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A SURVEY ON HILBERT SPACE METHODS FOR NON-HOMOGENEOUS ELLIPTIC BOUNDARY VALUE PROBLEMS

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A survey on Hilbert space methods for non-homogeneous elliptic boundary value problems.

by

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ABSTRACT

A survey is given on the Hilbert space approach to various kinds of non-homogeneous boundary value problems. The exposition is limited to boundary value problems of variational type.

KEY WORDS & PHRASES: *non-homogeneous boundary value problems, elliptic partial differential equations, Sobolev spaces, generalized (or weak) solutions, coercivity, trace theorems.*

CONTENTS

PREFACE

In an earlier report (TW 140/74) a survey on Hilbert space methods for *homogeneous* elliptic boundary value problems was given. The present report intends to give an overall picture in which way *non*-homogeneous boundary value problems are dealt with in the Hilbert space approach.

The concept of boundary condition itself raises a number of important problems. In case that u is to satisfy homogeneous conditions, one requires that u belongs to a certain subspace (e.g. $H_0^m(\Omega)$) of $H^m(\Omega)$. If one wishes to deal with non-homogeneous conditions one must extend the notion of the restriction of a function to the boundary, evident for continuous functions, to all elements of a Sobolev space. This is done in section 2. Before, in section 1, Sobolev spaces of functions that have their domain of definition on the boundary of a certain domain, are defined. In section 3 the theory is applied to the non-homogeneous Dirichlet problem for elliptic equations of arbitrary order. Sections 4,5 and 6 are devoted to other boundary conditions of various kinds, although only problems of the so-called variational type are considered.

The aim of this report does not differ from that of the former one on homogeneous problems, namely to give a general idea of the Hilbert space approach without going into the details of all proofs. Of course, references to the literature are given. Furthermore, the present report should be seen as a sequel to the former one, which in the text is referred to by [H].

1. SOBOLEV SPACES OF FUNCTIONS DEFINED ON THE BOUNDARY OF A DOMAIN

The definition of $H^s(\partial\Omega)$, where $\partial\Omega$ is the (sufficiently smooth) boundary of a bounded domain Ω, is discussed. For standard notations and preliminaries the reader is referred to [H]. Some of what is needed most is recalled here.

1.1. STANDARD NOTATIONS AND PRELIMINARIES. The usual notations are used. Throughout this report $x = (x_1,\ldots,x_n)$ is a point in the real n-dimensional Euclidean space $\mathbb{R}^n$; $|x| = (x_1^2+\ldots+x_n^2)^{\frac{1}{2}}$, $D_i = \partial/\partial x_i$, $D^p = D_1^{p_1}\ldots D_n^{p_n}$, where $p_1,\ldots,p_n$ are non-negative integers, and $p = (p_1,\ldots,p_n)$, $|p| = p_1+\ldots+p_n$. Given an open set (domain), we shall denote by $C^m(\Omega)$, $m = 0,1,2,\ldots,\infty$, the set of all complex valued functions that have continuous derivatives up to order m, and by $C^m(\overline{\Omega})$ the set of functions which have this property uniformly in Ω. The subset $C^m(\Omega)$ consisting of all functions that have a compact support in Ω is denoted by $C_0^m(\Omega)$.

The linear space $C_0^\infty(\Omega)$ can be regarded as a topological vector space $\mathcal{D}(\Omega)$. A distribution in Ω is a continuous linear functional on $\mathcal{D}(\Omega)$; elements of $\mathcal{D}(\Omega)$ are usually denoted by φ, and a distribution is written as $\langle f,\varphi\rangle$.

If in a point x of the boundary $\partial\Omega$ the normal to $\partial\Omega$ exists, one can speak of the derivative in that direction, $\partial u/\partial\nu$. Usually, ν will be chosen in the outward direction.

The m-th Sobolev space (m being a non-negative integer) $H^m(\Omega)$ is defined as the linear space of all functions $u \in L^2(\Omega)$, of which the distributional derivatives up to order m are functions belonging to $L^2(\Omega)$, equipped with the Hilbert space structure that goes along with the scalar product

$$(u,v)_{m,\Omega} = \sum_{|p|\le m} \int_\Omega D^p u \, D^p \bar{v} \, dx.$$

The associated norm is indicated by $\|\cdot\|_{m,\Omega}$. In case confusion is unlikely, the index Ω is often surpressed. From the definitions it follows that the injective mappings

$$H^k(\Omega) \hookrightarrow H^m(\Omega) \hookrightarrow L^2(\Omega) = H^0(\Omega), \quad k > m \geq 0,$$

are continuous. When $m \neq 0$, the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$ is a proper subspace of $H^m(\Omega)$, denoted by $H_0^m(\Omega)$.

Finally, the definition of smoothness of a boundary is given. The boundary $\partial\Omega$ of an open set $\Omega \subset \mathbb{R}^n$ is said to be of class C^k in the neighbourhood of $z \in \partial\,\Omega$, if there exists an open set Ω_0 with $z \in \Omega_0$, such that for some i each point $x \in \partial\,\Omega \cap \Omega_0$ can be uniquely represented in the form

$$x = (x_1,\dots,x_{i-1},h(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n),x_{i+1},\dots,x_n)$$

and that

$$x_i > h(x_1,\dots,x_{i-1},x_{i+1},\dots.x_n) \qquad \text{in } \Omega_0 \cap \Omega,$$

$$x_i < h(x_1,\dots,x_{i-1},x_{i+1},\dots.x_n) \qquad \text{in } \Omega_0 \setminus \Omega,$$

where h is a k times differentiable function. Note that $z_i = h(z_1,\dots,z_{i-1},z_{i+1},\dots,z_n)$. By changing the coordinate system, one can always choose this i to be equal to n. Write $x' = (x_1,\dots,x_{n-1})$, then the boundary points within Ω_0 are given by $(x',h(x'))$, where x' remains in the projection Δ_0 of $\partial\,\Omega \cap \Omega_0$ onto $\mathbb{R}^{n-1}$. The mapping $\kappa_z\, x = y$, $x \in \Omega_0$, given by

$$(1.1) \qquad \begin{cases} y_n = x_n - h(x') \\ y' = x' \end{cases}$$

maps Ω_0 one-to-one onto an open set in $\mathbb{R}^n$, in such a manner that $\kappa_z(\Omega_0 \cap \Omega)$ is a set that has a portion of its boundary on $y_n = 0$, which

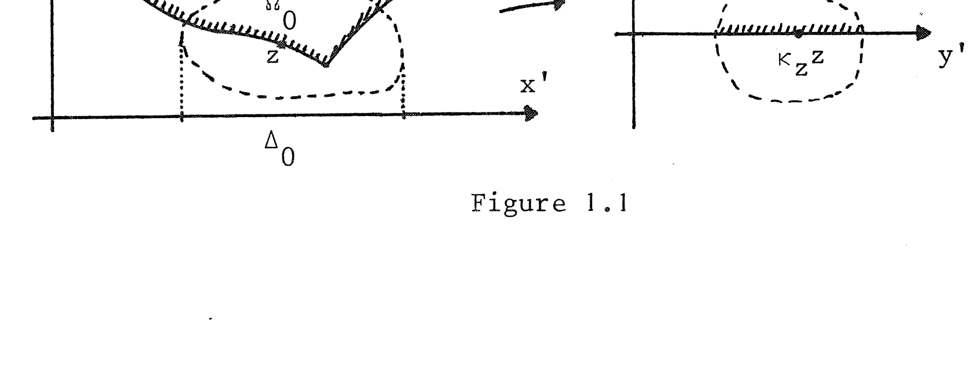

Figure 1.1

may be identified with Δ_0; κ_z has the same differentiability as h.

In case that the function h has Lipschitz continuous highest order derivatives, the boundary is said to belong to $C^{k,1}$ in the neighbourhood of z. If one of the two properties holds for all $z \in \partial\,\Omega$, one simply says that $\partial\Omega$ belongs to C^k (or $C^{k,1}$).

1.2. FOURIER TRANSFORM. Let S be the linear space of all complex valued C^∞-functions $\varphi(x)$, defined on $\mathbb{R}^n$, with the property that $\varphi(x)$ together with all its derivatives decay for $|x| \to \infty$ faster than any negative power of $|x|$. For such φ one can define the Fourier transform

$$(1.2) \qquad F\varphi = \hat{\varphi}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(x) e^{-i(\xi,x)} dx,$$

where, of course, $\xi = (\xi_1,\ldots,\xi_n)$ and $(\xi,x) = \xi_1 x_1 + \ldots + \xi_n x_n$. It is well-known that F is an isomorphism of S onto itself (see e.g. DE JAGER [1970], p.95 ff. for more details on the subject of Fourier transforms). For any *tempered distribution* (i.e. $f \in S'$) the Fourier transform f can be defined by

$$<\hat{f},\varphi> = <f,\hat{\varphi}> \text{ for all } \varphi \in S.$$

1.3. THEOREM. *An equivalent definition of the* m-th *Sobolev space for the domain* $\mathbb{R}^n$ *is that it is the space of all tempered distributions* u *in* $\mathbb{R}^n$ *for which the expression* $(1 + |\xi|^2)^{m/2}\, \hat{u}(\xi)$ *is square integrable. The norm corresponding to the scalar product*

$$(1.2) \qquad (u,v)'_{m,\mathbb{R}^n} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^m\, \hat{u}(\xi)\, \bar{\hat{v}}(\xi) d\xi$$

is equivalent to the one described in 1.1.

PROOF. Using the fact that the Fourier transform is an isometry of $L^2(\mathbb{R}^n)$ onto itself, we have $\|D^p u\|_0 = \|\xi^p \hat{u}\|_0$, so that

$$\|u\|_m^2 = \int_{\mathbb{R}^n} \sum_{|p|\le m} \xi^{2p} |\hat{u}(\xi)|^2 d\xi.$$

The proof is easily completed. □

1.4. DEFINITION. The preceding theorem brings us into the position of being able to define Sobölev spaces for arbitrary non-integer values of s, as follows: for $s \in \mathbb{R}$

$$H^s(\mathbb{R}^n) = \{u \in S' \mid (1 + |\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n)\}.$$

The scalar product $(.,.)_{s,\mathbb{R}^n}$ is defined in the same manner as in (1.2).

Any function $u \in C_0^\infty(\Omega)$ $[H_0^m(\Omega)]$, where Ω is an arbitrary open set of $\mathbb{R}^n$, can be considered as an element of $C^\infty(\mathbb{R}^n)$ $[H^m(\mathbb{R}^n)]$, by extending u by zero outside Ω. To avoid unncessary complicated speech, one usually does not make a strict distinction between u and its trivial extension. The space $H_0^s(\Omega)$, Ω being a bounded open set of $\mathbb{R}^n$, can be defined as the closed subspace of elements $u \in H^s(\mathbb{R}^n)$ that have compact support within Ω.

The following theorem states for a special case the existence of a so-called *partition of unity*, which is needed in the definition of Sobolev spaces of functions defined on the boundary $\partial\Omega$ of a domain Ω.

1.5. THEOREM. *Let* K *be a compact set of* $\mathbb{R}^n$, *and let* $\Omega_1,\dots,\Omega_k$ *be open sets of* $\mathbb{R}^n$ *that cover* K, *i.e.* $K \subset \bigcup_{j=1}^k \Omega_j$. *Then there exist* C^∞ *functions* φ_j, $j = 1,\dots,k$, *having compact supports contained in* Ω_j, *such that*

(i) $\varphi_j(x) \ge 0$ and $\sum_{j=1}^k \varphi_j(x) \le 1$ in $\mathbb{R}^n$;

(ii) $\sum_{j=1}^k \varphi_j(x) = 1$ for $x \in K$.

PROOF. See FRIEDMAN [1969], p.9 ff. or NEČAS [1967], p.27. □

Consider a bounded domain $\Omega \subset \mathbb{R}^n$, of which the boundary is a C^m manifold $(m \ge 0)$. For each point $z \in \partial\Omega$ one can then find an open neighbourhood

Ω_z of z in $\mathbb{R}^n$ and a m times differentiable mapping κ_z which can be described in the same way as in (1.1), such that $\Omega_z \cap \Omega$ is mapped one-to-one onto an open set of $\mathbb{R}^n$ having a portion of its boundary on $\mathbb{R}^{n-1}$. Now because $\partial\Omega$ is compact, a finite number, say k, of such neighbourhoods $\Omega_{z_j} = \Omega_j$ of z_j are sufficient to cover $\partial\Omega$; call the corresponding maps κ_j. If $\Omega_j \cap \Omega_\ell \neq \emptyset$, then, of course, $\kappa_\ell\kappa_j^{-1}$ is a C^m one-to-one mapping of $\kappa_j(\Omega_j \cap \Omega_\ell)$ onto $\kappa_\ell(\Omega_j \cap \Omega_\ell)$.

As in 1.1, for each j one can change the order of the coordinates, such that i becomes equal to n; doing so, $\Omega_j \cap \partial\Omega$ will consist of the points $(x^{j\prime},h(x^{j\prime}))$, and Ω_j of $(x^{j\prime},x_n^j)$ with $x_n^j > h(x^{j\prime})$ whenever x belongs to Ω too. The domain of $x^{j\prime}$, which is the projection onto $\mathbb{R}^{n-1}$ of $\Omega_j \cap \partial\Omega$ will be denoted by Δ_j.

Now let $(\varphi_j)_{j=1}^k$ be a partition of unity with respect to Ω_j, such that for $x \in \partial\Omega$ $\sum_{j=1}^k \varphi_j(x) = 1$. If u is a function defined for almost all $x \in \partial\Omega$, then for each j, $1 \le j \le k$, the function $(\varphi_j u)(x^{j\prime},h(x^{j\prime})) = \varphi_j(x^{j\prime},h(x^{j\prime}))u(x^{j\prime},h(x^{j\prime}))$ is defined almost everywhere in Δ_j.

After all these preparations we are able to give the definitions of $L^2(\partial\Omega)$ and $H^s(\partial\Omega)$, $s > 0$.

1.6. DEFINITION. The space $L^2(\partial\Omega)$ consists of all functions u defined almost everywhere on $\partial\Omega$ that fulfil the requirement $(\varphi_j u)(x^{j\prime},h(x^{j\prime})) \in L^2(\Delta_j)$, $j = 1,\dots,k$. This linear space is equipped with a Hilbert space structure by means of the inner product

$$(1.3)\qquad (u,v)_{0,\partial\Omega} = \sum_{j=1}^k (\varphi_j u(x^{j\prime},h(x^{j\prime})),\ \varphi_j v(x^{j\prime},h(x^{j\prime})))_{0,\Delta_j}.$$

It is easily shown that a different choice of the covering by Ω_j or of the partition of unity φ_j leads to an inner product of which the corresponding norm is equivalent to the one corresponding to (1.3). The proof of the completeness of $L^2(\partial\Omega)$ is also very simple.

By $H^s(\partial\Omega)$ we denote the space of functions belonging to $L^2(\partial\Omega)$, such that for each j, $1 \le j \le k$, $\varphi_j u(x^{j\prime},h(x^{j\prime})) \in H^s(\mathbb{R}^{n-1})$. Because $\varphi_j = 0$ near the boundary of Δ_j, this essentially has the same meaning as being an element of $H_0^s(\Delta_j)$ (see definition 1.4). In this definition it is assumed that $0 \le s \le m$. NEČAS [1967], p.88, shows that in order to define $H^m(\partial\Omega)$ it is sufficient to have a $C^{m-1,1}$ boundary. The inner product in $H^m(\partial\Omega)$ is, quite naturally, defined as

$$(1.4) \qquad (u,v)_{s,\partial\Omega} = \sum_{j=1}^{k} (\varphi_j u(x^{j\prime},h(x^{j\prime})),\ \varphi_j v(x^{j\prime},h(x^{j\prime})))_{s,\Delta_j}.$$

Again, if we change the choice of the covering Ω_j or that of the partition of unity φ_j, then we merely replace the Hilbert space structure imposed on by (1.4) by an equivalent one.

2. TRACE THEOREMS

Let $\Omega \subset \mathbb{R}^n$ be a *bounded* domain, and u a function defined in Ω. If u is continuous in Ω up to the boundary, i.e. $u \in C^0(\overline{\Omega})$, it is obvious what we mean by the restriction of u to the boundary $\partial\Omega$. This is not the case when u is merely an element of some Sobolev space $H^m(\Omega)$. Nevertheless, when dealing with non-homogeneous boundary value problems we must be able to speak in some sense of the values on the boundary of a generalized solution of the elliptic differential equation. In this section, therefore, the problem of extending the notion "restriction of a function to the boundary" to non-continuous functions in $\partial\Omega$ is discussed. First of all, the following result is of great importance.

2.1. THEOREM. *Let* m *be any non-negative integer and let the boundary of* Ω *be* C^m. *The restrictions to* Ω *of the infinitely differentiable functions defined on* $\mathbb{R}^n$ *form a dense subspace of* $H^m(\Omega)$. □

This theorem can also be stated in the following form: $C^\infty(\overline{\Omega}) \cap H^m(\Omega)$ is dense in $H^m(\Omega)$. For a function $u \in C^\infty(\overline{\Omega})$ its values on $\partial\Omega$ are easily defined; these values are called the *trace* of u on $\partial\Omega$, which simply is its restriction to $\partial\Omega$.

The sort of theorem we are out for in this section is:

2.2. THEOREM. *Let* m *be an integer* ≥ 1. *Let further* Ω *have a* C^m *boundary. Then the trace on* $\partial\Omega$, *first defined in the above trivial manner for functions of* $C^\infty(\overline{\Omega})$ *can be extended in a unique fashion to a bounded linear* *mapping*

$$(2.1) \qquad \gamma : H^m(\Omega) \xrightarrow[\text{onto}]{} H^{m-\frac{1}{2}}(\partial\Omega). \qquad \square$$

The first step in the proofs of statements like those above consists of simplifying the geometrical situation for which the statement is to be proved. As in the preceding section we choose a finite covering $\{\Omega_j\}_{j=1}^k$ of $\partial\Omega$, such that for each j there exists a m-times differentiable one-to-one mapping κ_j of Ω_j onto an open set in $\mathbb{R}^n$ that has the property that $\kappa_j(\Omega_j \cap \Omega)$ has a portion of its boundary on $x_n = 0$. Additionally, we choose an open subset Ω_0 of Ω containing $\Omega \backslash \bigcup_{j=1}^k \Omega_j$. Let $\{\varphi_j\}_{j=0}^k$ be a partition of unity with respect to $\{\Omega_j\}_{j=0}^k$, such that for $x \in \overline{\Omega}$ $\sum_{j=0}^k \varphi_j(x) = 1$.

Let u be an arbitrary function of $H^m(\Omega)$; it can be written as $u = \sum_{j=0}^k \varphi_j u$. Suppose that the following version of theorem 2.1 is true:

2.3. LEMMA. *For each* $j = 0,\ldots,k$ *there exists a sequence of functions* $u_{j,\nu} \in C_0^\infty(\Omega_j)$ *which converge to* $\varphi_j u$ *in* $H^m(\Omega_j \cap \Omega)$. $\square$

From this lemma it would easily follow that $\sum_{j=0}^k u_{j,\nu}$ converges to u in $H^m(\Omega)$.

Now assume that $u \in C^\infty(\overline{\Omega})$, and let γu be its trace on $\partial\Omega$. Suppose further that the following lemma is valid:

2.4. LEMMA. *For each* $j = 1,\ldots,k$ *there exists a positive constant* C_j *independent of* u *such that*

$$\|\gamma(\varphi_j u)\|_{m-\frac{1}{2},\partial\Omega\cap\Omega_j} \le C_j \|\varphi_j u\|_{m,\Omega_j}$$

for all $u \in C^\infty(\overline{\Omega})$. $\square$

Since $\varphi_j u$ has a support contained in Ω_j, this lemma would imply that

$$\|\gamma u\|_{m-\frac{1}{2},\partial\Omega} \le C\|u\|_{m,\Omega}$$

for all $u \in C^\infty(\overline{\Omega})$, where C is some positive constant. Combination with theorem 2.1 would then establish that γ can be extended as a mapping of $H^m(\Omega) \longrightarrow H^{m-\frac{1}{2}}(\partial\Omega)$.

If all this was carried out, it would remain to show that the trace map γ is *onto*. This means that we would still have to prove the following lemma:

2.5. LEMMA. *Let* $w \in H^{m-\frac{1}{2}}(\partial\Omega)$. *For each* $j = 1,\dots,k$ *there exists a* $u_j \in H^m(\Omega_j \cap \Omega)$, *the support of which is contained in a compact subset of* Ω_j, *such that the trace of* u_j *is* $\varphi_j w$. □

Since the trace of $\sum_{j=1}^k u_j$ on $\partial\Omega$ is equal to w, lemma 2.5 would then imply that γ is a surjective mapping of $H^m(\Omega)$ onto $H^{m-\frac{1}{2}}(\partial\Omega)$.

So far, we have reduced the proofs of theorems 2.1 and 2.2 to the question whether lemmas 2.3, 2.4 and 2.5 are true. The next step is to see that it is sufficient to prove these lemmas for *flat* boundaries. This follows from the fact that for each $j = 1,\dots,k$ the C^m mapping κ_j maps $\Omega_j \cap \Omega$ one-to-one onto an open set in $\mathbb{R}^n$ that has a portion of its boundary on $x_n = 0$; or, which has precisely the same meaning, that the restrictions of the functions $\varphi_j u$ to $\partial\Omega$ may be considered as functions defined on $\Delta_j \subset \mathbb{R}^{n-1}$ in the following manner: $\varphi_j u = (\varphi_j u)(x^{j\prime}, h(x^{j\prime}))$; here, of course, $h \in C^m(\Delta_j)$.

So we have to do no more than to prove the lemmas for the case that $\Omega_j \cap \Omega$ is replaced by a subset V of $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$ that has a portion of its boundary, say Γ, on $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n \,|\, x_n = 0\}$.

At this point one more observation should be made. In the argument above all the elements of $H^m(\Omega \cap \Omega_j)$ and of $H^{m-\frac{1}{2}}(\Omega_j \cap \partial\Omega)$ involved had supports contained in compact subsets of Ω_j. Of course, the C^m mappings κ_j do not disturb this situation. As a consequence of this, we might as well prove the lemmas with V replaced by $\mathbb{R}^n_+$ and Γ by $\mathbb{R}^{n-1}$, since all functions considered may be extended by zero outside their supports.

So it turns out to be sufficient to prove the following three lemmas:

2.3'. LEMMA. *The restrictions to* $\mathbb{R}^n_+$ *of functions belonging to* $C_0^\infty(\mathbb{R}^n)$ *form a dense subspace in* $H^m(\mathbb{R}^n_+)$.

2.4'. LEMMA. *There exists a constant* $C > 0$ *such that for all* u *that are restrictions to* $\mathbb{R}^n_+$ *of functions belonging to* $C_0^\infty(\mathbb{R}^n)$, (2.2) $\qquad \|\gamma u\|_{m-\frac{1}{2},\mathbb{R}^{n-1}} \le C\ \|u\|_{m,\mathbb{R}^n_+},$

where γu *denotes the trace on (or restriction to)* $\mathbb{R}^{n-1}$ *of* u. *(This implies, in combination with the previous lemma, that* γ *may be extended as a continuous linear map of* $H^m(\mathbb{R}^n_+)$ *into* $H^{m-\frac{1}{2}}(\mathbb{R}^{n-1})$.)

2.5'. LEMMA. *Let* v *be any element of* $H^{m-\frac{1}{2}}(\mathbb{R}^{n-1})$, *then there exists a* $u \in H^m(\mathbb{R}^n_+)$ *such that* $\gamma u = v$.

For the proof of lemma 2.3' one is referred to e.g. WLOKA [1969] p.9 or TRÈVES [1975], p.241. With regard to lemma 2.4' and 2.5' we shall leave our policy to omit proofs. The main reason for this is not that their proofs are simple - which they are -, but that then the loss of ½ in (2.1) and (2.2) is explained.

PROOF OF LEMMA 2.4'. Let u be an arbitrary functions that is a restriction of a function of $C_0^\infty(\mathbb{R}^n)$. This function u may be considered as the restriction of a function $\tilde{u} \in C_0^\infty(\mathbb{R}^n)$ such that

$$\|\tilde{u}\|_{m,\mathbb{R}^n} \le \text{const.}\ \|u\|_{m,\mathbb{R}^n_+}$$

For example, one can extend u given in $\mathbb{R}^n_+$ to the lower half space by setting for $x_n < 0$

$$\tilde{u}(x',x_n) = \sum_{\ell=1}^{m} \lambda_\ell u(x', -\ell x_n),$$

where the coefficients λ_ℓ satisfy the system of equations

$$\sum_{\ell=1}^{m} (-\ell)^p \lambda_\ell = 1,\ p = 0,\dots,m-1.$$

In the argument below we shall drop the distinction between u and $\tilde{u}$.

Denote by $F'u(\xi',x_n)$ the Fourier transform with respect to the first n-1 variables $x' = (x_1,\dots,x_{n-1})$ (by ξ' we of course mean $(\xi_1,\dots,\xi_{n-1})$), and by $Fu(\xi) = Fu(\xi',\xi_n)$ the Fourier transform with respect to all variables. Then one has

$$F'u(\xi',0) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} Fu(\xi',\xi_n)d\xi_n$$

as a consequence of the inverse Fourier transform formula with respect to ξ_n. Using the Cauchy-Schwarz inequality one finds

$$(2\pi)^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} |F'u(\xi',0)|^2 (1+|\xi'|^2)^{m-\frac{1}{2}} d\xi' = \int_{\mathbb{R}^{n-1}} \Big| \int_{-\infty}^{\infty} Fu(\xi',\xi_n) d\xi_n \Big|^2 (1+|\xi'|^2)^{m-\frac{1}{2}} d\xi'$$

$$= \int_{\mathbb{R}^{n-1}} \Big| \int_{-\infty}^{\infty} Fu(\xi)(1+|\xi|^2)^{m/2}(1+|\xi|^2)^{-m/2} d\xi_n \Big|^2 (1+|\xi'|^2)^{m-\frac{1}{2}} d\xi'$$

(2.3)

$$\le \int_{\mathbb{R}^{n-1}} \Big[\int_{-\infty}^{\infty} |Fu(\xi)|^2 (1+|\xi|^2)^m d\xi_n \int_{-\infty}^{\infty} (1+|\xi|^2)^{-m} d\xi_n \Big] (1+|\xi'|^2)^{m-\frac{1}{2}} d\xi'.$$

Relatively simple calculations lead to an identity of the form

(2.4) $$\int_{-\infty}^{\infty} (1+|\xi'|^2 + \xi_n^2)^{-m} d\xi_n = \mu(m)(1+|\xi'|^2)^{-m+\frac{1}{2}},$$

where $\mu(m)$ is some constant depending on m. In view of this identity and of theorem 1.3 one then sees that (2.3) is equivalent to

$$\|u(x',0)\|_{m-\frac{1}{2},\mathbb{R}^{n-1}} \le \text{const. } \|u(x)\|_{m,\mathbb{R}^n_+}. \qquad \square$$

<u>PROOF OF LEMMA 2.5'</u>. Let $v(x')$ be any element of $H^{m-\frac{1}{2}}(\mathbb{R}^{n-1})$. Denote by $(F')^{-1}$ the inverse Fourier transform with respect to the n-1 variables ξ'. Then

(2.5) $$u(x',x_n) = (F')^{-1}[F'v(\xi')\exp(-(1+|\xi'|^2)^{\frac{1}{2}} x_n)]$$

is a function that fulfils the requirements. Clearly, for $x_n = 0$ the function is defined by (2.5) is equal to v. It remains to show that $u \in H^m(\mathbb{R}^n_+)$. In order to have u defined for all x_n, we extend u as an even function in x_n for $x_n < 0$.

Then we have

$$Fu(\xi',\xi_n) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} F'v(\xi')(1+|\xi'|^2)^{\frac{1}{2}}(1+|\xi'|^2 + \xi_n^2)^{-1},$$

so using (2.4) with m = 1 we find

$$\int_{\mathbb{R}^n} |Fu(\xi)|^2(1+|\xi|^2)^m = \frac{2}{\pi}\int_{\mathbb{R}^{n-1}} |F'v(\xi')|^2(1+|\xi'|^2) \int_{-\infty}^{\infty} (1+|\xi|^2)^{m-2}d\xi_n d\xi'$$

$$\leq \frac{2}{\pi}\int_{\mathbb{R}^{n-1}} |F'v(\xi')|^2(1+|\xi'|^2)^m \int_{-\infty}^{\infty} (1+|\xi|^2)^{-1}d\xi_n d\xi'$$

$$\leq \text{const.} \int_{\mathbb{R}^{n-1}} |F'v(\xi')|^2(1+|\xi'|^2)^{m-\frac{1}{2}}d\xi'.$$

In fact, we have shown that the mapping ε: $H^{m-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H^m(\mathbb{R}^n_+)$ defined by

(2.6) $$\varepsilon v(x) = (F')^{-1}[F'v(\xi')\exp(-(1+|\xi'|^2)^{\frac{1}{2}}x_n)]$$

is continuous; this mapping ε, of course, is a *right inverse* to γ, i.e. $\gamma(\varepsilon v) = v$ for all $v \in H^{m-\frac{1}{2}}(\mathbb{R}^{n-1})$. □

In view of former considerations, this completes the proof of theorem 2.2.

2.6. REMARK. Note that the condition that $\partial\Omega$ is a C^m boundary is only used where the portion of $\partial\Omega$ within Ω_j is flattened. NEČAS [1967], p.88 ff. shows that it would have been sufficient to require $C^{m-1,1}$ smoothness.

2.7. DEFINITION. We now may define *traces of higher order* $\ell < m$. Clearly, D_n^ℓ maps $H^m(\mathbb{R}^n_+)$ into $H^{m-\ell}(\mathbb{R}^n_+)$, and γ the latter onto $H^{m-\ell-\frac{1}{2}}(\mathbb{R}^n_+)$. Define the ℓ-th order trace as the continuous linear map

$$\gamma_\ell : H^m(\mathbb{R}^n_+) \longrightarrow H^{m-\ell-\frac{1}{2}}(\mathbb{R}^{n-1})$$

$$\gamma_\ell u = \gamma(D_n^\ell u).$$

In a way similar to the proof of lemma 2.5' one can show that

$$\varepsilon_\ell v(x) := D_n^\ell \varepsilon v(x) = (F')^{-1}[F'v(\xi')(1+|\xi'|^2)^{-\ell/2}\exp(-(1+|\xi'|^2)^{\frac{1}{2}}x_n)]$$

defines a continuous right inverse $\varepsilon_\ell\colon H^{m-\ell-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H^m(\mathbb{R}^n_+)$ to γ_ℓ. This leads us to the following theorem.

2.8. THEOREM. *Let* m *be a positive integer. Denote by* $\tilde{\gamma}$ *the mapping*

$$H^m(\mathbb{R}^n_+) \to \prod_{\ell=0}^{m-1} H^{m-\ell-\frac{1}{2}}(\mathbb{R}^{n-1})$$

$$u \mapsto (\gamma_0 u, \ldots, \gamma_{m-1} u).$$

This mapping is continuous, linear and surjective; furthermore, the kernel of $\tilde{\gamma}$ *in* $H^m(\mathbb{R}^n_+)$ *is exactly equal to* $H^m_0(\mathbb{R}^n_+)$. □

Of course, one has the following theorem.

2.9. THEOREM. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with* C^m*-boundary. Then the mapping*

$$\tilde{\gamma}\colon H^m(\Omega) \to \prod_{\ell=0}^{m-1} H^{m-\ell-\frac{1}{2}}(\partial\Omega)$$

$$u \to \left(\gamma u,\ \gamma\frac{\partial u}{\partial\nu}, \ldots, \gamma\frac{\partial^{m-1} u}{\partial\nu^{m-1}}\right),$$

$\partial u/\partial\nu$ *denoting the derivative along the outwardly oriented normal to* $\partial\Omega$*, is continuous, linear and surjective.*

PROOF. By localization and flattening of the boundary as was set forth at the beginning of this section. □

3. APPLICATION TO THE NON-HOMOGENEOUS DIRICHLET PROBLEM

In this section the trace theorems are used to deal with the non-homogeneous Dirichlet problem for elliptic equations. It is assumed that the reader is familiar with the Hilbert space theory for homogeneous elliptic boundary value problems, a survey of which may be found in [H].

3.1. THE GENERALIZED NON-HOMOGENEOUS DIRICHLET PROBLEM. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^m boundary $\partial\Omega$, and let

$$L\cdot = \sum_{|p|,|q|\le m} (-1)^{|p|} (D^p\, a_{pq}(x) D^q \cdot) \tag{3.1}$$

be a uniformly strongly elliptic operator in Ω. Let $B(u,v)$ be the bilinear form associated with L, i.e.

$$(3.2)\qquad B(u,v) = \sum_{|p|,|q|\le m} (a_{pq}(x)\, D^q u,\ D^p v)_{0,\Omega},$$

defined for all $u,\ v \in H^m(\Omega)$. It is easily seen that B is bounded on $H^m(\Omega)$. Further, according to Gårding's inequality, under certain mild conditions for which the reader is referred to [H], B then is coercive over $H_0^m(\Omega)$. For convenience, we here assume that B is *strongly* coercive.

Assume that are given arbitrary $f \in L^2(\Omega)$ and $g_j \in H^{m-j-\frac{1}{2}}(\partial\Omega)$, $j = 0,\dots,m-1$ and consider the Dirichlet problem

$$(3.3)\qquad \begin{cases} Lu = f & \text{in } \Omega \\ \partial^j u/\partial\nu^j = g_j & \text{on } \partial\Omega,\quad j = 0,\dots,m-1. \end{cases}$$

By the latter of course is meant $\gamma(\partial^j u/\partial\nu^j) = g_j$. By theorem 2.9 one can find a function $F \in H^m(\Omega)$, such that F depends continuously on g_j in the sense that the map

$$\begin{aligned} H^{m-j-\frac{1}{2}}(\partial\Omega) &\to H^m(\Omega) \\ g_j &\to F \end{aligned}$$

is continuous. Let $w = u-F$. Then w will satisfy the homogeneous boundary value problem

$$(3.4)\qquad \begin{cases} Lw = f - LF & \text{in } \Omega \\ \partial^j w/\partial\nu^j = 0 & \text{on } \partial\Omega,\quad j = 0,\dots,m-1. \end{cases}$$

Here LF has only a meaning in distributional sense. Nevertheless, this will not cause any difficulties if we make use of the bilinear form B, since then differentiations up to order m "are brought to the other side". The generalized problem for (3.4), or the generalized Dirichlet problem for non-vanishing boundary data is formulated in the following manner:

$$(3.5)\qquad \begin{cases} \text{find } w = u - F \in H_0^m(\Omega), \text{ such that} \\ B(w,v) = (f,v)_0 - B(F,v) \text{ for all } v \in H_0^m(\Omega). \end{cases}$$

The Lax-Milgram theorem guarantees a uniquely determined solution to (3.5) in case that B is strongly coercive. If one has coercivity only, a Fredholm type existence theorem applies (see [H]).

Next, we wish to discuss the dependency of the solution w of 3.5 on the data f and g_j, $j = 0,\dots,m-1$. In view of theorem 2.9 we have

$$(3.6)\quad \begin{aligned} B(w,v) &\le \|f\|_{0,\Omega}\|v\|_{0,\Omega} + \text{const.}\ \|F\|_{m,\Omega}\|v\|_{m,\Omega} \\ &\le \text{const.}\ (\|f\|_{0,\Omega} + \sum_{j=0}^{m-1} \|g_j\|_{m-j-\frac{1}{2},\partial\Omega})\|v\|_{m,\Omega} \end{aligned}$$

for all $v \in H_0^m(\Omega)$. Choosing $v = w$ in (3.6), we find, assuming strong coervity of B,

$$\|w\|^2_{m,\Omega} \le \text{const.}\ (\|f\|_{0,\Omega} + \sum_{j=0}^{m-1} \|g_j\|_{m-j-\frac{1}{2},\partial\Omega})\|w\|_{m,\Omega}.$$

By dividing both sides of this inequality by $\|w\|_{m,\Omega}$ we get the following result.

3.2. THEOREM. *Let Ω be a bounded domain of $\mathbb{R}^n$ with C^m boundary, and let B be a bounded and strongly coercive bilinear form over $H_0^m(\Omega)$. Then the linear mapping*

$$T:\ L^2(\Omega) \times \prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\partial\Omega) \to H^m(\Omega)$$

$$(f, g_0, \dots, g_{m-1}) \to w$$

where w is the (unique) solution of the generalized Dirichlet problem (3.5) for this set of data, is continuous. A different way to state this result is to say that the generalized solution of the non-homogeneous elliptic boundary value problem depends continuously on the data. □

3.3. REGULARITY. There is no significant difference between the regularity theory for homogeneous and non-homogeneous boundary value problems. All we have to do is to require that F is sufficiently regular, namely that $F \in H^{2m}(\Omega)$, which, of course, means that we have to assume that the boundary data $g_j \in H^{2m-j-\frac{1}{2}}(\partial\Omega)$, $j = 0,\ldots,m-1$. Then $B(F,v) = (LF,v)_{0,\Omega}$ for all $v \in H_0^m(\Omega)$, so that all we have to do is to deal with the regularity of a homogeneous boundary value problem with righthand side f-LF, which type of problem was considered in section 4 of [H].

4. GENERAL REMARKS ON OTHER BOUNDARY VALUE PROBLEMS

So far, only Dirichlet boundary conditions were considered. We now wish to turn our attention to boundary value problems of other type. In [H], section 5, some very brief remarks were made on the homogeneous Neumann and mixed problems. Here we shall deal with non-homogeneous boundary value problems of various types. Before passing on to a more general discussion of the existence and regularity of generalized solutions of such boundary value problems, we first examine a simple example.

4.1. THE NEUMANN PROBLEM FOR $-\Delta+\lambda$. Let Ω be bounded, $\partial\Omega$ of class C^2, $\lambda \in \mathbb{R}$, $\lambda > 0$, $f \in L^2(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$. We wish to "solve"

(4.1.a) $\quad (-\Delta+\lambda)u = f \quad \text{in } \Omega,$

(4.1.b) $\quad \partial u/\partial\nu|_{\partial\Omega} = g,$

in a way similar to the treatment of the Dirichlet problem. The boundary condition should be read as

(4.2) $\quad \gamma(\partial u/\partial\nu) = g \quad (\text{in } H^{\frac{1}{2}}(\partial\Omega)).$

The trace theory provides us the existence of a function $h \in H^2(\Omega)$ such that $\gamma(\partial h/\partial\nu) = g$. Hence, by setting $u = w+h$, (4.1) can be replaced by a problem with homogeneous boundary data, namely

(4.3)
$$(-\Delta+\lambda)w = (\Delta-\lambda)h+f \quad \text{in } \Omega,$$
$$\partial w/\partial\nu|_{\partial\Omega} = 0.$$

We now introduce the bilinear form

$$D_\lambda(u,v) = D(u,v) + \lambda(u,v)_{0,\Omega} \quad \text{for } u,\ v \in H^1(\Omega),$$

where D(u,v) is the *Dirichlet integral*

(4.4) $$D(u,v) = \sum_{i=1}^{n} \int_\Omega D_i u\, \overline{D_i v}\, dx.$$

Since

$$v \longmapsto (f,v)_{0,\Omega} + (g,\gamma v)_{0,\partial\Omega}$$

is a continuous bilinear - or rather antilinear - functional on $H^1(\Omega)$, and since D_λ is strongly coercive over $H^1(\Omega)$ (see [H]), the Lax-Milgram theorem implies the existence of a unique element $u \in H^1(\Omega)$ such that

(4.5) $$D_\lambda(u,v) = (f,v)_{0,} + (g,\gamma v)_{0,\partial\Omega} \quad \text{for all } v \in H^1(\Omega).$$

Let us, for the moment, restrict the set of functions v to the subspace $H_0^1(\Omega)$. We then find that

$$D_\lambda(u,v) = (f,v)_{0,\Omega} \quad \text{for all } v \in H_0^1(\Omega),$$

since now the boundary integral becomes zero. So the unique solution u defined by (4.5) satisfies in a generalized sense equation (4.1.a) in any Ω' with $\overline{\Omega}' \subset \Omega$.

It will be shown later on that the regularity theory in [H], section 4 remains true for other than Dirichlet boundary conditions. Making use of this fact, we find, since f and $\Delta h \in L^2(\Omega)$, that $u \in H^2(\Omega)$. Employing the well-known *Green's formula*

$$\int_\Omega \nabla u \nabla \bar{v} dx = \int_\Omega -\Delta u \bar{v} dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \bar{v} d\sigma$$

we find that

(4.6) $$D_\lambda(u,v) = ((-\Delta+\lambda)u,v)_{0,\Omega} + (\gamma \frac{\partial u}{\partial \nu}, \gamma v)_{0,\partial\Omega}.$$

The first term on the right hand side is equal to $(f,v)_{0,\Omega}$ so that comparison of (4.5) and (4.6) shows that

$$(g,\gamma v)_{0,\partial\Omega} = (\gamma \frac{\partial u}{\partial \nu}, \gamma v)_{0,\partial\Omega}.$$

Since the functions v are essentially arbitrary, the conclusion must be $\gamma \frac{\partial u}{\partial \nu} = g$ (in $H^{\frac{1}{2}}(\partial\Omega)$).

4.2. ABSTRACT BOUNDARY VALUE PROBLEM. The above example should give us enough motivation to study the following type of problems, that we shall call *abstract boundary value problems.*

Let V be a given closed subspace of $H^m(\Omega)$, such that $H_0^m(\Omega) \subset V$. Let B be a bilinear form associated to the uniformly strongly elliptic operator (3.1), i.e.

(4.7) $$B(\varphi,\psi) = (L\varphi,\psi)_{0,\Omega} \quad \text{for all } \varphi,\ \psi \in C_0^\infty(\Omega).$$

Assume that B is coercive over V. Find for given $f \in L^2(\Omega)$ the solution $u \in V$ of

(4.8) $$B(u,v) = (f,v)_{0,\Omega} \quad \text{for all } v \in V.$$

If $H_0^m(\Omega)$ is a proper subspace of V, then one does not consider the Dirichlet problem, but some other type of problem. In the example above $m = 1$, $V = H^1(\Omega)$ was chosen, which led to the Neumann problem.

In this section and the next we shall give answers to the following questions.

(i) How is (unique) *existence* of a solution $u \in V$ established in the case that B is not strongly coercive, but merely coercive over V?

(ii) What can one say about the *regularity* of the solution $u \in V$ of (4.8)?

(iii) Which *"concrete" interpretations* should one adhere to the abstract boundary value problem for different choices of V and of B?

The bilinear form B associated with the differential operator L

$$B(u,v) = \sum_{|p|,|q|\le m} (a_{pq}(x)\, D^p u,\ D^q v)_{0,\Omega}$$

is coercive over $H_0^m(\Omega)$, if L satisfies certain mild conditions (see [H], theorem 2.15, Gårding's inequality). This result does not extend to coercivity over $H^m(\Omega)$, as we shall see later on. That is why we have to state the coercivity of B explicitly amongst the conditions of the following theorems.

In part the proof of the Fredholm type existence theorem for the generalized Dirichlet problem (see theorem 3.5 of [H]) was based on the compact imbedding I: $H_0^m(\Omega) \to L^2(\Omega)$. For the general abstract boundary value problem given above we need the compactness of the imbedding I: $V \to L^2(\Omega)$. If the boundary is Lipschitz continuous, this is simply a consequence of Rellich's theorem (theorem 1.11 of [H]). So it remains possible to apply the Riesz-Schauder theory that sustained the existence theory for the Dirichlet problem. Therefore, the following theorem holds.

4.3. THEOREM. *Let* Ω *have a Lipschitz continuous boundary, and let* B *and* V *be as above. Assume that* B *is coercive over* V *and bounded on* V. *Then for any* $f \in L^2(\Omega)$ *either there exists a unique solution* $u \in V$ *of*

$$(4.9) \qquad B(u,v) = (f,v)_{0,\Omega} \quad \text{for all} \quad v \in V,$$

or the homogeneous equation

$$(4.10) \qquad B(v,u) = 0 \quad \text{for all} \quad v \in V$$

has a finite number of linearly independent solutions u_j, $j = 1,\dots,\ell$, *all belonging to* V, *in which case equation* (4.9) *has a (non-unique) solution if and only if*

$$(f,u_j)_{0,\Omega} = 0 \qquad j = 1,\dots,\ell. \qquad \square$$

The next question is that of regularity. As for the interior regularity, there is no difference with the case of Dirichlet boundary conditions, since boundary conditions do not play a role at all here. To proof regularity, up to the boundary, one first has to carry out the procedure of localization and flattening of the boundary, as was described in the beginning of section 2. The compact boundary $\partial\Omega$, which is assumed to be of class C^k, is covered by a finite number of open sets Ω_j, in such a way that $\Omega \cap \Omega_j$ is mapped one-to-one and k times differentiably onto an open set in the upper half space, that has a portion of its boundary on $x_n = 0$. Further, a partition of unity $\{\zeta_j\}$ with respect to Ω_j is introduced. In this way the problem of regularity of the solution u up to the boundary is reduced to the regularity of each $\zeta_j u$ in the closed upper half space $\{x \in \mathbb{R} | x_n \geq 0\}$.

The next step in the proof of regularity is to extend the function ζu - we shall drop the subscript j in the rest of our exposition - by some reflection principle into the lower half plane (how this can be achieved is described in the proof of lemma 2.4'). In consequence of this the proof of regularity of ζu up to the boundary $\mathbb{R}^{n-1}$ boils down to the proof of *interior regularity* of the extended ζu in $\mathbb{R}^n$, or, which is the same, in some compact subset of $\mathbb{R}^n$. All this can be carried through for the case $V \neq H_0^m$ if the following to conditions hold:

(V1) if $v \in V$, then $\zeta v \in V$ for all $\zeta \in C_0^\infty(\mathbb{R}^n)$

(V2) if $v \in V$, then sufficiently small translations of ζv in directions tangent to $x_n = 0$ give elements of V.

The first condition is necessary to allow a proof by partition of unity, the second is needed to guarantee that the difference quotients of ζv parallel to $x_n = 0$, and thus their limits, remain in V. Both conditions are, as is easily seen, fulfilled in the case of Dirichlet boundary conditions, i.e. $V = H_0^m$.

On account of these considerations it can be shown that the regularity theory for the Dirichlet case (as summarized in [H], section 4) may be extended to more general boundary conditions. For details see AGMON [1965], p.142 f.

4.4. THEOREM *(Regularity). Let* Ω *be bounded,* $\partial\Omega$ *of class* C^{2m}, *let* V *be a subspace of* $H^m(\Omega)$ *containing* $H_0^m(\Omega)$, *such that conditions* (V1) *and* (V2) *are satisfied. Let further the bilinear form* B *be coercive and bounded on* V, *and also j-smooth, i.e. the coefficients* $a_{pq}(x)$, *for* $|p|+j-m \geq 0, |q| \leq m$ *belong to* $C^{|p|+j-m}(\bar{\Omega})$. *Let, finally,* f *be any element of* $L^2(\Omega)$ *and let* $u \in V$ *solve the equation*

$$B(u,v) = (f,v)_{0,\Omega} \quad \textit{for all } v \in V.$$

Then $u \in H^{m+j}(\Omega)$. □

After having settled existence and regularity, we, lastly, must furnish the means to find a "concrete" interpretation of the abstract boundary value problem for different choices of B and V. In the rest of this section we will always assume the coefficients of B (or of L) to be sufficiently differentiable to allow the manipulations performed.

4.5. DEFINITION. Let L be a uniformly strongly elliptic operator in Ω and V a closed subspace of $H^m(\Omega)$ containing $H_0^m(\Omega)$. A bilinear form $B: V \times V \to \mathbb{C}$ is said to be *associated to* L if

$$(4.11) \qquad B(\varphi,\psi) = (L\varphi,\psi)_{0,\Omega} \quad \text{for all } \varphi,\ \psi \in C_0^\infty(\Omega).$$

In connection with Dirichlet boundary conditions we always spoke of *the* bilinear form associated with L. Apparently more bilinear forms satisfying (4.11) are possible. Later on, though, it will become clear that in the Dirichlet case, i.e. in the case that $V = H_0^m(\Omega)$, different choices for B lead to the same boundary value problem. This explains our former way of speech.

4.6. DEFINITION. Let $\partial\Omega$ be of class C^k, and let M_j, $j = 0,\dots,k-1$ be k *boundary operators* of order j defined by

$$(4.12) \qquad M_j\varphi = \sum_{|r|\le j} c_{jr}(x)\, D^r\varphi$$

for all $\varphi \in C^k(\bar{\Omega})$, with $c_{jr} \in C^k(\partial\Omega)$. More exactly, $M_j\varphi$ is defined by the mapping

$$C^k(\bar{\Omega}) \longrightarrow H^{k-j-\frac{1}{2}}(\partial\Omega)$$

$$\varphi \longrightarrow \sum_{|r|\le j} c_{jr}(x)\, \gamma\, (D^r\varphi),$$

where γ is the trace mapping from theorem 2.2. These boundary operators can be extended in an evident manner to all $\varphi \in H^k(\Omega)$. The boundary $\partial\Omega$ is said to be *non-characteristic for* M_j if

$$(4.13) \qquad \sum_{|r|\le j} c_{jr}(x)\, \nu^r \neq 0 \qquad \text{on } \partial\Omega,$$

where $\nu = (\nu_1,\dots,\nu_n)$ is the unit outer normal vector at x to $\partial\Omega$. This has the meaning that all along the boundary the direction of the highest order derivative in M_j is nowhere tangent to $\partial\Omega$.

4.7. THEOREM. *(Generalized Green's formula.) Let* Ω *be bounded and have* C^{2m} *boundary,* L *be a uniformly strongly elliptic operator of order* 2m*,* B *a bilinear form associated with* L*. Let further be given a system of linear boundary operators* $\{M_j\}_{j=0}^{m-1}$*, each exactly of order* j *with the property that* $\partial\Omega$ *is nowhere characteristic for these operators. Then there exist* m *linear differential operators* N_{2m-1-j} *of order* $2m-1-j$*,* $j = m,\dots,2m-1$*, defined on* $\partial\Omega$*, which is non-characteristic for these operators, such that for all* $u, v \in C^{2m}(\bar{\Omega})$

$$(4.14)\qquad B(u,v) = (Lu,v)_{0,\Omega} + \sum_{j=0}^{m-1} \int_{\partial\Omega} (N_{2m-1-j}u)\overline{M_j v}\, d\sigma.$$

Here the operators N_{2m-1-j} *do not only depend on* M_j*, but also on the bilinear form* B *that is chosen.*

PROOF. See LIONS & MAGENES [1968], p.127 f. or AGMON [1965], p.134 f. Again, the proofs make use of the localization and flattening of the boundary. □

4.8. REMARKS. Formula (4.14) remains true under less restrictive smoothness conditions for $\partial\Omega$. Further, in practice the exact forms of N_{2m-1-j} are found by integration by parts. Of course the formula extends to $u, v \in H^{2m}(\Omega)$ by interpreting the boundary integrals in the trace sense. Choosing m = 1, $M_0 v = v$ and writing L in the form

$$(4.15)\qquad L = -\sum_{i,j=1} D_i(a_{ij}D_j\cdot) + \sum_{i=1}^{n} a_i D_i\cdot + a_0\cdot$$

one gets instead of (4.12) the more familiar formula

$$(4.16)\qquad B(u,v) = (Lu,v)_{0,\Omega} + \int_{\partial\Omega} N_1 u\bar{v}\, d\sigma$$

where B is the bilinear form

$$(4.17)\qquad B(u,v) = \sum_{i,j=1}^{n} (a_{ij}D_j u, D_i v)_{0,\Omega} + \sum_{i,j=1} (a_i D_i u, v)_{0,\Omega} + (a_0 u, v)_{0,\Omega}$$

and

$$(4.18)\qquad N_1 u = \sum_{i,j=1}^{n} a_{ij} \frac{\partial x_i}{\partial \nu} D_j u = \nu \cdot A \text{ grad } u,$$

where A is the matrix of coefficients a_{ij}.
Condition (4.13) in the second order case reads

$$\sum_{j=1}^{n} c_j \nu_j + \sum_{j=1}^{n} d_j \neq 0 \quad \text{on } \partial\Omega$$

so that the strong ellipticity condition implies immediately that $\partial\Omega$ is non-characteristic for N_1:

$$\sum_{j=1}^{n} c_j \nu_j = \sum_{i,j=1}^{n} a_{ij} \nu_i \nu_j \geq E|\nu|^2.$$

4.9. NATURAL BOUNDARY CONDITIONS. Let $k < m$, and let $M_{i(j)}$, $j = 0,\dots,k-1$ be given boundary operators all of different order $i(j) \leq m-1$, such that $\partial\Omega$ is nowhere characteristic for each $M_{i(j)}$. Let V be the closure in $H^m(\Omega)$ of all functions $\varphi \in C^\infty(\overline{\Omega})$ such that $M_{i(j)}\varphi = 0$, $j = 0,\dots,k-1$. In addition choose m-k boundary operators $M_{i(j)}$, $j = k,\dots,m-1$ each of order $i(j)$ different among themselves and different from the orders of the first k operators $M_{i(j)}$. (In fact, this may be expressed as $\{i(j)\,|\,0 \leq j \leq k-1\} \cap \{i(j)\,|\,k \leq j \leq m-1\} = \emptyset$ and $\{i(j)\,|\,0 \leq j \leq m-1\} = \{i\,|\,0 \leq i \leq m-1\}$). These m-k operators should be chosen such that $\partial\Omega$ is nowhere characteristic for them, but further the choice is arbitrary. Now, if $u \in V$ solves

$$B(u,v) = (f,v)_{0,\Omega} \quad \text{for all } v \in V,$$

then by (4.14) one has

$$(Lu,\varphi)_{0,\Omega} = (f,\varphi)_{0,\Omega} \quad \text{for all } \varphi \in C_0^\infty(\Omega),$$

or, in other words, $Lu = f$ in a generalized sense in any sub-domain of Ω. Making use of this knowledge, and again applying (4.14), one finds that this solution u satisfies

$$\int_{\partial\Omega} \sum_{j=0}^{m-1} N_{2m-1-i(j)}u \, \overline{M_{i(j)}v} \, d\sigma = 0 \quad \text{for all } v \in V.$$

Thus, since $M_{i(j)}v$, $j = k,\ldots,m-1$ can be chosen essentially arbitrarily, one obtains besides the given boundary conditions $M_{i(j)}u = 0$, $j = 0,\ldots,k-1$, the so-called *natural boundary conditions*

$$N_{2m-1-i(j)}u = 0, \qquad j = k,\ldots,m-1.$$

Observe that the space V is defined such that it reflects the *given* boundary conditions. If $k = m$, then natural boundary conditions do not occur.

5. VARIATIONAL BOUNDARY VALUE PROBLEMS FOR SECOND ORDER EQUATIONS

In this section we wish to show that certain boundary value problems for elliptic equations of second order can be treated in a unified manner. Before starting off the discussion of these boundary value problems, it should be emphasized that not all types of boundary value problems allow a reformulation in terms of a (strongly) coercive bilinear form over a "right" Hilbert space V that is intermediate between $H_0^1(\Omega)$ and $H^1(\Omega)$. Those which do are often referred to as *variational* boundary value problems. The reason for this is that it can be shown (see e.g. TREVES [1967], p.196,197) that the solution u of the abstract boundary value problem

$$B(u,v) = (f,v)_{0,\Omega} \quad \text{for all } v \in V$$

at the same time minimizes strictly the non-linear functional

$$Q(v) = \tfrac{1}{2}B(v,v) - (f,v)_{0,\Omega}$$

on V.

We shall now discuss a variety of examples, using the general knowledge on existence and regularity of solutions that was summed up in the preceding section. Throughout this section Ω will always be a *bounded* domain with sufficiently smooth boundary. Further, we shall not bother to state the smoothness requirements the coëfficients of the operators are to fulfil.

5.1. THE NEUMANN PROBLEM. Let L be the general uniformly strongly elliptic operator (4.15), and B the bilinear form (4.17) associated with L. According to Gårding's inequality (see [H], theorem 2.7), B is coercive over $H^1(\Omega)$. Now assume that we are in the situation that the abstract boundary value problem

$$B(u,v) = (f,v)_{0,\Omega} \quad \text{for all } v \in H^1(\Omega), \tag{5.1}$$

where $f \in L^2(\Omega)$, has a *unique* solution. Since the conditions (V1) and (V2) certainly hold for $V = H^1(\Omega)$, we may apply the regulatity theory. Hence, $f \in L^2(\Omega)$ implies $u \in H^2(\Omega)$. Arguing as in example 4.1, we first restrict v to $H_0^1(\Omega)$, thus finding

$$B(u,v) = L(u,v)_{0,\Omega} = (f,v)_{0,\Omega} \text{ for all } v \in H_0^1(\Omega),$$

so that in the interior of Ω $Lu = f$. Taking subsequently the whole of $H^1(\Omega)$ as the range for the functions v, and using Green's formula (4.16), we find

$$\int_{\partial\Omega} (\nu .\ A \ \ \text{grad } u)\bar{v}\, d\sigma = 0 \text{ for all } v \in H^1(\Omega),$$

so, because the functions v are essentailly arbitrary on $\partial\Omega$, we obtain the boundary conditions

$$\frac{\partial u}{\partial \nu_L} := \nu .\ A \ \ \text{grad } u = 0 \text{ on } \partial\Omega,$$

which, of course, should be understood in the sense of traces, i.e.

$$\gamma(\nu .\ A \ \ \text{grad } u) = 0.$$

Consequently, the abstract problem (5.1) has the concrete interpretation

$$Lu = f \text{ in } \Omega, \tag{5.2}$$

$$\frac{\partial u}{\partial \nu_L} = 0 \text{ on } \partial\Omega. \tag{5.3}$$

Observe that, if $L = -\Delta$, $\partial u/\partial\nu_L$ is exactly $\partial u/\partial\nu$. Treatment of a non-homogeneous Neumann boundary value problem

$$(5.4) \qquad \gamma \frac{\partial u}{\partial \nu_L} = g \text{ on } \partial\Omega$$

causes no trouble if $g \in H^{\frac{1}{2}}(\partial\Omega)$, because then, by a straightforward generalization of the trace theory, one can find a function $h \in H^2(\Omega)$ such that $\gamma(\partial h/\partial\nu_L) = g$. Then one proceeds in the same fashion as in example 4.1.

Higher regularity of the solution is obtainable in case the boundary, the boundary condition g and the right hand size f satisfy stronger regularity requirements. From Sobolev's theorem (see [H], theorem 1.3) it then follows that the equation and the boundary conditions are satisfied in the classical sense provided that the solution u belongs to a Sobolev space of sufficiently high order. □

5.2. THE NEUMANN PROBLEM FOR $-\Delta$. The Dirichlet integral (4.4) is *not* strongly coercive over $H^1(\Omega)$, though it is coercive over $H^1(\Omega)$. So if we wish to deal with the existence of a solution of the problem

$$(5.5) \qquad -\Delta u = f \quad \text{in } \Omega \qquad f \in L^2(\Omega),$$

$$(5.6) \qquad \partial u/\partial\nu = \quad \text{in } \partial\Omega, \quad g \in H^{\frac{1}{2}}(\partial\Omega),$$

by Hilbert space methods we cannot use the Lax-Milgram theorem, but have to make use of the more complicated Fredholm alternative. It is easily seen that the homogeneous adjoint equation

$$D(v,u) = 0 \quad \forall\, v \in H^1(\Omega)$$

has a one-dimensional solution space, namely the constant functions in Ω. Hence

$$(5.7) \qquad D(u,v) = (f,v)_{0,\Omega} \text{ for all } v \in H^1(\Omega)$$

has solutions if and only if f is orthogonal to the constant functions. In the same way as in the preceding example it is found that (5.7) is equivalent to (5.5) and (5.6) with $g \equiv 0$. Regularity of a solution is also settled as in the preceding example. In case of non-homogeneous boundary conditions one, again, chooses a $h \in H^2(\Omega)$ such that $\gamma(\partial h/\partial\nu) = g$ – such a function h exists by the trace theory – , and studies

$$(5.8) \qquad D(w,v) = (f,v)_{0,\Omega} + (\Delta h,v)_{0,\Omega} \text{ for all } v \in H^1(\Omega),$$

instead of (5.7), where $w = u-h$. By Green's formula, (5.8) is equivalent to

$$(5.9) \qquad D(w,v) = (f,v)_{0,\Omega} + \int_{\partial\Omega} g\bar{v}d\sigma \quad \text{for all } v \in H^1(\Omega).$$

Fredholm's alternative tells us that there exist solutions if the right hand side of (5.8) or (5.9) is orthogonal to the function 1. Explicitly, this *compatibility condition*, as it is called, reads

$$\int_\Omega f dx + \int_{\partial\Omega} g d\sigma = 0,$$

a well-known expression in the classical approach to this type of boundary value problems. □

5.3. THE MIXED PROBLEM FOR $-\Delta+\lambda$.

Let $\partial\Omega = \partial_0\Omega \cup \partial_1\Omega \cup \Lambda$, $\Lambda = \overline{\partial_0\Omega} \cap \overline{\partial_1\Omega}$, $\partial_0\Omega \cap \partial_1\Omega = \emptyset$; Λ considered as a subset of $\partial\Omega$ has measure 0. Let further be given $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial_1\Omega)$. Choose the subspace V of $H^1(\Omega)$ to be the closure of the sets of functions $\varphi \in C^\infty(\Omega)$ that vanish in a neighbourhood in $\bar{\Omega}$ of $\partial_0\Omega$. The elements of V all have trace 0 on $\partial_0\Omega$. Consider the abstract boundary value problem

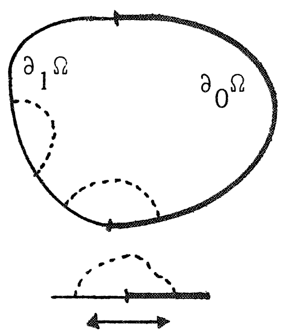

Figure 5.1

$$(5.10) \qquad D(u,v) + \lambda(u,v)_{0,\Omega} = (f,v)_{0,\Omega} + \int_{\partial\Omega} g_1\bar{v}d\sigma \text{ for all } v \in V,$$

the solution of which exists uniquely in V by the Lax-Milgram theorem. By a reasoning quite similar to the one used in the preceding examples, (5.10) may be interpreted as

$$(5.11)\qquad \begin{cases} (-\Delta+\lambda)u = f & \text{in } \Omega, \\ \gamma(\partial u/\partial\nu) = g_1 & \text{on } \partial_1\Omega, \\ \gamma u = 0 & \text{on } \partial_0\Omega. \end{cases}$$

Now we wish to replace the last condition by a non-homogeneous one, g_0. We make the assumptions that g_0 is the restriction to $\partial_0\Omega$ of a function $\tilde{g}_0 \in H^{3/2}$, and that g_1 is the restriction to $\partial_1\Omega$ of a $\tilde{g}_1 \in H^{\frac{1}{2}}(\partial\Omega)$. Then by the trace theory there exists a $h \in H^2(\Omega)$ such that $\gamma h = \tilde{g}_0$ and $\gamma(\partial h/\partial\nu) = \tilde{g}_1$. Writing again $u = w+h$, we apparently have to consider the abstract boundary value problem with homogeneous boundary conditions

$$D(w,v) + \lambda(w,v)_{0,\Omega} = (f,v)_{0,\Omega} + ((\Delta-\lambda)h,v)_{0,\Omega} \text{ for all } v \in V,$$

the solution of which may be interpreted as the generalized solution of (5.11) with 0 replace by g_0.

Regularity of the solution is a rather delicate matter in the neighbourhood of where the two portions of the boundary $\partial_0\Omega$ and $\partial_1\Omega$ meet. There the condition (V2) is not necessarily fulfilled. A proof of regularity of the solution in the neighbourhood of Λ certainly fails if Λ is not sufficiently smooth, and if one allows less regular boundary conditions (e.g. $g_1 \in H^{\frac{1}{2}}(\Gamma)$ for all open subsets $\Gamma \subset \partial_1\Omega$ instead of being extendable to a function belonging to $H^{\frac{1}{2}}(\partial\Omega)$). □

5.4. <u>OBLIQUE BOUNDARY CONDITION</u>. For simplicity, we restrict ourselves to the case $L = -\Delta+\lambda$, $\lambda \in \mathbb{R}$, $\lambda > 0$. Let $\Omega \subset \mathbb{R}^2$, $V = H^1(\Omega)$, $f \in L^2(\Omega)$ and $a \in C^0(\partial\Omega)$; consider the abstract boundary value problem

$$(5.12)\qquad B(u,v) := D(u,v) + \lambda(u,v)_{0,\Omega} - (a\, D_1u, D_2v)_{0,\Omega} + (a\, D_2u, D_1v)_{0,\Omega} = (f,v)_{0,\Omega} \text{ for all } v \in V.$$

As is seen by integration by parts, the bilinear form occuring in the left hand side of (5.12) is associated with $-\Delta+\lambda$. Further, it is strongly coercive over V. By the standard argument of the preceding examples, i.e.

temporary restriction of the v to $H_0^1(\Omega)$, once again it is shown that in the interior of Ω (5.12) is equivalent to $(-\Delta+\lambda)u = f$. Green's formula gives

$$B(u,v) = ((-\Delta+\lambda)u,v)_{0,\Omega} + \int_{\partial\Omega} (\nu.\ A \ \ \text{grad}\ u)\bar{v}d\sigma \tag{5.13}$$

where

$$A = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}.$$

So the solution $u \in V$ of (5.12) must satisfy the boundary condition

$$\int_{\partial\Omega} (\nu.\ A \ \ \text{grad}\ u)\bar{v}d\sigma = 0,$$

from which it follows that

$$\nu.\ A \ \ \text{grad}\ u = (\nu_1 - a\nu_2)D_1u + (a\nu_1+\nu_2)D_2u = 0 \tag{5.14}$$

since the v are essentially arbitrary. Another way of writing (5.14) is

$$\frac{\partial u}{\partial \nu} + a\frac{\partial u}{\partial \tau} = 0, \tag{5.15}$$

where $\partial u/\partial\tau$ signifies the derivative in the direction tangent to $\partial\Omega$, i.e. $\tau = (-\nu_2,\nu_1)$ at every point of $\partial\Omega$.

This is a good moment to stop at the fact that the boundary operators considered are indeed such that $\partial\Omega$ is nowhere characteristic, as was required by the generalized Green's formula. Here, this condition means that the derivative in (5.15) is nowhere tangent to the boundary, which is so if a remains finite for all $x \in \partial\Omega$.

The treatment of the corresponding non-homogeneous problem, as well as the application of the regularity theory, are straightforward.

5.5. INTERFACE CONDITIONS. Let $\Omega = \Omega_1 \cup \Omega_2 \cup (\partial_1\Omega \cap \partial_2\Omega)$, $\Omega_1 \cap \Omega_2 = \emptyset$, $V = H_0^1(\Omega)$ and write $u = u_j$ in Ω_j, $j = 1,2$. Let further a and b be two positive constants. Consider the abstract boundary value problem

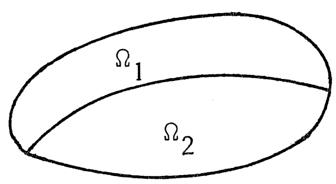

Figure 5.2

$$(5.16)\qquad B(u,v):= a \sum_{i=1}^{n} (D_i u, D_i v)_{0,\Omega_1} + b \sum_{i=1}^{n} (D_i u, D_i v)_{0,\Omega_2}$$

$$= (f,v)_{0,\Omega} \text{ for all } v \in V.$$

By restricting the choice of v to $H_0^1(\Omega_1)$ and $H_0^1(\Omega_2)$ respectively, one finds that in the interior of Ω_1 and Ω_2 respectively (5.16) is equivalent to

$$(5.17)\qquad \begin{cases} -\Delta u = f/a & \text{in } \Omega_1, \\ -\Delta u = f/b & \text{in } \Omega_2. \end{cases}$$

Green's formula takes here the form

$$B(u,v) = -a(\Delta u,v)_{0,\Omega_1} - b(\Delta u,v)_{0,\Omega_2}$$

$$+ a \int_{\partial\Omega\cap\partial\Omega_1} \frac{\partial u_1}{\partial\nu} \bar{v} d\sigma + b \int_{\partial\Omega\cap\partial\Omega_2} \frac{\partial u_2}{\partial\nu} \bar{v} d\sigma$$

$$+ \int_{\partial\Omega_1\cap\partial\Omega_2} (a \frac{\partial u_1}{\partial\nu_1} \bar{v} + b \frac{\partial u_2}{\partial\nu_2} \bar{v}) d\sigma$$

where ν_j is the outward normal to $\partial\Omega_1 \cap \partial\Omega_2$ with respect to Ω_j, $j = 1,2$. In view of (5.17), the interpretation of the abstract boundary value problem near the boundary must be

$$(5.18)\qquad u = 0 \text{ on } \partial\Omega \Longleftrightarrow u_j = 0 \text{ on } \partial\Omega_j \cap \partial\Omega, \quad j= 1,2$$

and

$$(5.19)\qquad a \frac{\partial u_1}{\partial\nu_1} = b \frac{\partial u_2}{\partial\nu_1} .$$

The last condition is called the *interface condition*. Since the bilinear form defined in (5.16) is strongly coercive over $H_0^1(\Omega)$, existence and regularity of the solution are dealt with as in previous examples. Regularity in the interior implies that $u_1 = u_2$ on $\partial\Omega_1 \cap \partial\Omega_2$ in the sense of traces.

5.6. GENERAL REMARKS ON BOUNDARY CONDITIONS. We now give a more general treatment of boundary conditions for second order elliptic operators. As we have seen in the previous examples, boundary conditions are represented by the choice of the space V, of the bilinear form B associated with the operator L, and of the "boundary part" of the functional F(v) in the right hand side of e.g. (5.9). Here these three choices are discussed in a more general setting.

First, consider the elliptic operator

$$L = -\sum_{i,j=1}^{n} D_i\, a_{ij}\, D_j \cdot + \sum_{i=1}^{n} a_i\, D_i \cdot + a_0 \cdot \, . \tag{5.20}$$

Introduce the bilinear form on $H^1(\Omega)$

$$\begin{aligned} B(u,v) = {} & \sum_{i,j=1}^{n} (b_{ij}\, D_j u,\, D_i v)_{0,\Omega} + \sum_{i=1}^{n} (b_i\, D_i u, v)_{0,\Omega} \\ & + \sum_{i=1}^{n} (u,\, D_i \beta_i\, v)_{0,\Omega} + (b_0 u,\, v)_{0,\Omega}. \end{aligned} \tag{5.21}$$

For the form B to be associated to L one must have

$$B(\varphi,\psi) = (L\varphi,\psi)_{0,\Omega} \quad \text{for all } \varphi,\ \psi \in C_0^\infty(\Omega).$$

This amounts to the conditions

$$\begin{cases} a_{ij} + a_{ji} = b_{ij} + b_{ji}, \\ a_i = b_i - \beta_i, \\ a_0 = b_0. \end{cases} \tag{5.22}$$

For the time being, we choose $V = H^1(\Omega)$, and we wish to interpret the boundary conditions corresponding to B and V. Application of Gauss' theorem a number of times, and use of (5.22) leads to a Green's formula of the form

$$B(u,v) = (Lu,v)_{0,\Omega} + \int_{\partial\Omega} \left[\sum_{i,j=1}^{n} b_{ij}\, \nu_i\, D_j u\, \bar{v} + \sum_{i=1}^{n} \beta_i\, \nu_i\, u\, \bar{v} \right] d\sigma \, .$$

Hence, if u is a solution of the abstract boundary value problem

(5.23) $$B(u,v) = (f,v)_{0,\Omega} + F_{\partial\Omega}(g,v) \quad \text{for all} \quad v \in V,$$

then u satisfies the differential equation Lu = f in the generalized sense in the interior of Ω, and the boundary condition

(5.24) $$\sum_{i,j=1}^{n} b_{ij}\, \nu_i\, D_j u + \sum_{i=1}^{n} \beta_i\, \nu_i u = F^*_{\partial\Omega} g.$$

where $F^*_{\partial\Omega}$ is some functional on a Sobolev space of functions defined on the boundary $\partial\Omega$.

We first look at homogeneous boundary conditions. The ordinary situation of course is that the boundary condition is prescribed. Boundary conditions purely tangent to the boundary must be excluded, since $\partial\Omega$ may not be characteristic. We shall investigate the boundary condition

(5.25) $$\frac{\partial u}{\partial \nu_c} + p\,\frac{\partial u}{\partial \tau} + qu = 0$$

where

$$\frac{\partial u}{\partial \nu_c} = \sum_{i,j=1}^{n} c_{ij}\, \nu_i\, D_j u, \quad \frac{\partial u}{\partial \tau} = \sum_{i=1}^{n} \tau_i\, D_i u$$

$$\sum_{i=1}^{n} \tau_i\, \nu_i = 0, \quad \sum_{i=1}^{n} \tau_i^2 = 1, \quad \sum_{i=1}^{n} \nu_i^2 = 1.$$

Using these relations, we find that (5.25) is equivalent to

(5.26) $$\sum_{i,j=1}^{n} \nu_i\, D_j u[c_{ij}+p(\tau_j \nu_i - \tau_i \nu_j] + qu = 0$$

The functions ν_j, τ_j, c_{ij}, p and q are, in the first instance, only defined for $x \in \partial\Omega$. We now assume that they all have extensions into the interior of Ω that are sufficiently regular. With the help of the trace theory, one can find the conditions on ν_j, τ_j, c_{ij}, p and q such that this is indeed the case. We leave this to the reader. Comparing (5.26) with (5.24), we see that we should choose

(5.27) $$b_{ij} = c_{ij} + p(\tau_j \nu_i - \tau_i \nu_j)$$

From (5.22) it then follows that $c_{ij} + c_{ji} = a_{ij} + a_{ji}$, so that in this approach of boundary value problems it is impossible to choose c_{ij} arbitrarily. In essence, only $\partial u/\partial \nu_L$ can be dealt with. Furthermore, we find that

(5.28) $$\sum_{i=1}^{n} \beta_i \nu_i u = qu, \quad \text{e.g. } \beta_i = q\nu_i.$$

In the special case $a_{ij} = c_{ij} = \delta_{ij}$ (the Kronecker δ), $\beta_i = 0$, $n = 2$, condition (5.27) is the same as (5.14) or (5.15).

In applications it is practical to write (5.21) in the form

(5.29) $$B(u,v) = \sum_{i,j=1}^{n} (b_{ij} D_i u, D_j v)_{0,\Omega} + \sum_{i=1}^{n} ((b_i - \beta_i)D_i u, v)_{0,\Omega} + (b_0 u,v)_{0,\Omega} + \int_{\partial\Omega} \beta_i \nu_i u\bar{v}d\sigma.$$

The question of (unique) existence and regularity of the solution is answered in the same way as in the previous examples, as well as the question how to deal with non-homogeneous boundary conditions. The details are left to the reader. A direct application is given in the next example.

5.7. <u>NEWTON'S RADIATION CONDITION</u>. Take $L = -\Delta+\lambda$, $\lambda > 0$. Newton's radiation condition reads

(5.30) $$\frac{\partial u}{\partial \nu} + hu = g \quad \text{on } \partial\Omega.$$

Choose $V = H^1(\Omega)$, and the coefficients b_{ij}, b_i, and β_i in such a manner that the bilinear form

$$B(u,v) = D(u,v) + \lambda(u,v)_{0,\Omega} + \int_{\partial\Omega} hu\bar{v}d\sigma$$

is considered i.e. $b_{ij} = \delta_{ij}$, $b_0 = \lambda$, $\beta_i = h\nu_i$, $b_i = \beta_i$. The solution of the abstract boundary value problem

$$B(u,v) = (f,v)_{0,\Omega} \quad \text{for all } v \in V$$

satisfies $(-\Delta+\lambda)u = f$ in the interior of Ω in the generalized sense, and (5.30) on $\partial\Omega$, in the sense of traces with $g \equiv 0$. The non-homogeneous problem is treated in the same manner as was done in the Neumann problem. Unique existence of the solution is guaranteed in the case that B is strongly coercive. This condition is certainly satisfied if $h(x) \geq 0$ for all $x \in \partial\Omega$. □

5.8. CONCLUDING REMARKS. We have seen that as long as we pick out boundary value problems from a relatively restricted list, namely those problems which can be formulated in the variational form, a unified treatment is possible.

Whether or not a boundary value problem can be put in the variational form, depends strongly on the form of the boundary conditions. A very general class of boundary conditions that do not allow a variational formulation, but for which an approach by means of Hilbert space theory is nevertheless possible, is formed by the so-called Lopatinski conditions. The mathematics involved is far more sophisticated than was needed for the variational boundary value problems, and falls beyond the scope of this report. TREVES [1975] and FRIEDMAN [1969] give a brief description of this class of boundary conditions, LIONS & MAGENES [1968] give a more lenghty discussion.

6. VARIATIONAL BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER EQUATIONS

A striking difference between the application of Hilbert space methods to Dirichlet boundary value problems and to those of other type is, that the former can be treated in the same way no matter which order the elliptic equation has, whereas the latter do not allow such a uniform treatment. The reason for this is found in the simple fact that for second order elliptic operators it is relatively simple to prove that bilinear forms associated with these operators are coercive over $H^1(\Omega)$, which is not at all the case for higher order operators, where Gårding's inequality (see [H], p.18) only establishes coercivity over $H_0^m(\Omega)$. The situation is even worse: in general it is not true that a bilinear form associated to a higher order operator is coercive over $H^m(\Omega)$. The following example is illustrative.

6.1. EXAMPLE. Let Ω be bounded and have a Lipschitz continuous boundary. The bilinear form

$$B(u,v) = (\Delta u,\Delta v)_{0,\Omega} + \lambda(u,v)_{0,\Omega} \tag{6.1}$$

is *not* coercive over $H^2(\Omega)$, whatever the positive constant λ may be (it *is*, of course, coercive over $H_0^2(\Omega)$). The proof of this statement depends on the following lemma.

6.2. LEMMA. *Let* Ω *be bounded and have* $C^{0,1}$ *boundary. Let* $B(u,v)$ *be a coercive bilinear form over a closed subspace* V *of* $H^m(\Omega)$. *If for all* $u \in V$

$$|\mathrm{Re}\, B(u,u)| \le c\, \|u\|_{m-1}^2 \tag{6.2}$$

for some positive constant c, *then* V *is finite-dimensional.*

PROOF. We give the proof because of its clarifying nature. Coercivity of B and (6.2) lead to

$$c_0\|u\|_m^2 \le (c+\lambda)\, \|u\|_{m-1}^2 \quad \text{for all } u \in V. \tag{6.3}$$

where c_0 is some positive constant, and λ is chosen such that $B(u,v)+\lambda(u,v)_0$ is strongly coercive over V. Take any sequence $\{u_k\}_{k=1}^{\infty} \subset V$ with $\|u_k\|_{m-1} \le 1$ for all k. By (6.3) then also $\|u_k\|_m$ are bounded for all k. Then Rellich's theorem (theorem 1.11 in [H]) implies that $\{u_k\}$ has a convergent subsequence in $H^{m-1}(\Omega)$, and therefore in V since V is a closed subspace of $H^{m-1}(\Omega)$. So the unit ball is compact in V. But if that is so in a normed linear space, this space must be finite dimensional. □

CONTINUATION OF EXAMPLE 6.1. Let $V = \overline{\{u \in H^2(\Omega) \mid \Delta u = 0\}}$ then for these u there holds

$$|\mathrm{Re}\, B(u,u)| = \mathrm{Re}\, \lambda\, \|u\|_0^2.$$

If B(u,v) were coercive, then from lemma 6.2 (with m = 1) it would follow that the subspace of harmonic functions V is finite-dimensional, which is clearly not so. □

The question we are confronted with is: given a uniformly strongly operator L in $\Omega \subset \mathbb{R}^n$ of order 2m, $m \geq 2$, do there exist bilinear forms associated with L that are coercive over a subspace $V \subset H^m(\Omega)$, V being essentially "more" than $H_0^m(\Omega)$? Naturally, the case $V = H^m(\Omega)$ is the most interesting. The rest of this section is devoted to the formulation of conditions - mostly sufficient, but sometimes also necessary - on the coefficients of L under which the question posed here may be answered in a positive sense.

But let us first look at example 5.2, the Neumann problem for the second order elliptic operator $-\Delta$, from a different point of view than before. We saw that the bilinear form

$$(6.4) \qquad D(u,v) = \sum_{i=1}^{n} (D_i u, D_i v)_{0,\Omega}$$

is not strongly coercive over $H^1(\Omega)$, because D(u,v) cannot distinguish between two functions only differing a constant. But, as is easily seen, the Dirichlet integral is strongly coercive over the quotient space of $H^1(\Omega)$ modulo the constant functions. This idea can be generalized to higher order operators. We shall do this first, before we turn our attention to the question posed before.

6.3. <u>NOTATIONS</u>. Let Ω be a bounded domain, and let $P_{m-1}(\Omega)$ denote the set of polynomials in Ω of degree m-1. The quotient space $H^m(\Omega)/P_{m-1}(\Omega)$ consists of equivalence classes $\tilde{u}$ of functions $u \in H^m(\Omega)$: $u,v \in \tilde{u} \Leftrightarrow u-v \in P_{m-1}(\Omega)$. The canonical norm on this quotient space is defined in the usual manner:

$$(6.5) \qquad \|\tilde{u}\|_{H^m(\Omega)/P_{m-1}(\Omega)} = \inf_{u\in\tilde{u}} \|u\|_{m,\Omega}.$$

The following lemmas give another description of the norm on $H^m(\Omega)/P_{m-1}(\Omega)$.

6.4. <u>LEMMA</u>. *Let again Ω be bounded. Then there exits a family of functions* $f_p \in L^2(\Omega)$, $|p| \leq m-1$, *such that for all* $v \in P_{m-1}(\Omega)$

$$\sum_{|p|\leq m-1} (v,f_p)_{0,\Omega} = 0 \Leftrightarrow v \equiv 0.$$

PROOF. Very simple; choose e.g. $f_p = x^p$. □

6.5. LEMMA. *Let* Ω *be bounded and have a continuous boundary. Then*

$$(6.6)\qquad \left\{\sum_{|p|=m} \|D^p u\|^2_{0,\Omega} + \sum_{|p|\le m-1} |(u,f_p)_{0,\Omega}|^2\right\}^{1/2},$$

where $\{f_p\}$ *is as in lemma* 6.4, *is a norm on* $H^m(\Omega)$ *equivalent to the original one.*

PROOF. See NEČAS [1967], p. 111. □

6.6. THEOREM. *Suppose that* Ω *is bounded and* $\partial\Omega$ *of class* C^0. *Then there exist positive constants* c_1 *and* c_2 *such that*

$$(6.7)\qquad c_1\|\tilde{u}\|_{H^m(\Omega)/P_{m-1}(\Omega)} \le \left\{\sum_{|p|=m} \|D^p u\|^2_{0,\Omega}\right\}^{1/2} \le c_2\|\tilde{u}\|_{H^m(\Omega)/P_{m-1}(\Omega)},$$

where, of course, u *is a representative of the equivalence class* $\tilde{u}$. *In other words: the expression*

$$(6.8)\qquad (\tilde{u},\tilde{v})'_{H^m(\Omega)/P_{m-1}(\Omega)} = \sum_{|p|=m} (D^p u, D^p v)_{0,\Omega}$$

is an inner product on $H^m(\Omega)/P_{m-1}(\Omega)$ *that defines a norm which is equivalent to the canonical norm defined by* (6.5).

PROOF. The proof is based on two facts: first, the norm on $H^m(\Omega)$ can be characterized by (6.6); and second, the polynomials of degree less than or equal to m-1 are exactly those functions in $H^m(\Omega)$ that cause the first term in (6.6) to vanish. Functions only differing a polynomial $p \in P_{m-1}(\Omega)$ are therefore not distuinguished by the first term. For details see NEČAS [1967], p.112. □

6.7. APPLICATION. Applying this to example 5.2, the Neumann problem for the Laplace equation, we find that indeed the Dirichlet integral D(u,v) imposes on the quotient space $H^1(\Omega)/P_0(\Omega)$ a Hilbert space structure equivalent to the one defined by (6.5).

The characterization of the Hilbert space structure of the quotient space can be used to find strongly coercive forms over the quotient spaces. The following simple theorem gives a neat illustration.

6.8. <u>THEOREM</u>. *Let* Ω *be bounded and have* C^0 *boundary. Let further* B *be a bilinear form only having* m-th *order derivatives, i.e.*

$$(6.9) \qquad B(u,v) = \sum_{|p|=|q|=m} (a_{pq}(x)D^p u, D^q v)_{0,\Omega},$$

such that the coefficients a_{pq} *satisfy for arbitrary complex numbers* ζ_p *and some positive constant* c

$$(6.10) \qquad \mathrm{Re} \sum_{|p|=|q|=m} a_{pq}(x)\, \zeta_p \bar{\zeta}_q \geq c \sum_{|p|=m} |\zeta_p|^2$$

almost everywhere in Ω. *Then* $B(\tilde{u},\tilde{v})$ *is strongly coercive over* $H^m(\Omega)/P_{m-1}(\Omega)$.

<u>PROOF</u>. Indeed, for $u \in H^m(\Omega)$ *one has*

$$\mathrm{Re}\, B(u,v) = \mathrm{Re} \sum_{|p|=|q|=m} (a_{pq} D^p u, D^q u)_0 \geq \sum_{|p|=m} D^p u\ {}^2_0.$$

By theorem 6.6. the last expression defines an equivalent norm on $H^m(\Omega)/P_{m-1}(\Omega)$. □

6.9. <u>REMARK</u>. Observe that condition (6.10) is stronger that the uniform ellipticity condition (see [H], definition 2.10). This is seen by substituting in (6.10) $\zeta_p = \xi^p = \xi_1^{p_1} = \xi_n^{p_n}$. □

A more general theorem, the proof of which rests on theorem 6.8, is the following one.

6.10. <u>THEOREM</u>. *Let* Ω *be bounded and have* C^0 *boundary. Let* B *be the bilinear*

$$B(u,v) = \sum_{|p|,|q|\leq m} (a_{pq} D^p u, D^q v)_{0,\Omega}.$$

Suppose further that condition (6.10) *is satisfied.*

(i) *If*

(6.11) $$\mathrm{Re}[B(u,u) - \sum_{|p|=|q|=m} (a_{pq}D^p u, D^q u)_0] \geq 0 \quad \text{for all } u \in H^m(\Omega)$$

then $B(\tilde{u},\tilde{v})$ *is strongly coercive over* $H^m(\Omega)/P_{m-1}(\Omega)$.

(ii) *If* P *is a closed linear subspace of* $P_{m-1}(\Omega)$, *if* (6.11) *is satisfied, and if additionally*

(6.12) $$\mathrm{Re}\, B(u,v) = 0 \Longleftrightarrow u \in P \quad \text{for all } u \in P_{m-1},$$

then $B(\tilde{u},\tilde{v})$ *is strongly coercive over* $H^m(\Omega)/P$.

(iii) *Let* V *be a closed linear subspace of* $H^m(\Omega)$. *If condition* (6.11) *is true for all* $u \in V$, *then* $B(\tilde{u},\tilde{v})$ *is strongly coercive over* $V/P_{m-1}(\Omega)$.

(iv) *Let* V *be as in* (iii), *and let* P *be a closed linear subspace of* $V \cap P_{m-1}(\Omega)$. *If* (6.11) *holds for all* $u \in V$, *and* (6.12) *for all* $u \in P_{m-1}(\Omega) \cap V$, *then* $B(\tilde{u},\tilde{v})$ *is strongly coercive over* V/P.

PROOF. See NEČAS [1967], p.144 ff. and p.41 ff. □

6.11. EXAMPLE. THE BIHARMONIC OPERATOR. As we have seen before, the bilinear form $(\Delta u,\Delta v)_{0,\Omega}$ associated with the operator Δ^2 is not coercive over $H^2(\Omega)$. For the two-dimensional case we consider another bilinear form on $H^2(\Omega)$

(6.13) $$\begin{aligned} B(u,v) = {} & (D_1^2u, D_1^2v)_0 + 2(1-\sigma)(D_1D_2u, D_1D_2v)_0 \\ & + \sigma(D_1^2u, D_2^2v)_0 + \sigma(D_2^2u, D_1^2v)_0 + (D_2^2u, D_2^2v)_0. \end{aligned}$$

In case of sufficient smoothness of $\partial\Omega$ and of u and v, the generalized Green's formula (4.12) (i.e. integration by parts) yields

(6.14) $$B(u,v) = (\Delta^2 u, v)_0 + \int_{\partial\Omega} [(N_1u)\bar{v} + (N_2u)\frac{\partial\bar{v}}{\partial\nu}]d\sigma$$

where

(6.15) $$N_1u = \sigma\Delta u + (1-\sigma)(\nu_1^2D_1^2u + 2\nu_1\nu_2D_1D_2u + \nu_2^2D_2^2u)$$

and

$$\text{(6.16)} \qquad N_2u = -\frac{\partial}{\partial\nu}\Delta u+(1-\sigma)\frac{\partial}{\partial\tau}(\nu_1\nu_2D_1^2u-(\nu_1^2-\nu_2^2)D_1D_2u - \nu_1\nu_2D_2^2u).$$

Here $\nu = (\nu_1,\nu_2)$ is the normal to the boundary, $\partial/\partial\nu$ the normal derivative at the boundary, and $\partial/\partial\tau$ the derivative in the tangent direction. We wish to apply theorem 6.8. The bilinear form B at hand is examined as follows

$$|\zeta_{11}|^2 + 2(1-\sigma)|\zeta_{12}|^2 + \sigma(\zeta_{11}\bar{\zeta}_{22}+\zeta_{22}\bar{\zeta}_{11}) + |\zeta_{22}|^2 =$$
$$(1-\sigma)(|\zeta_{11}|^2 + |\zeta_{22}|^2 + 2|\zeta_{12}|^2) + \sigma|\zeta_{11}+\zeta_{22}|^2 \geq$$
$$(1-\sigma)(|\zeta_{11}|^2 + |\zeta_{22}|^2 + 2|\zeta_{12}|^2).$$

So for $0 \leq \sigma < 1$ the bilinear form B is strongly coercive over $H^2(\Omega)/P_1(\Omega)$. Let $f \in L^2(\Omega)$, $g_1,g_2 \in L^2(\partial\Omega)$ be given. If the equation

$$\text{(6.17)} \qquad B(u,v) = (f,v)_{0,\Omega} + (g_1,\gamma v)_{0,\partial\Omega} + (g_2,\gamma\frac{\partial v}{\partial\nu})_{0,\Omega}$$

is to be solvable, it must be required that the right hand side of (6.17), which is a continuous linear functional on $H^2(\Omega)$, defines a continuous linear functional on $H^2(\Omega)/P_1(\Omega)$. The condition under which this is true is obvious, it must be orthogonal to $P_1(\Omega)$, i.e.

$$\text{(6.18)} \qquad (f,v)_{0,\Omega} + (g_1,\gamma v)_{0,\partial\Omega} + (g_2,\gamma\frac{\partial v}{\partial\nu})_{0,\partial\Omega} = 0$$

for $v = 1$, $v = x_1$ and $v = x_2$

We now may say that we have established by Hilbert space methods the existence of a generalized solution of the boundary value problem

$$\text{(6.19)} \qquad \begin{cases} \Delta^2u = f & \text{in } \Omega \\ N_1u = g_1,\ N_2u = g_2 & \text{on } \partial\Omega, \end{cases}$$

where f, g_1 and g_2 satisfy (6.18). This is a Neumann type boundary value problem for Δ^2, though not the usual one, since we are not allowed to take $\sigma = 0$. □

Another approach to coercivity of bilinear forms is now mentioned, that is due originally to ARONSZAJN and later to SMITH [1961], and that involves deeper mathematics. We start with a simple lemma.

6.12. LEMMA. *Let* Ω *be bounded and have a Lipschitz continuous boundary. Let further* $B(u,v)$ *be a coercive over* V *and suppose that* $\operatorname{Re} B(u,u) \geq 0$ *for all* $u \in V$. *Then*

$$\{u \in V \mid \operatorname{Re} B(u,u) = 0\}$$

is a linear finite-dimensional subspace of V.

PROOF. That V is of finite dimension is an immediate consequence of lemma 6.1. The linearity is proved with aid of the real-valued bilinear form

$$B_1(u,v) = \tfrac{1}{2}\{B(u,v) + \overline{B(v,u)}\}. \qquad \square$$

6.13. THEOREM. *Let* $P_1(\zeta),\ldots,P_\ell(\zeta)$ *be homogeneous polynomials of degree* m, *having constant coefficients. Consider the bilinear form*

$$(6.20) \qquad B(u,v) = \sum_{k=1}^{\ell} (P_k(D)u, P_k(D)v)_{0,\Omega}$$

on $H_0^m(\Omega)$, *where* Ω *is supposed to be a bounded domain with Lipschitz continuous boundary.*

(i) *A necessary and sufficient condition for* B *to be coercive over* $H_0^m(\Omega)$ *is that the polynomials* $P_k(\zeta)$, $k = 1,\ldots,\ell$ *have no common non-zero real zero.*

(ii) *A necessary and sufficient condition for* B *to be coercive over* $H^m(\Omega)$ *is that the polynomials* $P_k(\zeta)$, $k = 1,\ldots,\ell$ *have no common non-zero complex zero.*

PROOF.

(i) The non-vanishing of $\sum_{k=1}^{\ell} |P_k(\zeta)|^2$ for non-zero real ζ is equivalent to the ellipticity of the differential operator associated with B. So the sufficiency is an immediate result of Gårding's (see [H], theorem 2.15), and the necessity follows from the converse of the theorem on Garding's inequality.

(ii) This is the hard part of the theorem, especially the sufficiency in the proof of which one needs Hilbert's "Nullenstellensatz". See AGMON [1965], p.160 ff. □

6.14. EXAMPLE. THE BIHARMONIC OPERATOR. Let $\Omega \subset \mathbb{R}^2$ bounded with $C^{0,1}$ boundary. Consider the bilinear form

$$B(u,v) = ((D_1^2-D_2^2)u,(D_1^2-D_2^2)v) + 4(D_1D_2u,D_1D_2v)_{0,\Omega}. \tag{6.21}$$

for all $u,v \in H^2(\Omega)$. This bilinear form is coercive over $H^2(\Omega)$, because the polynomials $\zeta_1^2 - \zeta_2^2$ and $2\zeta_1\zeta_2$ have no other common complex zeros than 0. By integration by parts one can show that a function u satisfying

$$B(u,v) = (f,v)_{0,\Omega} \quad \text{for all } v \in H^2(\Omega)$$

is a generalized solution of

$$\Delta^2 u = 0 \quad \text{in} \quad \Omega,$$

$$-\Delta u + 2(\nu_1^2D_1^2u + 2\nu_1\nu_2D_1D_2u + \nu_2^2D_2^2u) = 0 \text{ and}$$

$$-\frac{\partial}{\partial\nu}\Delta u + 2\frac{\partial}{\partial\tau}(\nu_1\nu_2D_1^2u - (\nu_1^2-\nu_2^2)D_1D_2u - \nu_1\nu_2D_2^2u) = 0 \quad \text{on } \partial\Omega.$$

Observe that the bilinear form (6.21) is merely coercive over $H^2(\Omega)$, whereas the former approach gave strong coercivity, but over a "smaller" space, $H^2(\Omega)/P_1(\Omega)$. Here the existence of a solution is not guaranteed by the Lax-Milgram theorem, but by a Fredholm type existence theorem. □

6.15. EXAMPLE. We still have not dealt with the boundary value problem

$$\begin{cases} \Delta^2 u + \lambda u = f \text{ in } \Omega \ (\lambda \in \mathbb{R},\ \lambda > o), \\ \Delta u = g_1,\ \frac{\partial}{\partial\nu}\Delta u = g_2 \text{ on } \partial\Omega. \end{cases} \tag{6.22}$$

Set $w = \Delta u$, then $\Delta^2 w + \lambda w = \Delta f$ in Ω, $w = g_1$ and $\partial w/\partial\nu = g_2$ on $\partial\Omega$. Hence (6.22) is reduced to a Dirichlet boundary value problem. □

We close this section by stating a theorem for non-constant coefficients.

6.16. <u>THEOREM</u>. *Let* Ω *be bounded and let* $\partial\Omega$ *be Lipschitz continuous. Let* $P_1(x\ D),\ldots,P_\ell(x\ D)$ *be differential operators of order* m, *having coefficients that are bounded in* Ω. *Assume further that coefficients of the highest order parts* $P_1'(x,D),\ldots,P_\ell'(x,D)$ *are continuous in* $\overline{\Omega}$. *If*

(i) *for each* $y \in \Omega$ *the polynomials* $P_j'(y,\zeta)$ *have no common non-zero real zero, and*

(ii) *for each* $y \in \partial\Omega$ *the* $P_j'(y,\zeta)$ *have no common non-zero complex zero,*

then the bilinear form

$$B(u,v) = \sum_{j=1}^{\ell} (P_j(x,D)u, P_j(x,D)v)_{0,\Omega}$$

is coercive over $H^m(\Omega)$.

<u>PROOF</u>. See AGMON [1967], p.167 ff. □

REFERENCES

AGMON, S., *Lectures on elliptic boundary value problems*. Van Nostrand, Princeton etc., 1965.

[H] COOLEN, T.M.T., *A survey on Hilbert space methods for homogeneous elliptic boundary value problems*. Report TW 140/74, Mathematisch Centrum, Amsterdam, 1974.

FRIEDMAN, A., *Partial differential equations*. Holt, Rinehart and Winston, New York etc., 1969.

JAGER, E.M. de, *Theory of distributions*. Ch. II in: E. Roubine (ed.), *Mathematics applied to physics*. Springer, Berlin etc./UNO, Paris, 1970.

LIONS, J.L. & E. MAGENES, *Problèmes aux limites non homogènes, I,II*, Dunod, Paris, 1970.

NEČAS, J., *Les méthodes directes en théorie des équations elliptiques*. Academia, Prague, 1967.

SMITH, K.T., *Inequalities for formally positive integro-differential forms.* Bull. Amer. Math. Soc. 67 (1961) 368-370.

TREVES, F. *Basic linear partial differential equations.* Academic Press, New York etc., 1975.

WLOKA, J., *Grundräume und verallgemeinete Funktionen.* Lecture Notes in Mathematics 82, Springer, Berlin etc., 1969.