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# A CLASS OF PROBLEMS IN HYDRODYNAMICS

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A class of problems in hydrodynamics

by

H.A. Lauwerier

Dedicated to I.N. Vekua at the occasion of his 70th birthday

ABSTRACT

A survey is given of a number of problems in hydrodynamics which lead to elliptic partial differential equations with oblique boundary conditions.

KEY WORDS & PHRASES: *hydrodynamics; elliptic partial differential equations.*

## 1. INTRODUCTION

The first contacts between the Amsterdam Mathematical Centre and the so-called Tiflis school of mathematicians was perhaps made in 1954 when at Amsterdam we were studying some mathematical problems raised by the flood disaster of 1953. In February 1st 1953 the South Western part of the Low Countries was struck by a flood which caused a loss of over 1800 human lives and an economic loss of almost two billions of guilders. The government appointed a committee to set up plans for preventing similar disasters in future. This so-called Delta committee took several scientific institutions as advisors. The Mathematical Centre was asked to consider the relevant statistical and hydrodynamical problems. The latter problem concerns the important question which heightening of the sea level at the Dutch coast is caused by a given windfield moving over the North Sea.

The research was initiated by D. van Dantzig and after his untimely death in 1959 continued by the present author. Very soon we discovered that even a rather idealistic mathematical model entailed the solving of elliptic partial differential equations with unusual boundary conditions. Fortunately the library of the Mathematical Centre contained a good number of books and journals in Russian language. In search of related subjects we came across a copy of Vekua's book on "New methods for solving elliptic equations" (1948) - at that time perhaps the only copy existing in Holland. This book and also the book of N.I. Muskhelishvili on "Singular integral equations" have dominated our research for many years to follow.

In this note I hope to give an idea of the most interesting aspects of this hydrodynamical problem. However, it should be noted already here that this research is far from completed. A number of problems of pure and applied mathematical interest are still unsolved. At present the interest of the hydrodynamics engineer is focussed upon numerical techniques exploiting the modern facilities of the computer rather than upon fundamental mathematical research. On the other hand mathematicians have turned to other fields of at least equal interest.

It is hoped therefore that this note may stimulate young mathematicians to study problems as those considered in this note.

2.

We consider a shallow sea as given by the domain D in the x,y -plane. The depth h can be a prescribed function of x and y but in most analytic pilot models it is taken as a constant. The boundary $\partial D$ may consist in two parts. The coastal part $\partial D_1$ represents the coast line. The oceanic part $\partial D_2$ is the imaginary line by which the shallow sea is separated from the deep ocean. The behaviour of the sea is described by the components $u(x,y,t)$ and $v(x,y,t)$ of the total stream, i.e. the stream integrated from bottom to the mean surface, and by the disturbance of the surface $\zeta(x,y,t)$ with $\zeta = 0$ for the undisturbed level. The (linearised) equations of motion can be written as

$$(2.1)\qquad \begin{cases} (\frac{\partial}{\partial t} + \lambda)u - \Omega v + gh\,\frac{\partial \zeta}{\partial x} = U, \\ (\frac{\partial}{\partial t} + \lambda)v + \Omega u + gh\,\frac{\partial \zeta}{\partial y} = V, \end{cases}$$

where $\lambda$ is a coefficient of friction, $\Omega = 2\omega_E \sin\phi$ the coefficient of the Coriolis force where $\omega_E$ is the velocity of the rotation of the Earth and $\phi$ the geographic latitude, g the acceleration of gravity, U and V the components of the traction force exerted on the watersurface by the windfield. We note that $\sqrt{U^2 + V^2}$ is proportional to the square of the velocity of the wind just above sea level.

To (2.1) we add the equation of continuity

$$(2.2)\qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \zeta}{\partial t} = 0.$$

The boundary conditions may be expressed as

$$(2.3)\qquad u\,dx + v\,dy = 0 \quad \text{at} \quad \partial D_1$$

i.e. no flow across the coast-line, and

$$(2.4)\qquad \zeta = 0 \quad \text{at} \quad \partial D_2.$$

The latter condition implies the assumption that the ocean is considered as a reservoir of infinite capacity with an undisturbed surface. Finally we have to add an initial condition of the kind

(2.5) $\quad U, V, \zeta \quad$ given at $\quad t = 0.$

The system (2.1), (2.2) has been set up on the basis of considerable simplifications but for seas or lakes which cover only a minor portion of the Earth's surface the description is fully adequate. Then for $\lambda$ and $\Omega$ numerical constants can be taken.

Of course the research at Amsterdam was carried out with the North Sea as the main field of application. However, seas like the Adriatic Sea and the Caspian Sea would offer equal opportunities for application. At the Mathematical Centre a great number of pilot models with simple geometries were considered. They can be listed as follows

$$\text{rectangle} \quad \begin{cases} \text{full plane} \\ \text{half-plane} \\ \text{strip} \end{cases}$$

$$\text{sector or wedge}$$

$$\text{circle} \quad \begin{cases} \text{full circle} \\ \text{semi-circle}\,. \end{cases}$$

Since the North Sea has roughly the shape of a rectangular bay bordering at one end to the Atlantic Ocean particular stress has been laid upon the rectangular model. As regards the time behaviour there are the following possibilities

stationary situation
free oscillations
periodic motion under influence of an external periodic force
initial value problem.

Let us consider the initial value problem for a sea with a still arbitrary geometry. For simplicity we take the initial condition (2.5) as

$$(2.6)\qquad u = v = \zeta = 0 \quad \text{for} \quad t = 0.$$

Then the system (2.1), (2.2) is subjected to a Laplace transformation according to

$$(2.7)\qquad \bar{\zeta}(x,y,p) = \int_0^\infty e^{-pt}\zeta(x,y,t)dt$$

etcetera. We obtain

$$(2.8)\qquad \begin{cases} (p+\lambda)\bar{u} - \Omega\bar{v} + gh\bar{\zeta}_x = \bar{U}. \\ (p+\lambda)\bar{v} + \Omega\bar{u} + gh\bar{\zeta}_y = \bar{V}. \\ \bar{u}_x + \bar{v}_y + p\bar{\zeta} = 0. \end{cases}$$

By elimination of $\bar{u}$ and $\bar{v}$ we arrive at an elliptic equation of the form

$$(2.9)\qquad \Delta\bar{\zeta} + a(x,y)\,\bar{\zeta}_x + b(x,y)\bar{\zeta}_y + c(x,y)\bar{\zeta} = \bar{F}.$$

Precisely this equation is the subject of the first chapter in Vekua's book.

Here we shall consider the special case of a sea with a uniform depth. Then $c = \sqrt{gh}$ is the constant speed of the propagation of long waves. We may take $c = 1$ by an appropriate time scale. Then (2.9) becomes a non-homogeneous Helmholtz equation of the form

$$(2.10)\qquad \Delta\bar{\zeta} - q^2\bar{\zeta} = \bar{F},$$

where

$$(2.11)\qquad q^2 = p(p+\lambda) + \Omega^2 p(p+\lambda)^{-1},$$

and

$$(2.12)\qquad \bar{F} = (\bar{U}_x+\bar{V}_y) + \frac{\Omega}{p+\lambda}(\bar{V}_x-\bar{V}_y).$$

The boundary conditions for $\bar{\zeta}$ are as follows

$$(2.13)\qquad \frac{\partial\bar{\zeta}}{\partial n} + \frac{\Omega}{p+\lambda}\frac{\partial\bar{\zeta}}{\partial s} = \bar{f} \quad \text{along} \quad \partial D_1,$$

and

$$(2.14) \qquad \bar{\zeta} = 0 \quad \text{along} \quad \partial D_2,$$

In (2.13) $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial s}$ denote the normal derivative in outward direction and the tangential derivative.
Further

$$(2.15) \qquad \bar{f} = \bar{W}_n + \frac{\Omega}{p+\lambda} \bar{W}_s,$$

where $W_n$ and $W_s$ are the normal and the tangential component of the external force (U,V).

The oblique boundary condition (2.13) is a particular case of the well-known Poincaré condition. The corresponding so-called Poincaré problem has been amply studied in the works of Vekua and Khvedelidze. However, in many problems this condition has to be combined with the Dirichlet condition at the ocean (2.14). This causes a further complication and has the effect of introducing some sort of singular behaviour at the points where the coastline and ocean meet.

In order to unravel these complexities it has been found useful to study a few pilot models in which a specific aspect of the general problem is prominent. It is then necessary to consider general representations of solutions of the homogeneous Helmholtz equation

$$(2.16) \qquad (\Delta+\sigma^2)\bar{\zeta} = 0.$$

One may start from simple solutions of the kind

$$\exp(\alpha x+\beta y) \quad , \quad \alpha^2+\beta^2+\sigma^2 = 0,$$

and apply linear superposition.
Vekua (l.c. 13.10) using a different approach shows that for star-shaped domains any solution of the Helmholtz equation (2.16) should be of the form

$$(2.17) \qquad h(r,\theta) - \int_0^r h(\rho,\theta) \frac{\partial}{\partial\rho} J_0(\sigma\sqrt{r(r-\rho)})d\rho$$

where $r,\theta$ are polar coordinates and $h$ is a harmonic function.
This representation may be considered as a transformation of the Helmholtz equation into the simpler potential equation. The usefulness of (2.17) and similar expressions depends of course on the relevant boundary conditions.
The present author has studied the case of a sectorial region $r > 0$, $0 < \theta < \theta_0$ in some detail. There the following representation has been found very useful

$$\int_L e^{i\sigma r \cosh \lambda} F(\lambda + i\theta) d\lambda, \tag{2.18}$$

where $F(\lambda)$ is an analytic function and $L$ is some suitable path in the complex $\lambda$-plane. The link between (2.17) and (2.18) is the well-known Sommerfeld representation

$$J_0(\sqrt{a^2+b^2}) = \frac{1}{2\pi i} \int e^{-a \sinh \lambda + ib \cosh \lambda} d\lambda. \tag{2.19}$$

3.

The following pilot model will be considered. Let the semi-infinite channel $0 < x < \pi$, $y > 0$ be bounded by coasts at $x = 0$, $x = \pi$ and an ocean at $y = 0$. In particular we study the reflection at the ocean boundary of a progressive so-called Kelvin wave of frequency $\omega$ ($\omega \geq 0$). This model is more or less representative for the behaviour of long waves in the North Sea near the Atlantic Ocean.

It is not our object to explore this problem in detail but rather to show that it results eventually in an expansion problem of considerable importance and of great interest in itself:

To expand a given function $f(x)$ in the interval $(0,\pi)$ as

$$f(x) = \sum c_k \sin(kx + \mu_k \pi), \tag{3.1}$$

where the summation starts with $k = 0$ or $k = 1$, with prescribed phases $\mu_k$. This problem has been solved in a few special cases only and it seems that

there is a good deal still to do.

The mathematical model is obtained from (2.8) by putting $u(x,y,t) = \bar{u}(x,y)\exp i\omega t$ etc. so that $p = i\omega$. Further, we take $\bar{U} = \bar{V} = 0$ and also $gh = 1$. Elimination of $\bar{u}$ and $\bar{v}$ gives the homogeneous Helmholtz equation (cf. eq. 2.10)

$$\Delta\zeta + \sigma^2\bar{\zeta} = 0, \quad 0 < x < \pi, \quad y > 0, \tag{3.2}$$

with

$$\sigma^2 = \omega(\omega - i\lambda) - \Omega^2\frac{\omega}{\omega - i\lambda}. \tag{3.3}$$

The boundary conditions are obtained from (2.13) and (2.14) as

$$(\lambda + i\omega)\bar{\zeta}_x + \Omega\bar{\zeta}_y = 0 \quad \text{for} \quad x = 0,\ x = \pi \tag{3.4}$$

and

$$\bar{\zeta} = 0 \quad \text{for} \quad y = 0. \tag{3.5}$$

The free waves in the channel are linear combinations of two so-called Kelvin waves and an infinity of Poincaré waves. The Kelvin waves are given by

$$\bar{\zeta}_+ = \exp\{\Omega(x - \tfrac{1}{2}\pi) - (\lambda + i\omega)y\} \tag{3.6}$$

travelling in the positive y-direction, and

$$\bar{\zeta}_- = \exp\{-\Omega(x - \tfrac{1}{2}\pi) + (\lambda + i\omega)y\} \tag{3.7}$$

travelling in the opposite direction from infinity to $y = 0$. The Poincaré waves are for $k = 1,2,3,\ldots$ given by

$$\bar{\zeta}_k = \{(\lambda + i\omega)\cos kx + \Omega\theta_k \sin kx\}\exp -k\theta_k y \tag{3.8}$$

with

$$\theta_k = \sqrt{1 + \sigma^2/k^2}.$$

They may be considered as local disturbances near the ocean boundary $y = 0$.

Our problem is to determine the effect of the reflection of a given Kelvin wave $\bar{\zeta}_-$ at the ocean boundary. Then the total elevation is given by

$$(3.9) \qquad \bar{\zeta} = -c\bar{\zeta}_+ + \bar{\zeta}_- - \sum_{k=1}^{\infty} c_k \bar{\zeta}_k .$$

The coefficients are determined by the ocean condition (3.5). This gives the following rather unorthodox expansion problem

$$(3.10) \qquad \exp \Omega(\tfrac{1}{2}\pi - x) = c \exp \Omega(x - \tfrac{1}{2}\pi) + \sum_{k=1}^{\infty} c_k \{(\lambda + i\omega)\cos kx + \Omega\theta_k \sin kx\}$$

The special case $\Omega = 0$ gives the rather trivial result $c = 1$, $c_k = 0$ for all k. The special case $\omega = 0$ has been solved in a satisfactory way by the present author. Then (3.10) is of the form

$$(3.11) \qquad f(x) = \sum_{k=1}^{\infty} c_k(\lambda \cos kx + \Omega \sin kx).$$

where f(x) contains an arbitrary coefficient c.
It has been found that the function

$$(3.12) \qquad h(x) = (\mathrm{tg}\tfrac{1}{2}x)^{\frac{2}{\pi}\operatorname{arctg}\frac{\Omega}{\lambda}}$$

is orthogonal with respect to all elements $\lambda\cos kx + \Omega\sin kx$. Thus the validity of the expansion (3.11) implies the following condition

$$(3.13) \qquad \int_0^{\pi} f(x)h(x)dx = 0.$$

In practical applications $\Omega/\lambda$ may be large. Then h(x) has a singularity at $x = \pi$ which is only just integrable. For the North Sea we have $\Omega/\lambda = 5$ which leads to the exponent 0.87. In that case the condition (3.13) may be approximated by $f(\pi) = 0$. With a similar reasoning for (3.10) we obtain the approximation $c \approx \exp - \pi\Omega$. This gives a good idea of the intensity of the reflected Kelvin wave.

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