+\gamma+\frac{1}{2}) \geq -\frac{1}{2}$ .

The main purpose of this paper is to prove the nonnegativity of the expansion coefficients on the parabolic boundary line. It is sufficient to give the proof for the polynomials  $R_{n,n}^{\alpha,\beta,\gamma}(\xi,\eta)$ ; the coefficients corresponding to  $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$  then follow from a simple recurrence relation.

The positivity result is proved in section 3. The method followed here is quite similar to that used by GASPER [3] to prove the nonnegativity of the linearization coefficients for the product of two Jacobi polynomials. He used a three terms recurrence relation which was obtained by HYLLERAAS [5] from a fifth order differential equation for the product of two Jacobi polynomials. Remarking that  $R_{n,n}^{\alpha,\beta,\gamma}(\xi,\eta)$  can be expressed as an Appell's function  $F_4$  (cf. KOORNWINDER & SPRINKHUIZEN [7]), we obtain a three terms recurrence relation from a third order differential equation which was derived by APPELL & KAMPÉ DE FÉRIET [1] for the restriction of Appell's function  $F_4$  to the parabolic singular line of its differential equations. The linearization coefficients of the product of two Jacobi polynomials

are included as a special case ( $\gamma = -\frac{1}{2}$ ) of the expansion coefficients of  $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$  on the parabolic boundary line of its region of orthogonality. So the nonnegativity of these coefficients is proved again and the proof given here has some minor simplifications with respect to that given by Gasper.

## 2. PRELIMINARIES

### 2.1. APPELL'S FUNCTION $F_4$

The hypergeometric function  $F_4(a,b;c,c';x,y)$  of two variables is defined by

$$(2.1) \quad F_4(a,b;c,c';x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a)_{i+j} (b)_{i+j}}{(c)_i (c')_j i! j!} x^i y^j,$$

$$a,b,c,c' \in \mathbb{C}, c,c' \neq 0,-1,-2,\dots, x,y \in \mathbb{C}, |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1,$$

cf. APPELL & KAMPÉ DE FÉRIET [1] or SLATER [8]. It is a solution of the set of partial differential equations

$$(2.2a) \quad x(1-x)z_{xx} - 2xy z_{xy} - y^2 z_{yy} + (c-(a+b+1)x)z_x - (a+b+1)yz_y - abz = 0,$$

$$(2.2b) \quad -x^2 z_{xx} - 2xy z_{xy} + y(1-y)z_{yy} - (a+b+1)xz_x + (c'-(a+b+1)y)z_y - abz = 0.$$

These differential equations admit four linearly independent solutions as long as we avoid the singular lines  $x = 0$ ,  $y = 0$ ,  $(1-x-y)^2 = 4xy$ ,  $x = \infty$ ,  $y = \infty$ . As remarked by APPELL & KAMPÉ DE FÉRIET [1, Part I, Chapter V], the function  $z(x,f(x))$  is a solution of a linear differential equation of at most fourth order. If the function  $y = f(x)$  is the equation of one of the singular lines a differential equation of lower order is obtained. In particular if  $y = 0$  then (2.2a) results in a second order hypergeometric

differential equation. For our purpose we need a differential equation for Appell's function  $F_4$  restricted to the parabolic singular line  $(1-x-y)^2 = 4xy$ , which can be parametrized by  $x = t^2$ ,  $y = (1-t)^2$ . For  $F_4(a, b; c, c'; t^2, (1-t)^2)$  APPELL & KAMPÉ DE FÉRIET [1, Note II] derived the following differential equation

$$(2.3) \quad t^2(1-t)^2 \frac{d^3 z}{dt^3} + t(1-t)(L-Mt) \frac{d^2 z}{dt^2} + (N-Pt+Qt^2) \frac{dz}{dt} + 4ab(R-ST)z = 0,$$

$$\begin{aligned} L &= a + b + 2c - c' & , & \quad M = 2 + 2a + 2b + c + c', & \quad N = (2c-1)(1+a+b-c') \\ P &= 4ab + 2c(2a+2b+1), & \quad Q &= 4ab + (2a+2b+1)(c+c'), \\ R &= \frac{1}{2} - c & , & \quad S = 1 - c - c'. \end{aligned}$$

## 2.2. PROPERTIES OF A SPECIAL CLASS OF ORTHOGONAL POLYNOMIALS IN TWO VARIABLES

In the following  $R_n^{(\alpha, \beta)}(x)$  denotes a Jacobi polynomial normalized such that  $R_n^{(\alpha, \beta)}(1) = 1$ . For Jacobi polynomials see SZEGÖ [10, Chapter 4] or ERDÉLYI [2].

Let  $\Omega$  be the region

$$\Omega := \{(\xi, \eta) \mid \eta > 0, 1 - \xi + \eta > 0, \xi^2 - 4\eta > 0, 0 < \xi < 2\}.$$

Let

$$w_{\alpha, \beta, \gamma}(\xi, \eta) := \eta^\alpha (1-\xi+\eta)^\beta (\xi^2-4\eta)^\gamma, \quad (\xi, \eta) \in \Omega.$$

**DEFINITION 2.1.** Let  $\alpha, \beta, \gamma > -1$ ,  $\alpha + \gamma + \frac{3}{2} > 0$ ,  $\beta + \gamma + \frac{3}{2} > 0$ . Let  $n, k$  be integers,  $n \geq k \geq 0$ . Then  $R_{n,k}^{\alpha, \beta, \gamma}(\xi, \eta)$  is a linear combination of monomials  $1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^3, \xi^2\eta, \dots, \xi^n, \xi^{n-1}\eta, \dots, \xi^{n-k}\eta^k$  such that

$$(i) \quad \iint_{\Omega} R_{n,k}^{\alpha, \beta, \gamma}(\xi, \eta) \xi^{m-\ell} \eta^\ell w_{\alpha, \beta, \gamma}(\xi, \eta) d\xi d\eta = 0$$

if  $m \geq \ell$  and if either  $m < n$  or  $m = n$ ,  $\ell < k$ ;

$$(ii) \quad R_{n,k}^{\alpha, \beta, \gamma}(0,0) = 1.$$

In definition 2.1 a special class of orthogonal polynomials in two variables is defined. This class of orthogonal polynomials is studied in KOORNWINDER [6], SPRINKHUIZEN [9] and, most recently, in KOORNWINDER & SPRINKHUIZEN [7]. Two formulas which give explicit expressions for  $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$  with some restriction on the parameters or the degree are (cf. [7, (3.4) and (7.15)])

$$(2.4) \quad R_{n,k}^{\alpha,\beta,-\frac{1}{2}}(X+Y,XY) = \frac{1}{2}\{R_n^{(\alpha,\beta)}(1-2X)R_k^{(\alpha,\beta)}(1-2Y) + R_k^{(\alpha,\beta)}(1-2X)R_n^{(\alpha,\beta)}(1-2Y)\},$$

$$(2.5) \quad R_{n,n}^{\alpha,\beta,\gamma}(X+Y,XY) = \frac{(-1)^{n(\beta+1)}}{(\alpha+\gamma+\frac{3}{2})_n} {}_nF_4(-n, n+\alpha+\beta+\gamma+\frac{3}{2}; \alpha+1, \beta+1; XY, (1-X)(1-Y)).$$

On the parabolic boundary line of the region of orthogonality  $\Omega$  the following Jacobi series expansion for the polynomial  $R_{n,k}^{\alpha,\beta,\gamma}(\xi,\eta)$  is quite natural (cf. [7, section 5])

$$(2.6) \quad R_{n,k}^{\alpha,\beta,\gamma}(2t,t^2) = \sum_{m=n-k}^{n+k} b_{n,k;m}^{\alpha,\beta,\gamma} R_m^{(\alpha+\gamma+\frac{1}{2}, \beta+\gamma+\frac{1}{2})}(1-2t).$$

For the coefficients  $b_{n,k,m}^{\alpha,\beta,\gamma}$  the following relation holds

$$(2.7) \quad b_{n,k;m}^{\alpha,\beta,\gamma} = b_{k,k;m-n+k}^{\alpha,\beta,\gamma+n-k}.$$

The aim of this paper is to prove the positivity of the coefficients  $b_{n,k;m}^{\alpha,\beta,\gamma}$ . By (2.7) it is clear that it is sufficient to prove the positivity of  $b_{n,n;m}^{\alpha,\beta,\gamma}$ .

Formulas (2.4) and (2.6) result in

$$(2.8) \quad R_n^{(\alpha,\beta)}(1-2t)R_k^{(\alpha,\beta)}(1-2t) = \sum_{m=n-k}^{n+k} b_{n,k;m}^{\alpha,\beta,-\frac{1}{2}} R_m^{(\alpha,\beta)}(1-2t).$$

Thus the coefficients  $b_{n,k;m}^{\alpha,\beta,-\frac{1}{2}}$  are just the linearization coefficients of the product of two Jacobi polynomials.

In the following section we will study the expansion of  $R_{n,n}^{\alpha,\beta,\gamma}(2t,t^2)$  in Jacobi polynomials of order  $(\alpha+\gamma+\frac{1}{2}, \beta+\gamma+\frac{1}{2})$ . We summarize the following formulas for



$$y_m(t) = R_m^{(\alpha+\gamma+\frac{1}{2}, \beta+\gamma+\frac{1}{2})}(1-2t),$$

which are needed in the remainder of this paper

$$(2.9) \quad y_m(t) = {}_2F_1(-m, m+\alpha+\beta+2\gamma+2; \alpha+\gamma+\frac{3}{2}; t) \\ = \sum_{k=0}^m \frac{(-m)_k (m+\alpha+\beta+2\gamma+2)_k}{(\alpha+\gamma+\frac{3}{2})_k k!} t^k,$$

$$(2.10) \quad t(1-t)y_m'' + (\alpha+\gamma+\frac{3}{2} - (\alpha+\beta+2\gamma+3)t)y_m' \\ + m(m+\alpha+\beta+2\gamma+2)y_m = 0,$$

$$(2.11) \quad t(1-t)y_m''' + (\alpha+\gamma+\frac{5}{2} - (\alpha+\beta+2\gamma+5)t)y_m'' \\ + (m-1)(m+\alpha+\beta+2\gamma+3)y_m' = 0,$$

$$(2.12) \quad t(1-t)y_m' = \frac{m(m+\beta+\gamma+\frac{1}{2})}{(2m+\alpha+\beta+2\gamma+1)} y_m - mt y_m - \frac{m(m+\beta+\gamma+\frac{1}{2})}{(2m+\alpha+\beta+2\gamma+1)} y_{m-1},$$

$$(2.13) \quad t y_m = - \frac{(m+\alpha+\gamma+\frac{3}{2})(m+\alpha+\beta+2\gamma+2)}{(2m+\alpha+\beta+2\gamma+2)(2m+\alpha+\beta+2\gamma+3)} y_{m+1} \\ + \frac{1}{2} \left( 1 + \frac{(\alpha-\beta)(\alpha+\beta+2\gamma+1)}{(2m+\alpha+\beta+2\gamma+1)(2m+\alpha+\beta+2\gamma+3)} \right) y_m \\ - \frac{m(m+\beta+\gamma+\frac{1}{2})}{(2m+\alpha+\beta+2\gamma+1)(2m+\alpha+\beta+2\gamma+2)} y_{m-1}.$$

### 3. THE POSITIVITY OF CERTAIN COEFFICIENTS RELATED TO THE LINEARIZATION COEFFICIENTS OF THE PRODUCT OF TWO JACOBI POLYNOMIALS

Let us consider

$$(3.1) \quad R_{n,n}^{\alpha,\beta,\gamma}(2t, t^2) = \sum_{m=0}^{2n} b_{n,n;m}^{\alpha,\beta,\gamma} R_m^{(\alpha+\gamma+\frac{1}{2}, \beta+\gamma+\frac{1}{2})}(1-2t).$$

LEMMA 3.1. *If  $m = 0$  or  $m = 2n$  the coefficients  $b_{n,n;m}^{\alpha,\beta,\gamma}$  are given by*

$$(3.2) \quad b_{n,n;0}^{\alpha,\beta,\gamma} = \frac{(\gamma+\frac{3}{2})_n (n+\alpha+\beta+1)_n (\beta+\gamma+\frac{3}{2})_n}{(\alpha+1)_n (\alpha+\beta+2\gamma+3)_{2n}},$$

$$(3.3) \quad b_{n,n;2n}^{\alpha,\beta,\gamma} = \frac{(n+\alpha+\gamma+\frac{3}{2})_n (n+\alpha+\beta+1)_n (n+\alpha+\beta+\gamma+\frac{3}{2})_n}{(\alpha+1)_n (2n+\alpha+\beta+2\gamma+2)_{2n}}.$$

PROOF. First observe that

$$\begin{aligned} b_{k,k;0}^{\alpha,\beta,n-k-\frac{1}{2}} &= b_{n,k;n-k}^{\alpha,\beta,-\frac{1}{2}} = \\ &= \frac{\int_0^1 R_n^{(\alpha,\beta)}(1-2t) R_k^{(\alpha,\beta)}(1-2t) R_{n-k}^{(\alpha,\beta)}(1-2t) t^\alpha (1-t)^\beta dt}{\int_0^1 (R_{n-k}^{(\alpha,\beta)}(1-2t))^2 t^\alpha (1-t)^\beta dt} \end{aligned}$$

can be evaluated. It follows that (3.2) holds if  $\gamma = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots$ . For general  $\gamma$  we have

$$b_{n,n;0}^{\alpha,\beta,\gamma} = \frac{\int_0^1 R_{n,n}^{\alpha,\beta,\gamma}(2t, t^2) t^{\alpha+\gamma+\frac{1}{2}} (1-t)^{\beta+\gamma+\frac{1}{2}} dt}{\int_0^1 t^{\alpha+\gamma+\frac{1}{2}} (1-t)^{\beta+\gamma+\frac{1}{2}} dt}.$$

If  $\alpha, \beta$  and  $n$  are fixed then  $b_{n,n;0}^{\alpha,\beta,\gamma}$  is rational in  $\gamma$  by (2.5) and the right hand side of (3.2) is also rational in  $\gamma$ . Hence (3.2) holds for all  $\gamma$ . Formula (3.2) also can be proved by restricting the recurrence relation

$$(1-2\xi+4\eta) R_{n+1,n-1}^{\alpha,\beta,\gamma}(\xi, \eta) = \sum_{m,\ell} c_{m,\ell} R_{m,\ell}^{\alpha,\beta,\gamma}(\xi, \eta)$$

(cf. [9, section 9,  $u = 1 - 2\xi + 4\eta$ ]) to the boundary line  $\xi = 2t, \eta = t^2$ .

By comparing the coefficients of  $R_0^{(\alpha+\gamma+\frac{1}{2}, \beta+\gamma+\frac{1}{2})}(1-2t)$  and by remarking that the right hand side only contributes when  $(m, \ell) = (n, n)$ , a two term recurrence relation is obtained which connects  $b_{n+1,n-1;2}^{\alpha,\beta,\gamma} = b_{n-1,n-1;0}^{\alpha,\beta,\gamma+2}$  with  $b_{n,n;0}^{\alpha,\beta,\gamma}$ .

This recurrence relation together with  $b_{0,0;0} = 1$  immediately results in (3.2). Formula (3.3) follows by comparing the coefficients of  $t^{2n}$  in (3.1) with use of (2.5) and (2.9).  $\square$

**THEOREM 3.2.** Let  $\alpha, \beta, \gamma$  satisfy the inequalities of definition 2.1. If  $\alpha + \beta + 1 \geq 0$  and  $\alpha \geq \beta$  then the coefficients  $b_{n,n;m}^{\alpha, \beta, \gamma}$  in the Jacobi series expansion (3.1) are nonnegative.

**PROOF.** Using (2.5) we have

$$R_{n,n}^{\alpha, \beta, \gamma}(2t, t^2) = \frac{(-1)^n (\beta+1)_n}{(\alpha+\gamma+\frac{3}{2})_n} F_4(-n, n+\alpha+\beta+\gamma+\frac{3}{2}; \alpha+1, \beta+1; t^2, (1-t)^2)$$

Thus  $R_{n,n}^{\alpha, \beta, \gamma}(2t, t^2)$  is a solution of the differential equation (2.3) with  $a = -n$ ,  $b = n + \alpha + \beta + \gamma + \frac{3}{2}$ ,  $c = \alpha + 1$  and  $c' = \beta + 1$ . Application of the left hand side of (2.3) on  $y_m(t) = R_m^{\alpha+\gamma+\frac{1}{2}, \beta+\gamma+\frac{1}{2}}(1-2t)$  results in (use (2.10), (2.11), (2.12) and (2.13))

$$\begin{aligned} & \frac{(m+\alpha+\beta+1)(m+\alpha+\gamma+\frac{3}{2})(m+\alpha+\beta+2\gamma+2)(2n-m)(2n+m+2\alpha+2\beta+2\gamma+3)}{(2m+\alpha+\beta+2\gamma+2)(2m+\alpha+\beta+2\gamma+3)} y_{m+1} \\ & + \frac{1}{2}(\alpha-\beta) \left[ \frac{(m+1)(m+2\gamma+2)(2n-m)(2n+m+2\alpha+2\beta+2\gamma+3)}{(2m+\alpha+\beta+2\gamma+3)} \right. \\ & \quad \left. - \frac{m(m+2\gamma+1)(2n-m+1)(2n+m+2\alpha+2\beta+2\gamma+2)}{(2m+\alpha+\beta+2\gamma+1)} \right] y_m \\ & - \frac{m(m+\beta+\gamma+\frac{1}{2})(m+2\gamma+1)(2n-m+\alpha+\beta+1)(2n+m+\alpha+\beta+2\gamma+2)}{(2m+\alpha+\beta+2\gamma+1)(2m+\alpha+\beta+2\gamma+2)} y_{m-1}. \end{aligned}$$

Thus in order that the right hand side of (3.1) is a solution of (2.3) the coefficients  $b_{n,n;m}^{\alpha, \beta, \gamma}$  must obey the following recurrence relation:

$$(3.4) \quad A_{n;m}^{\alpha, \beta, \gamma} b_{n,n;m-1}^{\alpha, \beta, \gamma} + B_{n;m}^{\alpha, \beta, \gamma} b_{n,n;m}^{\alpha, \beta, \gamma} = C_{n;m}^{\alpha, \beta, \gamma} b_{n,n;m+1}^{\alpha, \beta, \gamma},$$

with

$$A_{n,m}^{\alpha, \beta, \gamma} = \frac{(m+\alpha+\beta)(m+\alpha+\gamma+\frac{1}{2})(m+\alpha+\beta+2\gamma+1)(2n-m+1)(2n+m+2\alpha+2\beta+2\gamma+2)}{(2m+\alpha+\beta+2\gamma)(2m+\alpha+\beta+2\gamma+1)},$$

$$B_{n;m}^{\alpha,\beta,\gamma} = \frac{1}{2}(\alpha-\beta) \left[ \frac{(m+1)(m+2\gamma+2)(2n-m)(2n+m+2\alpha+2\beta+2\gamma+3)}{(2m+\alpha+\beta+2\gamma+3)} \right. \\ \left. - \frac{m(m+2\gamma+1)(2n-m+1)(2n+m+2\alpha+2\beta+2\gamma+2)}{(2m+\alpha+\beta+2\gamma+1)} \right],$$

$$C_{n;m}^{\alpha,\beta,\gamma} = \frac{(m+1)(m+\beta+\gamma+2)(m+2\gamma+2)(2n-m+\alpha+\beta)(2n+m+\alpha+\beta+2\gamma+3)}{(2m+\alpha+\beta+2\gamma+3)(2m+\alpha+\beta+2\gamma+4)}.$$

In addition with  $b_{n,n;-1}^{\alpha,\beta,\gamma} = b_{n,n;2n+1}^{\alpha,\beta,\gamma} = 0$  and  $b_{n,n;0}^{\alpha,\beta,\gamma}$  and  $b_{n,n;2n}^{\alpha,\beta,\gamma}$  as given by (3.2) and (3.3) this recurrence relation completely defines the coefficients  $b_{n,n;m}^{\alpha,\beta,\gamma}$ . In the remainder of the proof the following lemma is needed.

**LEMMA 3.3.** *Let  $\alpha, \beta$  and  $\gamma$  satisfy the inequalities of theorem 3.2. Then there exists an integer  $M_0$ ,  $0 < M_0 < 2n$ , such that  $B_{n;m}^{\alpha,\beta,\gamma} \geq 0$  for  $m = 0, 1, \dots, M_0$  and  $B_{n;m}^{\alpha,\beta,\gamma} < 0$  for  $m = M_0 + 1, \dots, 2n$ .*

**PROOF of lemma 3.3.** We will give two different proofs. The first is equivalent to GASPER's [3, pp. 174, 175]. The second proof, which only holds if  $\gamma \geq -\frac{1}{2}$ , is due to Koornwinder. It exploits the fact that  $B_{n;m}^{\alpha,\beta,\gamma}$  is given as the difference of two related rational functions.

If  $m = 0$  or  $m = 2n$  we have respectively

$$(3.5) \quad B_{n;0}^{\alpha,\beta,\gamma} = \frac{(\alpha-\beta)(2\gamma+2)n(2n+2\alpha+2\beta+2\gamma+3)}{(\alpha+\beta+2\gamma+3)} \geq 0,$$

$$(3.6) \quad B_{n;2n}^{\alpha,\beta,\gamma} = -\frac{2(\alpha-\beta)n(2n+2\gamma+1)(2n+\alpha+\beta+\gamma+1)}{(4n+\alpha+\beta+2\gamma+1)} \leq 0.$$

Let us write

$$(3.7) \quad B_{n;m}^{\alpha,\beta,\gamma} = \frac{1}{2}(\alpha-\beta) \frac{F(m-1)}{(2m+\alpha+\beta+2\gamma+1)(2m+\alpha+\beta+2\gamma+3)}$$

and temporarily use  $M = m - 1$ ,  $a = \alpha + \beta + 1$ ,  $c = \gamma + 1$ , then  $M = -1, \dots, 2n-1$ ,  $a \geq 0$  and  $c > 0$ ,

$$F(M) = (M+2)(M+2c+1)(2n-M-1)(2n+M+2a+2c)(2M+a+2c) \\ - (M+1)(M+2c)(2n-M)(2n+M+2a+2c-1)(2M+a+2c+2) \\ = -6M^4 - 12(a+2c+1)M^3 + 2[4n(n+a+c-\frac{1}{2}) - (3a^2 + (16c+11)a + 16c^2 + 20c+3)]M^2 + \\ + 2(a+2c+1)[4n(n+a+c-\frac{1}{2}) - (4ac+4c^2+5a+8c)]M + \\ + 4[n(n-1)(4c^2+2(c+1)a) + (n-1)(a+c+\frac{1}{2})(4c^2+2(c+1)a) + a^2 + a].$$

Notice that the coefficients of  $M^4$  and  $M^3$  are negative and the constant term is positive (for  $a > 0$ , and  $n = 1, 2, \dots$ ; the case  $n = 0$  is trivial, and the case  $a = 0$  follows from analytical continuation). Denoting the coefficient of  $M^k$  in  $F(M)$  by  $\text{coef}(M^k)$  we obtain

$$(3.8) \quad \text{coef}(M) - (a+2c+1) \text{coef}(M^2) = 6(a+2c+1)^3 > 0.$$

If  $\text{coef}(M^2) \leq 0$ , then it is obvious that  $F(M)$  has only one variation of sign for  $M > 0$ . If  $\text{coef}(M^2) > 0$ , then, by (3.8),  $\text{coef}(M) > 0$  and thus again  $F(M)$  has only one variation of sign for  $M > 0$ . Consequently  $F(M)$  has exactly one positive root in the interval  $(0, 2n-1)$  (temporarily considering  $M$  as a real variable) and hence there exists a positive integer  $M_0 \in (0, 2n-1)$  depending on  $n$ ,  $a$  and  $c$  such that  $F(M) \geq 0$ ,  $M = 0, 1, \dots, M_0-1$  and  $F(M) < 0$ ,  $M = M_0, \dots, 2n-1$ . Therefore by (3.5), (3.6) and (3.7) the lemma is proved.

For the second proof observe that

$$B_{n;m}^{\alpha, \beta, \gamma} = -6(\alpha - \beta) \left\{ \frac{G(m+1)}{G'''(m+1)} - \frac{G(m)}{G'''(m)} \right\},$$

where

$$\begin{aligned} G(x) &:= x(x+2\gamma+1)(x-2n-1)(x+2n+2\alpha+2\beta+2\gamma+2), \\ G'''(x) &= 24(x+\frac{1}{2}\alpha+\frac{1}{2}\beta+\gamma+\frac{1}{2}). \end{aligned}$$

Since

$$\left( \frac{G(x)}{G'''(x)} \right)' = \frac{G'''(x)G'(x) - 24G(x)}{(G'''(x))^2}$$

it is clearly sufficient to prove that the fourth degree polynomial

$$(*) \quad H(x) := G'''(x)G'(x) - 24G(x)$$

changes sign at most once in  $(1, 2n+1)$ . Suppose that  $\gamma \geq -\frac{1}{2}$ ,  $\alpha + \beta + 1 \geq 0$ ,  $n \geq 1$ . Then  $G(x)$  has zeros  $x_1 < x_2 \leq x_3 < x_4$  with  $x_1 := -2n - 2\alpha - 2\beta - 2\gamma - 2$ ,  $x_2 := -2\gamma - 1$ ,  $x_3 := 0$ ,  $x_4 := 2n + 1$  and  $G'''(x)$  has zero  $y := -\frac{1}{2}(\alpha + \beta + 2\gamma + 1)$ . Observe that  $x_1 < y < 1$ . Now suppose that  $H(x)$  changes sign at least twice on  $(1, 2n+1)$ .

Since  $H'(x) = G'''(x)G''(x)$  there exists  $A > 0$  such that

$$H(x) = \frac{1}{2}(G''(x)-A)(G''(x)+A)$$

Hence, if  $H(z) = 0$  then  $H(2y-z) = 0$ . Thus  $H(x)$  has four distinct real zeros  $z_1, z_2, z_3, z_4$  such that  $z_1 < z_2 < y < 1 < z_3 < z_4$  and

$$z_{1,2} = -z_{4,3}^{-\alpha-\beta-2\gamma-1} < -2\gamma-1 = x_2.$$

Now using that  $H(x_1) > 0$ ,  $H(x_4) > 0$  (by (\*)) and that  $H'(x) = G'''(x)G''(x)$  has three zeros on  $(x_1, x_4)$  we conclude that  $x_1 < z_1 < z_2 < x_2$ . We already supposed that  $x_3 < z_3 < z_4 < x_4$ . It follows that  $H(x_i) > 0$  for  $i = 1, 2, 3, 4$ . However, by (\*),  $H(x_i)$  and  $H(x_{i+1})$  can have the same sign only if  $x_i < y < x_{i+1}$ . This is clearly impossible for all three cases  $i = 1, 2, 3$ .  $\square$

PROOF of theorem 3.2 (continuation).

Considering the coefficients in the recurrence relation (3.4) we can remark the following. It is clear that  $A_{n;1}^{\alpha,\beta,\gamma} \geq 0$  and  $A_{n;m}^{\alpha,\beta,\gamma} > 0$  for  $m = 2, 3, \dots, 2n$  and  $C_{n;m}^{\alpha,\beta,\gamma} > 0$  for  $m = 0, 1, \dots, 2n-2$  and  $C_{n;2n-1}^{\alpha,\beta,\gamma} \geq 0$ . From lemma 3.3 it

follows that  $B_{n;m}^{\alpha,\beta,\gamma} \geq 0$ ,  $m = 0, 1, \dots, M_0$  and  $B_{n;m}^{\alpha,\beta,\gamma} < 0$ ,  $m = M_0+1, \dots, 2n$ .

From lemma 3.1 it is clear that  $b_{n,n;0}^{\alpha,\beta,\gamma} > 0$  and  $b_{n,n;2n}^{\alpha,\beta,\gamma} > 0$ . By successive applications of (3.4) we obtain  $b_{n,n;m}^{\alpha,\beta,\gamma} \geq 0$  if  $m = 1, 2, \dots, M_0$  and (trans-

posing the term with  $b_{n,n;m}^{\alpha,\beta,\gamma}$  to the other side of the equal sign)

$$b_{n,n;m}^{\alpha,\beta,\gamma} \geq 0, \quad m = 2n-1, 2n-2, \dots, M_0+1. \quad \square$$

Remark 3.4. The recurrence relation (3.4) is equivalent to that used by GASPER [3, (5)] to prove the positivity of the linearization coefficients of the product of two Jacobi polynomials: At the one hand (3.4) results in Gasper's recurrence relation because use of (2.7) yields

$$A_{k;m-n+k}^{\alpha,\beta,n-k-\frac{1}{2}} b_{n,k;m-1}^{\alpha,\beta,-\frac{1}{2}} + B_{k;m-n+k}^{\alpha,\beta,n-k-\frac{1}{2}} b_{n,k;m}^{\alpha,\beta,-\frac{1}{2}} = C_{k;m-n+k}^{\alpha,\beta,n-k-\frac{1}{2}} b_{n,k;m+1}^{\alpha,\beta,-\frac{1}{2}},$$

where  $b_{n,k;m}^{\alpha,\beta,-\frac{1}{2}}$  denotes the linearization coefficient for the product of two Jacobi polynomials (cf. (2.8)). At the other hand the recurrence relation

for the linearization coefficients of the Jacobi polynomials gives the relation (3.4) for the coefficients  $b_{n,n;m}^{\alpha,\beta,\gamma}$  for  $\gamma = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots$ . Now remarking that the coefficients  $b_{n,n;m}^{\alpha,\beta,\gamma}$  are rational with respect to  $\gamma$  (by (2.5)), and that the same holds for the coefficients generated by (3.4), it follows by analytical continuation that the recurrence relation (3.4) holds for all values of  $\gamma$ .

COROLLARY 3.5. *The coefficients  $b_{n,k}^{\alpha,\beta,\gamma}$  in the Jacobi series expansion (2.6) are nonnegative if  $\alpha \geq \beta$  and  $\alpha + \beta + 1 > 0$ .*

PROOF. Use theorem 3.2 and (2.7).  $\square$

A special case of corollary 3.5 is (cf. (2.8)).

COROLLARY 3.6. (=GASPER [3, theorem]).

*The linearization coefficients for the product of two Jacobi polynomials of order  $(\alpha, \beta)$  are nonnegative if  $\alpha \geq \beta$  and  $\alpha + \beta + 1 \geq 0$ .*

COROLLARY 3.7. *If  $\alpha \geq -\frac{1}{2}$ ,  $\alpha \geq \beta$ ,  $\alpha + \beta + 1 \geq 0$  and  $\gamma \geq -\frac{1}{2}$ , then the polynomials  $R_{n,k}^{\alpha,\beta,\gamma}(\xi, \eta)$  satisfy*

$$|R_{n,k}^{\alpha,\beta,\gamma}(\xi, \eta)| \leq R_{n,k}^{\alpha,\beta,\gamma}(0,0) = 1, \quad (\xi, \eta) \in \partial\Omega,$$

where  $\partial\Omega$  denotes the boundary of the region of orthogonality  $\Omega$  (cf. definition 2.1).

PROOF. This corollary follows from [7, corollary 6.11 and its proof] and the nonnegativity of the coefficients  $b_{n,k;m}^{\alpha,\beta,\gamma}$ .  $\square$

Remark 3.8. It is possible to refine the condition  $\alpha + \beta + 1 \geq 0$  in theorem 3.3 as is done by GASPER [4] in the case of the Jacobi polynomials ( $\gamma = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \dots$ ).

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