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THE ASYMPTOTIC EXPANSION OF THE INCOMPLETE GAMMA FUNCTIONS

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The Asymptotic Expansion of the Incomplete Gamma Functions *)

by

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ABSTRACT

Earlier investigations on uniform asymptotic expansions of the incomplete gamma functions are reconsidered. The new results include estimations for the remainder and the extension of the results to complex variables. Furthermore asymptotic expansions of the inverse functions are given.

KEY WORDS & PHRASES: *incomplete gamma function, asymptotic expansion, inverse incomplete gamma functions.*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

We consider the incomplete gamma functions ratios P and Q defined by

$$(1.1)\qquad P(a,x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1}\, e^{-t}\, dt, \qquad Q(a,x) = \frac{1}{\Gamma(a)} \int_x^\infty t^{a-1}\, e^{-t}\, dt.$$

We suppose first that x and a are real with

$$(1.2)\qquad x \ge 0, \quad a > 0.$$

In TEMME [5] we derived asymptotic expansions of P and Q for $a \to \infty$, uniformly valid for $x \ge 0$. In this paper we reconsider these expansions. Our new results concern the representations of the remainder in the asymptotic expansion, representations for the coefficients of the expansion for numerical applications, numerical upper bounds for the remainder of the case of real variables, and extension of the asymptotic expansions to the case of complex variables. Furthermore we give an asymptotic expansion of the inverse function.

To describe the expansions given in [5] we introduce the following parameters

$$(1.3)\qquad \lambda = x/a, \quad \mu = \lambda - 1, \quad \eta = \{2[\mu - \ln(1+\mu)]\}^{\frac{1}{2}},$$

with the convention that the square root has the sign of $\mu (\mu > -1)$. As a function of μ, η is monotonous and infinitely differentiable on $(-1,\infty)$. Analytic properties of $\eta(\mu)$ for complex μ are considered in §5.

The asymptotic expansions of P and Q derived in [5] follow from the representations

$$(1.4)\qquad P(a,x) = \tfrac{1}{2}\,\mathrm{erfc}[-\eta(a/2)^{\frac{1}{2}}] - R_a(\eta), \qquad Q(a,x) = \tfrac{1}{2}\,\mathrm{erfc}[\eta(a/2)^{\frac{1}{2}}] + R_a(\eta)$$

with

$$(1.5) \qquad R_a(\eta) \sim (2\pi a)^{-\frac{1}{2}} e^{-\frac{1}{2}a\eta^2} \sum_{k=0}^{\infty} c_k(\eta)\, a^{-k}$$

for $a \to \infty$, uniformly valid with respect to $\eta \in \mathbb{R}$; erfc is the incomplete error function defined by

$$(1.6) \qquad \mathrm{erfc}(x) = 2\pi^{-\frac{1}{2}} \int_x^{\infty} e^{-t^2}\, dt.$$

The expansion (1.5) was derived by using saddle point methods. In § 2 we use a different method which yields recurrence relations for the coefficients c_k and a representation for the remainder of (1.5). In §3 we discuss representations for c_k that can be used for numerical applications. In §4 numerical error bounds are constructed for the remainder of the series in (1.5) when the first n terms in the series are used. Bounds are given up to $n = 10$. As a side result this section gives bounds for the remainder of the asymptotic expansion of the reciprocal gamma function $1/\Gamma(x)$ for real x. In § 5 the results are extended to complex values of a and x. In § 6 a new asymptotic expansion for the inverse of the incomplete gamma functions is derived. To describe this, let $q \in [0,1]$. Then the function $x(q,a)$ implicitly defined by the equation $Q(a,x) = q$ is called the inverse. We give an asymptotic expansion of the form

$$(1.7) \qquad x(q,a) \sim a(x_0 + x_1 a^{-1} + x_2 a^{-2} + \ldots)$$

for $a \to \infty$. This expansion is based on inversion of the uniform asymptotic expansion for Q. The analysis is formal but it appears that (1.7) is valid in $q \in [0,1]$. Some information is given about the first coefficients in (1.7).

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2. RECURRENCE RELATIONS FOR THE COEFFICIENTS AND REPRESENTATION OF THE REMAINDER.

First we remark that the asymptotic expansion for $a \to \infty$, of $dR_a(\eta)/d\eta$ may be obtained by formal differentiation of (1.5). This is not proved here, but it follows from the representation of $R_a(\eta)$ in our previous paper (formula (2.10) of TEMME [5]). The result is

$$\frac{dR_a(\eta)}{d\eta} \sim a(2\pi a)^{-\frac{1}{2}} e^{-\frac{1}{2}a\eta^2} \sum_{k=0}^{\infty} c_k^{(1)}(\eta)\, a^{-k} \qquad (2.1)$$

with

$$\begin{aligned} c_0^{(1)}(\eta) &= -\eta c_0(\eta) \\ c_k^{(1)}(\eta) &= -\eta c_k(\eta) + \frac{d\,c_{k-1}(\eta)}{d\eta}\,; \qquad k \geq 1. \end{aligned} \qquad (2.2)$$

Secondly, we need the coefficients of the asymptotic expansion of the complete gamma function. Let us define

$$\Gamma^*(a) = (a/2\pi)^{\frac{1}{2}}\, e^a\, a^{-a}\, \Gamma(a), \qquad a > 0. \qquad (2.3)$$

Then Γ^* and $1/\Gamma^*$ have the well-known asymptotic expansions for $a \to \infty$

$$\begin{aligned} \Gamma^*(a) &\sim \sum_{k=0}^{\infty} (-1)^k \gamma_k\, a^{-k} \\ 1/\Gamma^*(a) &\sim \sum_{k=0}^{\infty} \gamma_k\, a^{-k}\,. \end{aligned} \qquad (2.4)$$

The first few coefficients are

$$\gamma_0 = 1, \quad \gamma_1 = -\frac{1}{12}, \quad \gamma_2 = \frac{1}{288}, \quad \gamma_3 = \frac{139}{51840}\,.$$

Further coefficients follow from SPIRA [3] and WRENCH [6]. Wrench gives $(-1)^k\gamma_k$ up to $k = 20$ in rational form, Spira the remaining up to $k = 30$. Decimal representations are also given in both references.

With these preparations we have

THEOREM 1. *Let* $\{\gamma_k\}$ *be defined by* (2.4). *Then the coefficients* c_k *of* (1.5) *satisfy the recurrence relation*

$$(2.5)\qquad \begin{aligned} c_0(\eta) &= \frac{1}{\mu} - \frac{1}{\eta}, \\ \eta c_k(\eta) &= \frac{d\, c_{k-1}(\eta)}{d\eta} + \frac{\eta}{\mu}\gamma_k, \quad k \geq 1. \end{aligned}$$

PROOF. By differentiating one of the formulas in (1.4) with respect to η and by using (2.1) it follows that

$$(2.6)\qquad \frac{d}{d\eta}R_a(\eta) = (a/2\pi)^{\frac{1}{2}}\left(1 - \frac{1}{\mu+1}\,\frac{1}{\Gamma^*(a)}\,\frac{d\mu}{d\eta}\right) e^{-\frac{1}{2}a\eta^2}.$$

From (1.4) we have

$$(2.7)\qquad \frac{d\mu}{d\eta} = \frac{(\mu+1)\eta}{\mu},$$

and substituting (2.1) and the second relation of (2.4) we obtain (2.5) by collecting equal powers of a^{-1} and using (2.2). □

As follows from [5], the coefficients c_k are holomorphic in a neighbourhood of $\eta = 0$. In fact the singularities of $1/\mu$ and $1/\eta$ in c_0 cancel each other. So the limiting value of c_0 for $\eta \to 0$ is well defined.

Owing to the derivative of c_{k-1} in (2.5) this formula cannot be handled easily from a numerical point of view. Further, the above mentioned cancellation of singular parts in c_0 occurs in all c_k when working with (2.5). Therefore other representations are given for these coefficients. In the next section we discuss some aspects of the Taylor expansions for small $|\eta|$-values, while for larger $|\eta|$-values a recurrence relation is constructed from which the coefficients can be computed directly. But first we give representations of the remainder in the asymptotic expansion (1.5).

From (1.4) it follows that $R_a(\infty) = R_a(-\infty) = 0$. Hence, integration of (2.6) gives

$$(2.8)\qquad R_a(\zeta) = (a/2\pi)^{\frac{1}{2}} \int_{-\infty}^{\zeta} \left[1 - \frac{\eta}{\mu}\,\frac{1}{\Gamma^*(a)}\right] e^{-\frac{1}{2}a\eta^2}\, d\eta =$$

$$= -(a/2\pi)^{\frac{1}{2}} \int_{\zeta}^{\infty} [1 - \frac{\eta}{\mu} \frac{1}{\Gamma^*(a)}] e^{-\frac{1}{2}a\eta^2} d\eta,$$

where μ as a function of η is defined implicitly in (1.4). From these representations and the recurrence relations for c_k a simple expression for the remainder follows. For this purpose we introduce the notation

$$R_a(\eta) = (2\pi a)^{-\frac{1}{2}} e^{-\frac{1}{2}a\eta^2} [\sum_{k=0}^{N-1} c_k(\eta) a^{-k} + a^{-N} G_N(\eta;a), \tag{2.9}$$

$a > 0$, $\eta \in \mathbb{R}$, $N = 0,1,2,\ldots$. Furthermore, we need a notation for the remainder in the asymptotic expansion of $1/\Gamma^*(a)$, which is written as

$$1/\Gamma^*(a) = \sum_{k=0}^{N-1} \gamma_k a^{-k} + a^{-N} H_N(a), \qquad a > 0,\ N = 0,1,2,\ldots . \tag{2.10}$$

<u>THEOREM</u> 2. *Let* G_N *and* H_N *be defined by (2.9) and (2.10). Then*

$$e^{-\frac{1}{2}a\zeta^2} G_N(\zeta;a) = a \int_{\zeta}^{\infty} \eta\, c_N(\eta)\, e^{-\frac{1}{2}a\eta^2} d\eta \tag{2.11}$$

$$+ H_{N+1}(a) \int_{\zeta}^{\infty} \frac{\eta}{\mu} e^{-\frac{1}{2}a\eta^2} d\eta.$$

<u>PROOF</u>. Follows immediately from substitution of (2.9) and (2.10) in (2.6) (and by using (2.5) and (2.7)). □

The second integral in (2.11) can be expressed in $Q(a,x)$. From representations of c_k to be given in the following sections it follows that $|c_k(\eta)|$ is a bounded function of $\eta \in \mathbb{R}$. For numerical applications the following is important.

<u>COROLLARY</u> 1. *If* $|c_k(\eta)|$ *is bounded on* $\mathbb{R}$ *then for* $N = 0,1,2,\ldots$

$$|Q(a,x) - \tfrac{1}{2}\,\mathrm{erfc}[\eta(a/2)^{\frac{1}{2}}] - e^{-\frac{1}{2}a\eta^2} \sum_{k=0}^{N-1} c_k(\eta)\, a^{-k}| \le Q_N(\eta;a) \tag{2.12}$$

with

$$(2.13)\qquad Q_N(\eta;a) = \begin{cases} C_N\, e^{-\frac{1}{2}a\eta^2} \\ C_N\,(2-e^{-\frac{1}{2}a\eta^2}) \end{cases} + |H_{N+1}(a)|\; e^a a^{-a}\Gamma(a)Q(a,x)$$

where the upper term is for $\eta \geq 0$*, the lower one for* $\eta \leq 0$*, and where*

$$(2.14)\qquad C_N = \sup_{\eta\in\mathbb{R}} |c_N(\eta)|.$$

In §4 we give numerical values of C_N and bounds for H_N, $0 \leq N \leq 10$. With these values we have strict and realistic error bounds for the remainder of the uniform asymptotic expansion of $Q(a,x)$. Similar results hold for the function $P(a,x)$. For $N = 0,1,2,\ldots$ we have

$$(2.15)\qquad |P(a,x) - \tfrac{1}{2}\,\mathrm{erfc}[-\eta(a/2)^{\frac{1}{2}}] + e^{-\frac{1}{2}a\eta^2}\sum_{k=0}^{N-1} c_k(\eta)\, a^{-k}| \leq P_N(\eta;a)$$

with

$$(2.16)\qquad P_N(\eta;a) = \begin{cases} C_N(2-e^{-\frac{1}{2}a\eta^2}) \\ C_N \end{cases} + |H_{N+1}(a)|\; e^a a^{-a}\Gamma(a)\; P(a,x),$$

where the upper term is for $\eta \geq 0$, the lower one for $\eta \leq 0$.

<u>REMARK</u> 1. The functions multiplying the constants C_N in (2.13) and (2.16) have quite different behaviour for $\eta < 0$ and $\eta > 0$. This, however, is in agreement with the behaviour of the functions P and Q in the same formula. In fact, the bounds P_N and Q_N give a measure for the relative accuracy for the error in the uniform expansions.

<u>REMARK</u> 2. The asymptotic expansion (1.5) and the representation for the remainder is easily obtained by partial integration of one of the integrals in (2.8) and by using the recursions (2.5) and $H_k(a) = \gamma_k + \frac{1}{a} H_{k+1}(a)$.

3. REPRESENTATIONS OF c_k.

Using (2.5) with k = 1 we obtain

$$\eta c_1(\eta) = -\frac{1}{\mu^2}\frac{d\mu}{d\eta} + \frac{1}{\eta^2} + \frac{\eta}{\mu}\gamma_1, \tag{3.1}$$

and using (2.7) we have

$$c_1(\eta) = \frac{1}{\eta 3} - \frac{1+\mu+\mu^2/12}{\mu^3} \tag{3.2}$$

Computing higher order coefficients we notice the following structure

$$c_k(\eta) = (-1)^k\{\frac{Q_k(\mu)}{\mu^{2k+1}} - \frac{A_k}{\eta^{2k+1}}\}, \tag{3.3}$$

where Q_k is a polynomial in μ of degree 2k and $A_k = 2^k\Gamma(k+\frac{1}{2})/\Gamma(\frac{1}{2})$,

$$A_0 = 1, \quad A_k = (2k-1)A_{k-1}, \quad k \geq 1.$$

We obtain for Q_k a recurrence relation with respect to k by substituting (3.3) in (2.5). The result is

$$Q_k(\mu) = (1+\mu)[(2k-1)Q_{k-1}(\mu) - \mu\, Q'_{k-1}(\mu)] + (-1)^k\,\gamma_k\,\mu^{2k}, \tag{3.4}$$

where the derivative is with respect to μ. The first few polynomials are

$$\begin{aligned} Q_0(\mu) &= 1 \\ Q_1(\mu) &= 1 + \mu + \frac{1}{12}\mu^2 \\ Q_2(\mu) &= 3 + 5\mu + \frac{25}{12}\mu^2 + \frac{1}{12}\mu^3 + \frac{1}{288}\mu^4. \end{aligned} \tag{3.5}$$

Again, the recurrence relation (3.4) contains the derivative of Q_{k-1}, but now Q_{k-1} is a polynomial. In order to preserve accuracy for $\mu = -1$ we write

$$Q_k(\mu) = (1+\mu)\,P_k(\mu) + (-1)^k\,\gamma_k\,\mu^{2k}, \tag{3.6}$$

and we proceed with P_k. Writing

$$P_k(\mu) = p_0^{(k)} + p_1^{(k)}\mu \ldots p_{2k-2}^{(k)}\, \mu^{2k-2} \tag{3.7}$$

we have the relation (which is easily obtained by substituting (3.7) and (3.6) in (3.4))

$$\begin{aligned} p_0^{(k)} &= (2k-1)\, p_0^{(k-1)} \\ p_j^{(k)} &= (2k-1-j)[p_j^{(k-1)} + p_{j-1}^{(k-1)}], \quad j = 1,2,\ldots,2k-4 \\ p_{2k-3}^{(k)} &= 2p_{2k-4}^{(k-1)},\ p_{2k-2}^{(k)} = (-1)^{k-1}\, \gamma_{k-1}, \end{aligned} \tag{3.8}$$

with as starting polynomial $P_1(\mu) = 1$, or $p_0^{(1)} = 1$.

In Table I we give the coefficients $p_j^{(k)}$ of (3.7) for $k = 0,1,2,\ldots,5$, $j = 0,1,\ldots,2k-2$

TABLE I

k	$p_j^{(k)}$
1	1
2	3, 2, 1/12
3	15, 20, 25/4, 1/6, 1/288
4	105, 210, 525/4, 77/3, 49/96, 1/144, -139/51840
5	945, 2520, 9555/4, 1883/2, 12565/96, 149/72, 221/17280, -139/25920, -571/2488320.

At this stage, it is not clear for which η-values direct computation of c_k via (3.3) is safe. This depends of course on the desired accuracy. In applications, the desired accuracy in c_k will depend on k. For, when using the asymptotic expansion, first terms (i.e., terms $c_k(\eta)a^{-k}$ with k small) are needed with higher accuracy than late terms. Since the terms in the asymptotic expansion are decreasing in absolute value (if a is large) the coefficient c_0 is needed in good relative accuracy, while for the remaining terms a criterion based on absolute accuracy can be used. In Table II we give the μ-part and the η-part of c_k (cf.(3.3)) for $k = 0,1,2$, and $\eta = \pm 1$. The μ-values corresponding with $\eta = \pm 1$ are $\mu(-1) = -0.698\ldots$, $\mu(1) = 1.35\ldots$.

TABLE II

k	η	η-part	μ-part
0	1	-1	0.74
	-1	1	-1.43
1	1	1	-1.0034
	-1	-1	1.0054
2	1	-3	3.0022
	-1	3	2.9926

It turns out that $\eta = \pm 1$ are safe values for summing the asymptotic series as far as it concerns coefficients c_k up to and including $k = 2$. To give an indication for the c_k with $k \geq 2$, we notice that absolute accuracy in subtracting the η-part from the μ-part in (3.3) is preserved if both parts are in absolute value not larger than 1. From Stirling's approximation for A_k it follows that the η-part is in absolute value approximately $(2k/e\eta_1^2)^k$. This expression is smaller than 1 if $|\eta| > (2k/e)^{\frac{1}{2}}$. For $k = 10$ the righthand side is $2.71\ldots$.

If $|\eta|$ is small it is preferred to use expansions either in terms of η or in terms of μ. We advise expansions in η, since it gives better convergence properties. When expanding c_k in powers of μ we need (among others) the expansion of η in powers of μ. Due to the singularity of the logarithm in (1.3), the radius of convergence of this series is 1. Other singularities for η are zeros of $\mu - \ell n(1+\mu)$, but they are outside the domain $|\mu| \leq 1$. This follows from straightforward analysis. The reader may also consult an interesting note of DIEKMANN [2]. The expansion of μ in powers of η has radius of convergence $2\sqrt{\pi} \simeq 3.54$. This follows from the analysis of §5. From the recurrence relation (2.5) it is easily seen that the radius of convergence of the power series for c_k either in μ or in η is the same for all k.

We conclude this section with information on the construction of the coefficients for the expansion of c_k in powers of η.

It is convenient to start with the computation of the α_k in

$$(3.9) \qquad \mu(\eta) = \alpha_1\eta + \alpha_2\eta^2 + \ldots ,$$

where μ is defined implicitly in (1.3). Substitution of (3.9) in (2.7) yields the recurrence relation

$$(k+1)\ \alpha_k = \alpha_{k-1} - \sum_{j=2}^{k-1} j\alpha_j\ \alpha_{k-j+1}, \qquad k \geq 2.$$

The first few are

$$\alpha_1 = 1, \quad \alpha_2 = 1/3, \quad \alpha_3 = 1/36, \quad \alpha_4 = -1/270, \quad \alpha_5 = 1/4320.$$

With α_k we also have available the γ_k of (2.4), which are also needed in (2.5). The relation between α_k and γ_μ is

$$\gamma_k = (-1)^k\ 1.3.5.\ \ldots\ (2k+1)\ \alpha_{2k+1}, \qquad k = 0,1,2,\ldots .$$

In fact, the expansion (3.9) is of importance for the derivation of the expansion in (2.4) (see also §4), By using (2.7) and

$$(3.10) \qquad \tfrac{1}{2}\eta^2 = \mu - \ln(1+\mu),$$

it follows that the expansion of $\eta/\mu(\eta)$ occurring in (2.5) is given by

$$(3.11) \qquad \eta/\mu(\eta) = 1 + 2(\alpha_2 - \tfrac{1}{2})\eta + 3\alpha_3\ \eta^2 + 4\alpha_4\ \eta^3 + \ldots .$$

This gives the coefficients of c_0 of (2.5)

(3.12) $$c_0(\eta) = c_0^{(0)} + c_1^{(0)}\eta + c_2^{(0)}\eta^2 + \ldots$$

with

$$c_0^{(0)} = -\frac{1}{3}, \quad c_k^{(0)} = (k+2)\alpha_{k+2}, \quad k \geq 1.$$

By repeated use of (2.5) we obtain the recursion for the coefficients in

$$c_k(\eta) = c_0^{(k)} + c_1^{(k)}\eta + c_2^{(k)}\eta^2 + \ldots$$

(3.13) $$c_n^{(k)} = \gamma_k c_n^{(0)} + (n+2)\, c_{n+2}^{(k-1)} \qquad n \geq 0, \quad k \geq 1.$$

Of course, each $c_n^{(k)}$ can be expressed in terms of $c_n^{(0)}$. The relation is for $n \geq 0$ and $k \geq 1$

(3.14) $$c_n^{(k)} = \gamma_k\, c_n^{(0)} + \gamma_{k-1}(n+2)\, c_{n+2}^{(0)} + \ldots\, \gamma_0(n+2)\ldots(n+2k)\, c_{n+2k}^{(0)}.$$

Other functions may be used for expanding the coefficients c_k, for instance Chebyshev polynomials. In that case recurrence relations for corresponding coefficients can be constructed again. But the Taylor case gives simple relations and the coefficients can also be used for complex values of the parameters.

On account of the convergence properties of (3.12) (with radius $2\sqrt{\pi}$) successive terms in (3.14) are decreasing in absolute value. Hence no instability problems arise when using (3.14) for the computation of $c_n^{(k)}$.

4. BOUNDS FOR THE REMAINDER IN THE ASYMPTOTIC EXPANSION

In Table III we give the numbers C_k defined in (2.14). These bounds were obtained numerically by using representations of c_k given in the foregoing section. From (3.3) it follows that

$$\lim_{\eta\to+\infty} c_k(\eta) = 0, \quad \lim_{\eta\to-\infty} c_k(\eta) = (-)^{k+1} Q_k(-1) = -\gamma_k .$$

TABLE III

k	C_{2k}	C_{2k+1}
0	1	$3.5_{10}{-3}$
1	$9.2_{10}{-3}$	$6.9_{10}{-4}$
2	$2.1_{10}{-3}$	$3.5_{10}{-4}$
3	$1.3_{10}{-3}$	$3.5_{10}{-4}$
4	$1.7_{10}{-3}$	$6.0_{10}{-4}$
5	$3.4_{10}{-3}$	

Next we give details for computing the bounds H_k (defined in (2.10)) for $k = 0,1,\ldots,10$. It is convenient to start with details for obtaining the asymptotic expansion of $1/\Gamma(a)$. Again, a is a positive number. Starting point is Hankel's integral

$$(4.11) \qquad \frac{1}{\Gamma(a)} = \frac{1}{2\pi i} \int_{-\infty}^{(0^+)} e^t\, t^{-a}\, dt.$$

As in our previous paper [5] this can be written as

$$(4.2) \qquad \frac{1}{\Gamma(a)} = \frac{a^{1-a} e^a}{2\pi i} \int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} f(u)\, du$$

with

$$(4.3) \qquad f(u) = \frac{ut}{1-t}, \quad -\tfrac{1}{2}u^2 = t - 1 - \ln t.$$

The relation between u and t must be specified in more detail. Let L be the saddle point contour for (4.1) in the t-plane. That is

(4.4) $$L = \{t \mid t = \frac{\phi}{\sin\phi} e^{i\phi}, \quad -\pi < \phi < \pi\} .$$

Then u defined in (4.3) is real if $t \in L$, and sign(u) = sign [Im(t)]. The asymptotic expansion of $1/\Gamma(a)$ is obtained by expanding

(4.5) $$g(u) = \frac{1}{2i} [f(u) + f(-u)]$$

in powers of u and termwise integration. Let us define the function g_N by writing

(4.6) $$g(u) = \sum_{k=0}^{N-1} a_k u^{2k} + a_N u^{2N} g_N(u), \qquad N = 0,1,2,...,$$

with

(4.7) $$a_k = \frac{1}{(2k)!} g^{(2k)}(0) ;$$

all a_k are different from zero. Then the function H_N of (2.10) is given by

(4.8) $$H_N(a) = (a/2\pi)^{\frac{1}{2}} a_N a^N \int_{-\infty}^{\infty} e^{-\frac{1}{2}au^2} u^{2N} g_N(u)\, du.$$

Suppose that we have bounds

(4.9) $$G_N = \sup_{u \in \mathbb{R}} |g_N(u)|,$$

then a bound for H_N is given by

(4.10) $$|H_N(a)| \le |\gamma_N| G_N, \qquad a > 0,$$

where γ_k are the coefficients in (2.10).

As yet it is not clear that g_N is bounded on $\mathbb{R}$. But it follows from (4.6) that g_N is bounded if g is bounded. The function f of (4.3) is not bounded on $\mathbb{R}$, but its even part g is. This follows from using the representation of $t \in L$ as given in (4.4). In terms of u and ϕ, g is given by

$$(4.11)\qquad g(u) = \frac{u\phi\sin^2\phi}{\phi^2 + \sin^2\phi - \phi\sin 2\phi}\ , \qquad -\pi < \phi < \pi$$

with

$$\tfrac{1}{2}u^2 = 1 - \phi\ \mathrm{ctg}\ \phi + \ell n\,\frac{\phi}{\sin\phi}\ , \qquad \mathrm{sign}(u) = \mathrm{sign}(\phi)\ ,$$

from which it follows that g is bounded if $u \to \pm\infty$ or $\phi \to \pm\pi$. Table IV for M = 0,1,...,11.

Table IV

N	G_{2N}	G_{2N+1}
0	1	1
1	1.95	1
2	3.33	1
3	5.05	1
4	6.95	1
5	8.90	1

For N = 0,1,3,5,7,9,11 the maximal function values of $|g_N(u)|$ occur at u = 0; for M = 2,4,6,8,10 the maxima occur in the neighbourhood of $u = \pm 2\sqrt{\pi}$. These latter points are the points on the real axis marking the domain of convergence of the Taylor series of g.

With the data of Table III and Table IV and relation (4.10) the bounds Q_N and P_N defined in (2.13) and (2.16) are easily computed.

5. EXTENSION TO COMPLEX VARIABLES

In this section we will show that the asymptotic expansion for P and Q given by (1.4) and (1.5) are valid for $a \to \infty$ uniformly in $|\arg a| \le \pi - \varepsilon_1$, $|\arg x/a| \le 2\pi - \varepsilon_2$ where ε_1 and ε_2 are positive numbers, $0 < \varepsilon_1 < \pi$, $0 < \varepsilon_2 < 2\pi$.

The condition on the argument of a follows from the validity of the expansions in (2.4), which are known to be uniformly valid when $|\arg a| \leq \pi - \varepsilon_1$. As noticed in Remark 2 of §2 the asymptotic expansion of $R_a(\eta)$ can be obtained by partial integration of one of (2.8). If we consider the second integral, one of the assumptions by partial integration will be that $\exp(-\frac{1}{2}a\eta^2)$ vanishes at infinity in a certain direction of the η-plane. If $|\arg a| < \pi$ and if it is allowed to use η-values at infinity with $\arg(a\eta^2) < \frac{\pi}{2}$ then the convergence of the integral is established for $|\arg a| \leq \pi - \varepsilon_1$. From these inequalities it follows that it is sufficient to show that for large $|\eta|$ we can take arg η in $(-\frac{3\pi}{4}, \frac{3\pi}{4})$. A second aspect of using the second integral of (2.8) is the possibility of joining the point ζ with ∞ such that the function $\mu(\eta)$ of the integral is holomorphic along this path and such that the point ζ can be associated unequivocally with a point in the μ-plane. In order to settle this we discuss the relation between η and the parameter μ (or λ) for complex values.

It is convenient to consider

(5.1) $$\eta = [2(\lambda - 1 - \ln \lambda)]^{\frac{1}{2}}.$$

For $\lambda > 0$ the function η is to be interpreted as drawn in Figure 1. This implies a choice of the square root.

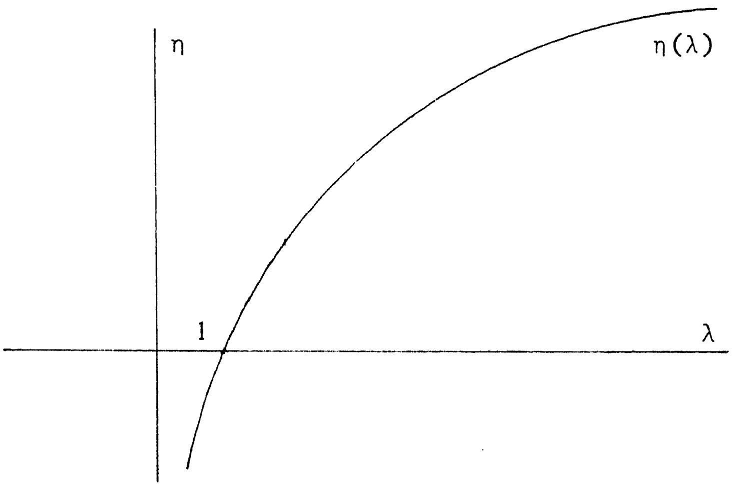

Figure 1.

We obtain a clear insight in the mapping $\lambda \to \eta(\lambda)$ and its inverse if we draw images of the half-lines ℓ_ϕ defined by

$$\ell_\phi = \{\lambda \mid \lambda = \rho e^{i\phi},\ \rho > 0\}$$

where ϕ is real, $|\phi| \le 2\pi$. Writing $\eta = \alpha + i\beta$ the image of ℓ_ϕ in the η-plane is governed by the equations

$$\tfrac{1}{2}(\alpha^2 - \beta^2) = \rho \cos\phi - 1 - \ln\rho$$

$$\alpha\beta = \rho \sin\phi - \phi.$$

Taking into account the convention about the choice of the square root in (5.1) we obtain Figure 2, which contains images of ℓ_ϕ for $0 \le \phi \le 2\pi$. The complete picture for $-2\pi \le \phi \le 2\pi$ is symmetric with respect to the α-axis.

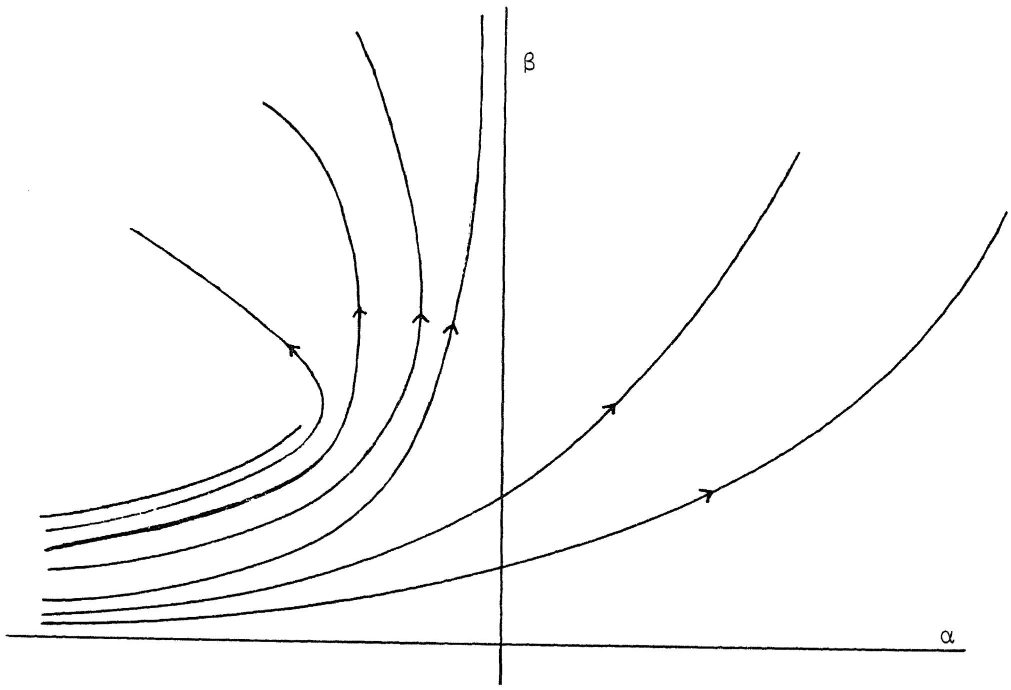

Figure 2.

The shown directions correspond to increasing values of ρ on ℓ_ϕ. The half-lines $\ell_{\pm 2\pi}$ are mapped on part of the hyperbolae $\alpha\beta = \mp 2\pi$. The points $\eta^{\pm} = e^{\pm 3\pi i/4}\, 2\sqrt{\pi}$ are singular points of the mapping. Other singular points are located in other Riemann sheets of the η-plane. Convenient branch-cuts for the function $\lambda(\eta)$ are the parts of the hyperbolae $\alpha\beta = \pm 2\pi$ with $\alpha \le -\sqrt{2\pi}$. With the η-plane cut along these curves, lines ℓ_ϕ with the values of ϕ outside the interval $[-2\pi, 2\pi]$ can be traced, but for our problem this is superfluous.

It is concluded that any point in the finite η-plane (not on the branch-cuts), corresponds to a point in the λ-plane with $|\arg \lambda| < 2\pi$. Consequently, if we integrate the second integral of (2.8) along a path that avoids the branch-cuts in the η-plane, the function $\mu(\eta) = \lambda(\eta) - 1$ is holomorphic. The conditions for allowing values of $\arg a$ in $(-\pi, \pi)$ are amply satisfied, since admissable directions in the η-plane can be found in the sector $-\pi < \arg \eta < \pi$.

<u>REMARK</u>. Singular points of the mapping $\eta \to \lambda(\eta)$ can also be found by considering the derivative $d\lambda/d\eta = \lambda\eta/(\lambda - 1)$; $\lambda = 1$ gives a regular point but $\lambda = e^{2\pi i n}$ $(n = \pm 1, \pm 2, \ldots)$ gives (due to the many-valuedness of the logarithm in (5.1)) singular points η_n satisfying $\frac{1}{2}\eta_n^2 = -2\pi i n$, $n = \pm 1, \pm 2, \ldots$.

The integration by parts procedure leads eventually to (2.9) and (2.11). From the properties of the coefficients c_k and by taking appropriate contours in (2.11) it follows that for $N = 0, 1, 2, \ldots$

$$e^{-\frac{1}{2}a\eta^2} G_N(\eta; a) = O(1), \qquad a \to \infty$$

uniformly in $|\arg a| \le \pi - \varepsilon_1$, $|\arg \lambda| \le 2\pi - \varepsilon_2$.

6. ASYMPTOTIC EXPANSION OF THE INVERSE FUNCTION

The incomplete gamma functions are basic for the chi-square probability function and the Poisson distribution. Applications in this field lead us to the investigations of reliable and accurate algorithms for the computation of the functions discussed in this paper. Existing methods are not efficient if both parameters x and a are large. In statistics the inverses of the probability functions are very important. In practice the inversion is usually carried out by Newton-like methods. However, if large parameters must be considered, they are not reliable and not efficient. Our numerical experiments with the inversion of the incomplete gamma functions by using the expansions of this paper are promising, especially if the parameters are large. Since our method is based on uniform expansions, the range of application is satisfactorily large. Owing to the uniform character of our results, the coefficients of the expansion are rather complicated. But for implementation in software packages this aspect is not very important.

In [5] we also derived asymptotic expansions for the incomplete beta function. This function can be inverted by the same methods as those for the incomplete gammma function described in this section.

6.1 We consider real values of x and a satisfying (1.2). Let $q \in [0,1]$. We describe a procedure for obtaining the asymptotic expansion of the function $x(q,a)$ implicitly defined by the equation

$$Q(a,x) = q. \tag{6.1}$$

We use the representaion of Q given in (1.4). If we have inverted Q then P is also inverted. The solution of $P(a,x) = p$, $0 \le p \le 1$, is simply $x(1-p,a)$, where $x(q,a)$ is the solution of (6.1), with $p + q = 1$.

First we describe the inversion in terms of the parameter η. Suppose we have available the value of η_0, which solves the equation $\frac{1}{2}\operatorname{erfc}[\eta_0(a/2)^{\frac{1}{2}}] = q$, where q is the same as in (6.1). This requires an inversion of the error function, but this problem is solved satisfactorily in the literature. See for instance BLAIR et.al.[1] or STRECOK [4].

The value for η implicitly defined by the equation

$$\tfrac{1}{2}\operatorname{erfc}[\eta(a/2)^{\frac{1}{2}}] + R_a(\eta) = q \tag{6.2}$$

is for large values of a approximated by η_0. Hence we write

(6.3) $$\eta = \eta_0 + \varepsilon(\eta_0, a)$$

and try to determine ε. From the previous section it follows that $R_a(\eta)$ is analytic for every $\eta \in \mathbb{R}$. Substituting (6.3), we find by expansion

(6.4) $$\tfrac{1}{2} \sum_{k=1}^{\infty} \frac{\varepsilon^k}{k!} \frac{d^k}{d\eta^k} \operatorname{erfc}[\eta(a/2)^{\frac{1}{2}}] + \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} \frac{d^k}{d\eta^k} R_a(\eta) = 0,$$

where the derivatives are evaluated at $\eta = \eta_0$. In this formula we substitute for the derivatives of $R_a(\eta)$ the derivatives of the asymptotic expansion (1.5). As remarked in §2, the series can be differentiated, giving (2.1). But it can be differentiated any number of times, giving

(6.5) $$\frac{d^k}{d\eta^k} R_a(\eta) \sim a^k (2\pi a)^{-\frac{1}{2}} e^{-\frac{1}{2}a\eta^2} \sum_{n=0}^{\infty} c_n^{(k)}(\eta)\, a^{-n},$$

with $c_n^{(0)}(\eta) = c_n(\eta)$, and for $k \geq 1$ (compare (2.2))

(6.6) $$c_0^{(k)}(\eta) = -\eta\, c_0^{(k-1)}(\eta), \quad c_n^{(k)}(\eta) = -\eta\, c_n^{(k-1)}(\eta) + \frac{d}{d\eta} c_{n-1}^{(k-1)}(\eta), \quad n \geq 1.$$

The derivatives of the error function in (6.4) can be replaced by Hermite polynomials, viz.

$$\tfrac{1}{2} \frac{d^k}{d\eta^k} \operatorname{erfc}[\eta(a/2)^{\frac{1}{2}}] = (-1)^k (a/2)^{k/2} \pi^{-\frac{1}{2}} e^{-\frac{1}{2}a\eta^2} H_{k-1}[\eta(a/2)^{\frac{1}{2}}].$$

For a similar series as in (6.5) let us write

(6.7) $$\tfrac{1}{2} \frac{d^k}{d\eta^k} \operatorname{erfc}[\eta(a/2)^{\frac{1}{2}}] = a^k (2\pi a)^{-\frac{1}{2}} e^{-\frac{1}{2}a\eta^2} \sum_{n=0}^{[(k-1)/2]} h_n^{(k)}(\eta) a^{-n}.$$

This series contains as much terms as the Hermite polynomials $H_{k-1}(x)$ when expanding it in powers of x. From well-known properties of these polynomials we derive for $k = 1, 2, \ldots,\ \ n = 0, 1, \ldots, [(k-1)/2]$

(6.8) $$h_n^{(k)}(\eta) = (-1)^{k+n}\,\eta^{k-1-2n}\,2^{-n}(k-1)!/[n!(k-1-2n)!].$$

Substituting (6.5) and (6.7) we obtain the asymptotic equality

(6.9) $$\sum_{n=0}^{\infty} c_n(\eta)a^{-n} + \frac{(\varepsilon a)}{1!}\sum_{n=0}^{\infty} e_n^{(1)}(\eta)a^{-n} + \frac{(\varepsilon a)^2}{2!}\sum_{n=0}^{\infty} e_n^{(2)}(\eta)a^{-n}\ldots \sim 0,$$

with $\eta = \eta_0$ and

(6.10) $$e_n^{(k)}(\eta) = c_n^{(k)}(\eta) + h_n^{(k)}(\eta), \qquad n \geq 0, \quad k \geq 1.$$

From this point the analysis is continued formally. We assume that ε in (6.3) can be developed in an asymptotic expansion. Let us make the "Ansatz"

(6.11) $$\varepsilon(\eta,a) \sim \frac{\alpha(\eta)}{a}\,[1 + \alpha_1(\eta)a^{-1} + \alpha_2(\eta)a^{-2} + \ldots],$$

where α and α_i are to be determined. This will be done in §6.2. With (6.11) an expansion for x defined in (6.1) can be obtained as follows. We have

(6.12) $$x(q,a) = a\lambda(\eta) = a[1+\mu(\eta)] = a[1+\mu(\eta_0)+\varepsilon\mu'(\eta_0)+\ldots]$$
$$\sim a[x_0(\eta_0) + x_1(\eta_0)a^{-1} + x_2(\eta_0)a^{-2} + \ldots], \qquad a \to \infty,$$

the first coefficients x_i being given by

(6.13)
$$x_0(\eta) = 1 + \mu(\eta),$$
$$x_1(\eta) = \mu'(\eta)\alpha(\eta)$$
$$x_2(\eta) = \tfrac{1}{2}\alpha^2(\eta)\mu''(\eta) + \alpha_1(\eta)\alpha(\eta)\mu'(\eta)$$
$$x_3(\eta) = \tfrac{1}{6}\alpha^3(\eta)\mu'''(\eta) + \alpha_1(\eta)\alpha^2(\eta)\mu''(\eta) + \alpha_2(\eta)\alpha(\eta)\mu'(\eta).$$

The accents denote differentiation with respect to η. The value of $\mu(\eta_0)$ can be obtained by the inversion of the relation between μ and η given in (1.3). Derivatives of $\mu(\eta)$ can be obtained via (2.7), but they also follow from the coefficients c_k and (2.5). For instance, $\mu'(\eta) = [1+\mu(\eta)][1+\eta c_0(\eta)]$. As will be seen in §6.2, the coefficients c_k are also needed in $\alpha(\eta)$, $\alpha_i(\eta)$. From the representations of α, α_1 and α_2 to be given in §6.2, the coefficients $x_0,\ldots,x_2$ of (6.13) can be determined.

6.2 The coefficients α and α_i of (6.11) are computed by substitution of (6.11) in (6.9) and by collecting equal powers of the large parameter a. By considering coefficients multiplying a^0 we obtain

$$c_0(\eta) + \sum_{k=1}^{\infty} \frac{\alpha^k(\eta)}{k!} e_0^{(k)}(\eta) = 0. \tag{6.14}$$

From (6.6) and (6.8) it follows that for $k \geq 1$

$$c_0^{(k)}(\eta) = (-\eta)^k c_0(\eta),\ h_0^{(k)}(\eta) = -(-\eta)^{k-1}. \tag{6.15}$$

So, $e_0^{(k)}$ of (6.14) defined in (6.10) is known and α is obtained by summation. The result is

$$\alpha(\eta) = \frac{1}{\eta} \ell n[1+\eta\ c_0(\eta)] = \frac{1}{\eta}\ell n(\eta/\mu). \tag{6.16}$$

From this representation we conclude that α is a well-defined bounded function of $\eta \in \mathbb{R}$ with $\alpha(\eta) \to 0$ if $\eta \to \pm\infty$.

For higher order coefficients α_i we need representations of $c_n^{(k)}$ in terms of c_k and their derivatives. For n = 0 this relation is given by the first of (6.15), for n = 1 it is given by

$$c_1^{(k)}(\eta) = (-1)^k[\eta^k\ c_1(\eta) - k\eta^{k-1} \frac{d}{d\eta} c_0(\eta) - \tfrac{1}{2}k(k-1)\eta^{k-2} c_0(\eta)] \tag{6.17}$$

and the general formula is

$$c_n^{(k)}(\eta) = (-1)^k\ k! \sum_{\mu=0}^{n} \sum_{\nu=0}^{\mu} \frac{(-1)^{\mu} 2^{-\nu} \eta^{k-\mu-\nu}}{(k-\mu-\nu)!(\mu-\nu)!} \frac{d^{\mu-\nu}}{d\eta^{\mu-\nu}} c_{n-\mu}(\eta). \tag{6.18}$$

These relations follow from induction. In the last formula the summations are carried out for those μ and ν such that $k-\mu-\nu \geq 0$.

Collecting in (6.9) coefficients of a^{-1} we obtain for α_1 the equation

$$c_1(\eta) + \sum_{k=1}^{\infty} \frac{\alpha^k(\eta)}{k!} [e_1^{(k)}(\eta) + k\alpha_1(\eta)\, e_0^{(k)}(\eta)] = 0,$$

which gives after summation

$$\text{(6.19)} \qquad \alpha_1(\eta) = \frac{c_1(\eta) + \alpha(\eta)c_0'(\eta) - \frac{1}{2}\alpha^2(\eta)c_0(\eta) + \alpha^3(\eta)r_2[\eta\alpha(\eta)]}{\alpha(\eta)[1+\eta\, c_0(\eta)]}\ .$$

The function r_n is for $n = 0,1,2,\ldots$ defined by

$$r_n(x) = [e^x - (1 + x + \frac{1}{2!}\, x^2 + \ldots \frac{x^n}{n!})]/x^{n+1}.$$

The result for α_2 is

$$\text{(6.20)} \qquad \alpha_2(\eta) = \{c_2 + \alpha c_1' + [c_0'' - c_1 + \eta(1+\eta c_0)\alpha_1^2 - \tfrac{1}{2}c_0'\alpha^3 + \tfrac{1}{4}\alpha^4 c_0 - 3\alpha^5 r_4\}/ [(1+\eta c_0)\alpha],$$

where r_4 has the argument $\eta\alpha(\eta)$ and the remaining functions have argument η. In (6.19) and (6.20) the primes denote differentiation with respect to η. Derivatives of c_k can be replaced by combinations of c_k by using (2.5). The first few relations are

$$c_0'(\eta) = \eta[c_1(\eta) + \frac{1}{12}\, c_0(\eta)] + \frac{1}{12}\ ,$$

$$c_1'(\eta) = \eta[c_2(\eta) - \frac{1}{288}\, c_0(\eta)] - \frac{1}{288}\ ,$$

$$c_0''(\eta) = \eta^2[c_2(\eta) - \frac{1}{288}\, c_0(\eta)] - \frac{1}{288}\,\eta + \frac{1}{12}[c_0(\eta) + 12\, c_1(\eta) + \frac{1}{12} + \eta[c_1(\eta) + \frac{1}{12}\, c_0(\eta)]\}.$$

6.3 Example for $q = \frac{1}{2}$.

If $q = \frac{1}{2}$ the relations are quite simple. In that case $\eta_0 = 0$ and x_i can be determined by computing limiting values of α and α_i for $\eta \to 0$. The following expression gives the values of x_0, x_1, x_2 of (6.12), viz.

(6.21) $$x(\tfrac{1}{2},a) \sim a(1 - \frac{1}{3} a^{-1} + \frac{8}{405} a^{-2} \ldots) .$$

Table V shows some results of numerical experiments. For the values of a indicated in the table we computed $x(\frac{1}{2},a)$ from (6.21). Then we computed $y = Q[a,x(\frac{1}{2},a)]$ with accuracy of about 12 significant digits. The table gives the difference $|y - \frac{1}{2}|$.

Table V

a	$\|y - \frac{1}{2}\|$
10	$0.93_{10}{-5}$
50	$0.16_{10}{-6}$
100	$0.29_{10}{-7}$
250	$0.29_{10}{-8}$
500	$0.51_{10}{-9}$
1000	$0.91_{10}{-10}$

Further experiments showed that for other q-values the results are of the same kind. In fact they show the uniform character of our expansion (6.12) with respect to $q \in [0,1]$.

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