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ELLIPTIC SINGULAR PERTURBATIONS OF FIRST ORDER
DIFFERENTIAL OPERATORS VANISHING AT AN INTERIOR
SURFACE

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Elliptic singular perturbations of first order differential operators
vanishing at an interior surface

by

J. Grasman

ABSTRACT

With the maximum principle asymptotic estimates are made for a class of linear elliptic singular perturbation problems with resonant turning-point behaviour in one or more independent variables.

KEY WORDS & PHRASES: *maximum principle, singular perturbation, asymptotic approximation, turning-point*

1. INTRODUCTION

We consider the Dirichlet problem for a function $\phi(x_1, \dots, x_k, y_1, \dots, y_m; \varepsilon)$ satisfying the linear elliptic differential equation

$$(1.1) \quad L_\varepsilon \phi \equiv \varepsilon L_2 \phi + L_1 \phi = f(x, y; \varepsilon) \quad \text{in } \Omega$$

with boundary conditions

$$(1.2) \quad \phi = h(x, y; \varepsilon) \quad \text{on } \partial\Omega.$$

The domain Ω is a bounded domain in \mathbb{R}^n ($n=k+m$) of such a form that

$$(1.3) \quad (x, y) \in \Omega \quad \text{implies} \quad (x, 0) \in \Omega.$$

The operator L_2 is a second order partial differential operator, uniformly elliptic in Ω , L_1 is a first order differential operator; both operators have coefficients being Hölder continuous in Ω ,

$$(1.4) \quad L_2 \equiv \sum_{i,j=1}^k \alpha_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^k \sum_{j=1}^m 2\beta_{ij} \frac{\partial^2}{\partial x_i \partial y_j} + \sum_{i,j=1}^m \gamma_{ij} \frac{\partial^2}{\partial y_i \partial y_j},$$

and

$$(1.5) \quad L_1 \equiv \sum_{j=1}^m b_j \frac{\partial}{\partial y_j}.$$

The differential operator

$$(1.6) \quad \sum_{i,j=1}^k \alpha_{ij}(x, 0) \frac{\partial^2}{\partial x_i \partial x_j}$$

is supposed to be uniformly elliptic in $\Gamma_x = \{x \mid (x, 0) \in \Omega\}$. Furthermore, it is assumed that

$$(1.7) \quad b(x, y) = 0 \quad \text{iff} \quad |y| = 0,$$

$$(1.8) \quad v(x,y) \cdot b(x,y) \leq 0 \quad \text{on } \partial\Omega,$$

$$(1.9) \quad \sum_{j=1}^m b_j(x,y)y_j \leq -L|y|^2 \quad \text{in } \bar{\Omega},$$

where $v(x,y)$ is the outer normal to $\partial\Omega$, L a positive constant independent of ε and $|y|$ the Euclidian length of y ($|y| = \sqrt{y_1^2 + \dots + y_m^2}$).

Condition (1.7) gives the problem its turning-point character; the coefficients of the first derivatives of (1.1) vanish on the set

$$(1.10) \quad \Gamma = \{(x,y) \mid (x,y) \in \Omega, \quad |y| = 0\}$$

in the interior of the domain Ω . Because of (1.9) we cannot apply the maximum principle in a way as carried out by DE JAGER [5], where $k = m = 1$ and $b_y(x,0) > 0$. For application of the maximum principle to elliptic singular perturbation problems without turning-points we refer to ECKHAUS and DE JAGER [2]. The class of elliptic problems with isolated turning-points was analyzed on its spectral properties by DE GROEN [4] and with formal methods by GRASMAN and MATKOWSKY [3].

It is our purpose to give asymptotic estimates of solutions of (1.1) - (1.9), which too are based on the maximum principle for elliptic differential equations. It turns out that the behaviour of the solution in Ω is mainly determined by the elliptic problem in Γ_x as described by (1.6).

2. MAXIMUM PRINCIPLE AND BARRIER FUNCTIONS

For the operator L_ε given by (1.1), (1.4) and (1.5) we formulate the maximum principle as follows: a twice continuously differentiable function ϕ satisfying $L_\varepsilon \phi > 0$ in a domain Ω cannot have a maximum in Ω , see PROTTER and WEINBERGER [8, p.61]. The following lemma is a direct consequence of the maximum principle.

LEMMA 2.1. *If the twice continuously differentiable functions ϕ and ψ satisfy*

$$(2.1) \quad |L_\varepsilon \phi| < -L_\varepsilon \Psi \quad \text{in } \Omega,$$

and if $|\phi| \leq \Psi$ on $\partial\Omega$, then $|\phi| \leq \Psi$ in $\bar{\Omega}$.

PROOF. From the maximum principle and (2.1) we deduce that $\phi - \Psi$ does not have a maximum in Ω and since $\phi - \Psi \leq 0$ on $\partial\Omega$, we conclude that $\phi - \Psi \leq 0$ in $\bar{\Omega}$. Similarly, $-\phi - \Psi$ does not have a maximum in Ω , and $-\phi - \Psi \leq 0$ at $\partial\Omega$, so that $-\phi - \Psi \leq 0$ in $\bar{\Omega}$. Combining these results we obtain $|\phi| \leq \Psi$ in $\bar{\Omega}$. \square

The uniform boundedness with respect to ε of solutions of (1.1) - (1.9) is established by the following theorem.

THEOREM 2.1. *Let the twice continuously differentiable function ϕ satisfy*

$$(2.2) \quad |L_\varepsilon \phi| \leq M|y|^2 \quad \text{in } \Omega$$

and $|\phi| \leq N$ on $\partial\Omega$ with M and N independent of ε . Then there exists a constant K independent of ε such that

$$(2.3) \quad |\phi| \leq K \quad \text{in } \bar{\Omega}.$$

PROOF. We introduce a so-called barrier function

$$(2.4) \quad \Psi(x,y) = -U(x) + R|y|^2 + S,$$

in which we choose $R > M/L$ with L given by (1.9) and $U(x)$ such that

$$(2.5) \quad \sum_{i,j=1}^k \alpha_{ij}(x,0) \frac{\partial^2 U}{\partial x_i \partial x_j} = P + 2R \sum_{i,j=1}^m \gamma_{ij}(x,0) \quad \text{in } \Gamma_x$$

and $U = 0$ on $\partial\Gamma_x$, where P is some positive constant independent of ε . This boundary value problem for U has a unique twice continuously differentiable solution, see [1, p.336].

Since the coefficients α_{ij} and γ_{ij} are Hölder continuous, there exists a positive constant F , such that

$$(2.6) \quad \sum_{i,j=1}^k \alpha_{ij}(x,y) \frac{\partial^2 U}{\partial x_i \partial x_j} - 2R \sum_{i,j=1}^m \gamma_{ij}(x,y) > F \text{ in } \Omega.$$

For $|y|^2 \geq \epsilon F/M$ we have

$$(2.7) \quad -L_\epsilon \Psi = \epsilon \left\{ \sum_{i,j=1}^k \alpha_{ij} \frac{\partial^2 U}{\partial x_i \partial x_j} - 2R \sum_{i,j=1}^m \gamma_{ij} \right\} - 2R \sum_{j=1}^m b_j y_j > \\ -\epsilon F + 2RL|y|^2 \geq M|y|^2.$$

Because of the Hölder continuity of α_{ij} and γ_{ij} at $y = 0$, we may replace (2.6) by the following estimate for $|y|^2 < \epsilon F/M$

$$(2.8) \quad \sum_{i,j=1}^k \alpha_{ij}(x,y) \frac{\partial^2 U}{\partial x_i \partial x_j} - 2R \sum_{i,j=1}^m \gamma_{ij}(x,y) = P + \delta(x,y)$$

with $\delta(x,y) \rightarrow 0$ as $\epsilon \rightarrow 0$. Thus, for $|y|^2 < \epsilon F/M$ we have

$$(2.9) \quad -L_\epsilon \Psi = \epsilon\{P+O(1)\} + 2RL|y|^2 > 2M|y|^2 > M|y|^2.$$

Finally, S of (2.4) is taken sufficiently large such that

$$(2.10) \quad \Psi \geq N \quad \text{on} \quad \partial\Omega.$$

From (2.7) and (2.9) we conclude that

$$(2.11) \quad |L_\epsilon \phi| < -L_\epsilon \phi \quad \text{in} \quad \Omega,$$

while from (2.10) it follows that $|\phi| \leq \Psi$ on $\partial\Omega$. Using lemma 2.1 we obtain $|\phi| \leq \Psi$ in $\bar{\Omega}$. Since the function $U(x)$ as well as the domain Ω is bounded, a positive constant K can be found such that $\Psi \leq K$ in $\bar{\Omega}$, which completes the proof of the theorem. \square

The next theorems show that the barrier function only needs to satisfy the differential equation upto $O(\epsilon)$ in a $O(\sqrt{\epsilon})$ neighbourhood of Γ .

THEOREM 2.2. *Let the twice continuously differentiable function ϕ satisfy*

$$(2.12) \quad |L_\epsilon \phi| \leq M\epsilon \quad \text{in} \quad \Omega$$

and $|\phi| \leq N$ on $\partial\Omega$ with M and N positive constants independent of ϵ . Then there exists a constant K independent of ϵ such that

$$(2.13) \quad |\phi| \leq K \quad \text{in} \quad \bar{\Omega}.$$

PROOF. As a barrier function we introduce the function

$$(2.14) \quad \Psi(x, y) = -U(x) + S,$$

with U satisfying

$$(2.15) \quad \sum_{i,j=1}^k \alpha_{ij}(x, 0) \frac{\delta^2 U}{\delta x_i \delta x_j} = M + 1$$

in Γ_x and $U = 0$ on $\partial\Gamma_x$. Because of property (1.3) Ψ is defined everywhere in Ω . We see that

$$(2.16) \quad |L_\epsilon \phi| \leq M\epsilon < (M+1)\epsilon = -L_\epsilon \Psi.$$

Taking S sufficiently large we also have $|\phi| \leq N \leq \Psi$. Inequality (2.13) follows then from lemma 2.1. \square

The proof of the following theorem as well as that of its corollary is the same as the proof of theorem 2.1 and will be omitted for that reason.

THEOREM 2.3. *Let the twice continuously differentiable function ϕ satisfy*

$$(2.17) \quad |L_\epsilon \phi| \leq M(|y|^2 + \epsilon) \quad \text{in} \quad \Omega$$

and $|\phi| \leq N$ on $\partial\Omega$ with M and N positive constants independent of ϵ . Then a constant K independent of ϵ exists such that $|\phi| \leq K$ in $\bar{\Omega}$.

COROLLARY 2.1. *Let the twice continuously differentiable function $\phi(x,y;\epsilon)$ satisfy*

$$(2.18) \quad |L_\epsilon \phi| \leq M(|y|^{2+\epsilon})f(\epsilon) \quad \text{in } \Omega$$

and $|\phi| \leq Nh(\epsilon)$ on $\partial\Omega$ with f and h continuous, positive functions for $0 < \epsilon \leq \epsilon_0$ (ϵ_0 sufficiently small) and with M and N independent of ϵ . Then a constant K independent of ϵ exists such that in $\bar{\Omega}$

$$(2.19a) \quad |\phi| \leq Kf(\epsilon) \quad \text{if } h/f \text{ is bounded for } \epsilon \rightarrow 0,$$

or

$$(2.19b) \quad |\phi| \leq Kh(\epsilon) \quad \text{if } f/h \text{ is bounded for } \epsilon \rightarrow 0.$$

3. ASYMPTOTIC APPROXIMATION

Let us assume that by some matched asymptotic approximation procedure we have found a formal uniformly valid asymptotic solution, say ϕ_{as} , of (1.1), (1.2). Its validity can be proved as follows. Substitution of

$$(3.1) \quad \phi(x,y;\epsilon) = \phi_{as}(x,y;\epsilon) + R(x,y;\epsilon)$$

into (1.1) and (1.2) gives the following problem for R

$$(3.2) \quad L_\epsilon R = f - L_\epsilon \phi_{as} \quad \text{in } \Omega$$

$$(3.3) \quad R = h - \phi_{as} \quad \text{on } \partial\Omega.$$

If we are able to show that the right-hand sides of (3.2) and (3.3) have the desired behaviour to apply corollary 2.1, the smallness of the remainder term R is established. It is to expect that the problem (1.1) - (1.9) has a boundary layer structure. This complicates the construction of a suitable ϕ_{as} , as the derivatives of ϕ_{as} may increase with some inverse power of ϵ in the boundary layer. This difficulty can be surmounted by adding (small) boundary layer corrections to the asymptotic approximation.

As an example we deal with the following problem. Let the function $\phi(x,y;\epsilon)$ satisfy

$$(3.4) \quad L_\epsilon \phi \equiv \epsilon \left\{ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right\} - y \frac{\partial \phi}{\partial y} = 0$$

in a bounded region Ω of \mathbb{R}^2 specified as

$$(3.5) \quad \Omega = \{(x,y) \mid -1 < x < 1, -p_-(x) < y < p_+(x)\}$$

with $p_\pm(x) \in C^3[-1,1]$, $p_\pm(-1) = p_\pm(1) = 0$ and $p_\pm(x) > 0$ for $-1 < x < 1$, see figure 1. Moreover, ϕ satisfies the boundary conditions

$$(3.6) \quad \phi(x, p_\pm(x)) = h_\pm(x), \quad -1 \leq x \leq 1,$$

with $h_\pm(x) \in C^2[-1,1]$ and $h_+(\pm 1) = h_-(\pm 1) = h(\pm 1)$.

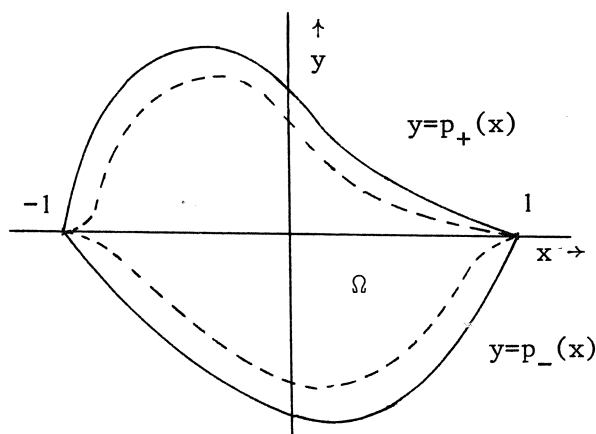


Fig. 1

THEOREM 3.1. *Let the function $\phi(x,y;\epsilon)$ satisfy the differential equation (3.4) in the domain Ω , as specified by (3.5), with boundary conditions (3.6). For ϵ sufficiently small there exists a positive constant K independent of ϵ such that*

$$(3.7) \quad |\phi - \phi_{as}| \leq K\epsilon \quad \text{in} \quad \bar{\Omega}$$

with

$$(3.8) \quad \phi_{as}(x, y; \varepsilon) = U(x) + V_0^+(x, y; \varepsilon) + V_0^-(x, y; \varepsilon),$$

$$(3.9) \quad U(x) = \frac{1}{2}\{h(1) - h(-1)\}(x+1) + h(-1),$$

$$(3.10) \quad V_0^\pm(x) = \bar{h}_\pm(x) \exp \left\{ -\alpha_\pm(x) \frac{p_\pm(x) \mp y}{\varepsilon} \right\},$$

$$\bar{h}_\pm(x) = h_\pm(x) - U(x), \quad \alpha_\pm(x) = \frac{p_\pm(x)}{p_\pm'(x)^2 + 1}.$$

PROOF. It is easily verified that in $y = p_\pm(x) \mp \varepsilon \eta$, $L_\varepsilon V_0^\pm$ does not decrease as $\varepsilon \rightarrow 0$ with η fixed. Therefore, and because of the fact that we need a higher accuracy near $y = 0$, we introduce additional boundary layer terms:

$$(3.11) \quad \begin{aligned} \bar{\phi}_{as}(x, y; \varepsilon) = & U(x) + V_0^+(x, y; \varepsilon) + V_0^-(x, y; \varepsilon) + \\ & + \varepsilon \{V_1^+(x, y; \varepsilon) + V_1^-(x, y; \varepsilon)\} + \\ & + \varepsilon^2 \{V_2^+(x, y; \varepsilon) + V_2^-(x, y; \varepsilon)\} \end{aligned}$$

with $V_i^\pm = V_i^\pm(x, \eta)$, $\eta = \{p_\pm(x) \mp y\} / \varepsilon$ satisfying the following recurrent system of equations with boundary conditions

$$(3.12) \quad M_0^\pm V_0^\pm = 0, \quad V_0^\pm(x, 0) = \bar{h}_\pm(x),$$

$$(3.13) \quad M_0^\pm V_1^\pm = -M_1^\pm V_0^\pm, \quad V_1^\pm(x, 0) = 0,$$

$$(3.14) \quad M_0^\pm V_2^\pm = -M_1^\pm V_1^\pm - M_2^\pm V_0^\pm, \quad V_2^\pm(x, 0) = 0,$$

where

$$M_0^\pm \equiv \{p_\pm'(x)^2 + 1\} \frac{\partial^2}{\partial \eta^2} + p_\pm(x) \frac{\partial}{\partial \eta},$$

$$M_1^\pm \equiv 2p'(x) \frac{\partial^2}{\partial x \partial \eta} + \{p_\pm''(x) - \eta\} \frac{\partial}{\partial \eta},$$

$$M_2^\pm \equiv \frac{\partial^2}{\partial x^2}.$$

Note that $L_\varepsilon \equiv \varepsilon^{-1} M_0^\pm + M_1^\pm + \varepsilon M_2^\pm$. In (3.10) we already gave the solution of (3.12). For V_1 we find

$$(3.15) \quad V_1^\pm(x, \eta) = [-\frac{1}{2}\bar{h}_\pm(x) \eta + \{\frac{1}{2}\bar{h}_\pm(x)p_\pm''(x) + \bar{h}_\pm'(x)p_\pm'(x)\}\eta^2] \exp\{-\alpha_\pm(x)\eta\},$$

while V_2 has the form

$$(3.16) \quad V_2^\pm(x, \eta) = \sum_{i=1}^4 A_i^\pm(x) \eta^i e^{-\alpha_\pm(x)\eta},$$

$$A_i^\pm = A_i^\pm(\bar{h}_\pm, \bar{h}_\pm', \bar{h}_\pm'', p_\pm, p_\pm', p_\pm'', p_\pm''').$$

From this we find by straight forward calculation that a constant M exists such that

$$(3.17) \quad |L_\varepsilon \tilde{R}| \leq M\varepsilon^2 \quad \text{in } \Omega,$$

while also a constant N can be found such that $|\tilde{R}| \leq N\varepsilon$ on $\partial\Omega$. From corollary 2.1 it is concluded that $|\tilde{R}| \leq \tilde{K}\varepsilon$ in $\bar{\Omega}$ for some \tilde{K} . Finally, we prove the validity of (3.7) by verifying that the additional terms V_i^\pm , $i = 1, 2$ are $O(\varepsilon)$ in $\bar{\Omega}$. \square

It is noted that if $p_\pm(x) \in C^2$ and/or $h_\pm(x) \in C^1$ outside a neighbourhood of $x = \pm 1$, the estimate (3.17) can be replaced by $|L_\varepsilon \tilde{R}| \leq (y^{2+\varepsilon})M\varepsilon$ in Ω . Then V_2^\pm have to be multiplied with a smooth cut-off function in x which is identically one near $x = \pm 1$ and zero away from these points.

4. SOME REMARKS

We presented a method for estimating the accuracy of asymptotic approximations of problems of the type (1.1)-(1.9). The construction of such asymptotic solutions is an aspect of the problem we did not study in depth and which may lead to interesting procedures of matching local

asymptotic solutions. Depending on the data and the required accuracy it may be necessary to construct corner layer solutions. Particularly for such approximations it is worthwhile to confine oneself to the lowest possible order of approximation needed to apply corollary 2.1. In this respect it is remarked that within a neighbourhood of the set Γ the differential equation has to be satisfied with a higher order accuracy than elsewhere. This plays a role in problems with boundaries $\partial\Omega$ being smooth near Γ , in which a new type of corner layer of thickness $O(\sqrt{\varepsilon})$ occurs near $\partial\Omega \cap \bar{\Gamma}$.

The results of the above investigations are important in the study of dynamical systems with small random perturbations. Let us consider the n -dimensional system

$$(4.1) \quad \frac{dz}{dt} = b(z)$$

with $b_i(z) = 0$ for $i = 1, \dots, k < n$ and with $(b_{j+k}(z), \dots, b_n(z))$ given by (1.7)-(1.8). Let the system be slightly disturbed by Gaussian white noise, then the trajectories of z in Ω satisfy the stochastic differential equation

$$(4.2) \quad dz_\varepsilon(t) = b(z_\varepsilon)dt + \varepsilon\sigma(z_\varepsilon)dw(t), \quad 0 < \varepsilon \ll 1,$$

where $w(t)$ is the n -dimensional Wiener process and $\sigma(z_\varepsilon)$ the diffusion matrix. A trajectory starting at a point $z_0 \in \Omega$ reaches the boundary $\partial\Omega$ with probability 1 (well-known result in probability theory). The probability distribution of points $z \in \partial\Omega$ where the trajectory first exits from the domain Ω is denoted by $p_\varepsilon(z, z_0)$. It can be shown (see MATKOWSKY and SCHUSS [7]) that

$$(4.3) \quad \phi(z; \varepsilon) = \int_{\partial\Omega} h(\bar{z}) p_\varepsilon(\bar{z}, z) dS_{\bar{z}},$$

where ϕ is the solution of the problem (1.1)-(1.7) with $z = (x, y)$ and

$$(4.4) \quad \begin{pmatrix} \alpha & \beta \\ \beta^* & \gamma \end{pmatrix} = \frac{1}{2} \sigma \times \sigma^*, \quad (* \text{ denotes the adjoint}).$$

As an example we mention a problem from genetics, namely the problem of correlations between loci and the possible extinction of certain genotypes, see LITTLER [6]. A population of diploid individuals, each characterized by its genotype with respect to two loci and with two alleles at each locus, is described by the fractions of types AB, Ab, aB and ab. Let these fractions be p_i , $i = 1, 2, 3, 4$ and let γ be a measure for the total number of individuals ($\gamma \gg 1$). Then in terms of the deterministic problem we have to deal with

$$(4.5) \quad \frac{1}{2} \sum_{i=1}^3 p_i (1-p_i) \frac{\partial^2 \phi}{\partial p_i^2} - \sum_{i=1}^3 \sum_{j=i+1}^3 p_i p_j \frac{\partial^2 \phi}{\partial p_i \partial p_j} - \gamma \{ p_1 (1-p_1-p_2-p_3) - p_2 p_3 \} \left(\frac{\partial \phi}{\partial p_1} - \frac{\partial \phi}{\partial p_2} - \frac{\partial \phi}{\partial p_3} \right) = 0$$

in a domain $S = \{(p_1, p_2, p_3) \mid p_1, p_2, p_3 > 0, p_1 + p_2 + p_3 < 1\}$. Substitution of

$$(4.6) \quad \begin{aligned} p_1 &= x_1 x_2 + y \\ p_2 &= x_1 (1-x_2) - y \\ p_3 &= (1-x_1) x_2 - y \end{aligned}$$

transforms equation (4.5) into

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^2 x_i (1-x_i) \frac{\partial^2 \phi}{\partial x_i^2} + y \frac{\partial^2 \phi}{\partial x_1 \partial x_2} + \sum_{i=1}^2 y (1-2x_i) \frac{\partial^2 \phi}{\partial x_i \partial y} + \\ \{ x_1 x_2 (1-x_1) (1-x_2) + y (1-2x_1) (1-2x_2) - y^2 \} \frac{\partial^2 \phi}{\partial y^2} - (\gamma+1) y \frac{\partial \phi}{\partial y} = 0. \end{aligned}$$

The domain S transforms into a domain of the type Ω satisfying (1.3) with $\Gamma_x = \{(x_1, x_2) \mid 0 < x_1, x_2 < 1\}$. However, there is one complication: the operator L_2 is not uniformly elliptic. Thus, we are only able to make estimates in subdomains Ω' of Ω bounded away from $\partial\Omega$ with ϕ given on $\partial\Omega'$.

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