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RUN FOR YOUR LIFE. A NOTE ON THE ASYMPTOTIC SPEED
OF PROPAGATION OF AN EPIDEMIC

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Run for your life. A note on the asymptotic speed of propagation of an epidemic *)

by

O. Diekmann

ABSTRACT

We study the large-time behaviour of the solution of a nonlinear integral equation of mixed Volterra-Fredholm type describing the spatio-temporal development of an epidemic. For this model it is known that there exists a minimal wave speed c_0 (i.e., travelling wave solutions with speed c exist if $|c| > c_0$ and do not exist if $|c| < c_0$). In this paper we show that c_0 is the asymptotic speed of propagation (i.e., for any c_1, c_2 with $0 < c_1 < c_0 < c_2$ the solution tends to zero uniformly in the region $|x| \geq c_2 t$, whereas it is bounded away from zero uniformly in the region $|x| \leq c_1 t$ for t sufficiently large).

KEY WORDS & PHRASES: *spread of infection in space and time; asymptotic speed of propagation; threshold phenomenon; hair-trigger effect; nonlinear integral equation of mixed Volterra-Fredholm type; comparison principle; construction of subsolutions.*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

In this paper, which may be regarded as a sequel to [4], we shall investigate the large-time behaviour of the solution u of the nonlinear integral equation of mixed Volterra-Fredholm type

$$(1.1) \qquad u(t,x) = \int_0^t A(t-\tau)\int_{\mathbb{R}^n} g(u(\tau,\xi))V(x-\xi)d\xi d\tau + f(t,x), \quad t \geq 0,\ x \in \mathbb{R}^n .$$

In [4] it was shown how an equation of this form arises as a model of the spatio-temporal development of an epidemic and, among other things, the existence, uniqueness and nonnegativity of a solution was established under some suitable assumptions concerning the given functions A, V, g and f.

Equation (1.1) corresponds to an initial value problem (the history up to $t = 0$ is prescribed; in fact it is incorporated in the function f). On the other hand, if one wants to describe an epidemic which has been evolving from the beginning of time then one arrives at the time-translation invariant homogeneous equation

$$(1.2) \qquad u(t,x) = \int_0^\infty A(t-\tau) \int_{\mathbb{R}^n} g(u(\tau,\xi))V(x-\xi)d\xi d\tau, \ -\infty < t < \infty,\ x \in \mathbb{R}^n .$$

When looking for travelling (plane) wave solutions (i.e., solutions of the form $u(t,x) = w(x\cdot\nu+ct)$, where ν is a fixed unit vector) one has to consider equation (1.2). The investigations in [4] and [5] have revealed that there exists $c_0 > 0$ such that (1.2) has a (for each ν unique modulo translation) travelling wave solution with speed c if $|c| > c_0$ and no such solution if $|c| < c_0$.

With this knowledge available several questions concerning the asymptotic behaviour (as $t \to \infty$) of solutions of (1.1) immediately present themselves. For instance, one might try to characterize those functions f for which the solution of (1.1) converges to a travelling wave of given speed c in some appropriate sense. Or one can investigate whether for a large class of functions f the solution develops into some structure of travelling waves. These questions have been successfuly studied for reaction-diffusion equations (see [6] for a survey of ideas, results and open problems) and one can acquire a lot of intuition by studying this theory. However, at present it seems that in order to deal with the *asymptotic form* of solutions of (1.1) one has to overcome some hard technical problems.

In two papers ([2],[3]) on Fisher's equation from population genetics, Aronson and Weinberger have introduced the concept of the *asymptotic speed* of propagation of disturbances from the rest state. Roughly speaking $c^* > 0$ is called the asymptotic speed if for any c_1, c_2 with $0 < c_1 < c^* < c_2$, the solution tends to zero uniformly in the region $|x| \geq c_2 t$, whereas it is bounded away from zero uniformly in the region $|x| \leq c_1 t$ for t sufficiently large (see Theorems 1 and 2 for a precise formulation). Any one who has read the papers of Aronson and Weinberger and the opening remarks above will immediately conjecture that for equation (1.1) c_0 is the asymptotic speed of propagation. It is the object of this paper to prove that this is indeed the case.

The organization of the paper is as follows. In section 2 we formulate the hypotheses concerning the functions A, V, g and f and we discuss in some more detail the known results for the equations (1.1) and (1.2). In section 3 we formulate our results in two theorems and we prove one of these, while the second one is proved in section 4 by means of a sequence of lemmas. Finally, in section 5 we review the epidemic model and we interpret our results biologically.

At this place we would like to acknowledge our thanks to Professors D.G. Aronson and H.F. Weinberger for the inspiration that came from their published works and for the stimulation received during a short visit.

NOTATION.

$\mathbb{R}_+ = \{y \in \mathbb{R} \mid y \geq 0\}$.

$BC(\mathbb{R}^n)$: the space of the bounded continuous functions on $\mathbb{R}^n$ equipped with the supremum norm.

$C(\mathbb{R}_+; BC(\mathbb{R}^n))$: the set of functions mapping $\mathbb{R}_+$ continuously into $BC(\mathbb{R}^n)$.

$B_R = \{x \in \mathbb{R}^n \mid |x| \leq R\}$.

supp ϕ: the support of the function ϕ.

$\phi \succ \psi$: we use this notation if ϕ and ψ are continuous functions defined on $\mathbb{R}^n$ and such that $\phi(x) \geq \psi(x)$ with strict inequality for $x \in \text{supp}\,\psi$.

2. THE CHARACTERISTIC EQUATION AND THE MINIMAL WAVE SPEED c_0

We consider the equation (1.1) under the following hypotheses:

H_A: $A: \mathbb{R}_+ \to \mathbb{R}$ is nonnegative, $A \in L_1(\mathbb{R}_+)$ and $\int_0^\infty A(\tau)d\tau = 1$.

H_V: $V: \mathbb{R}^n \to \mathbb{R}$ is nonnegative, $V \in L_1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} V(x)dx = 1$;
V is a radial function;
there exists $\delta > 0$ such that $\int_{\mathbb{R}^n} V(x)e^{\lambda x_1}dx < \infty$ for $\lambda \in (-\delta,\delta)$.

H_g: $g: \mathbb{R} \to \mathbb{R}$ is continuously differentiable;
$g(0) = 0$, $g'(0) > 1$ and there exists $C > 0$ such that $0 < g'(x) < C$ for $x \geq 0$; there exists $p > 0$ such that $g(x) > x$ for $0 < x < p$ and $g(p) = p$.

H_f: $f: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is nonnegative but not identically zero;
$f \in C(\mathbb{R}_+;BC(\mathbb{R}^n))$.

In [4] it was shown that the existence and the uniqueness of a solution $u \in C(\mathbb{R}_+;BC(\mathbb{R}^n))$ is guaranteed under less restrictive assumptions and that u is nonnegative.

If we look for a travelling wave solution $u(t,x) = w(x.\nu+ct)$ of equation (1.2), then w has to satisfy the following nonlinear convolution equation on the line

$$(2.1)\qquad w(y) = \int_{-\infty}^{\infty} g(w(\eta))\tilde{V}_c(y-\eta)d\eta, \quad -\infty < y < \infty,$$

where

$$(2.2)\qquad \tilde{V}_c(y) := \int_0^\infty \tilde{V}(y+c\tau)A(\tau)d\tau,$$

$$(2.3)\qquad \tilde{V}(y) := \int_{\mathbb{R}^{n-1}} V(y,x_2,\ldots,x_n)dx_2\ldots dx_n.$$

The equation does not depend on ν since V is a radial function and for the same reason we can restrict our attention to $c \geq 0$. For any c, equation

(2.1) has the constant solutions $w(y) = 0$ and $w(y) = p$, but the point is whether there exist nontrivial solutions with $0 < w(y) < p$.

In our analysis an important role is played by the characteristic equation

(2.4) $$L_c(\lambda) = 1,$$

where

(2.5) $$L_c(\lambda) := g'(0) \int_{-\infty}^{\infty} \tilde{V}_c(y) e^{\lambda y} dy$$

or, equivalently,

(2.6) $$L_c(\lambda) = g'(0) \int_0^{\infty} e^{-\lambda c \tau} A(\tau) d\tau \int_{\mathbb{R}^n} V(x) e^{\lambda x_1} dx.$$

This characteristic equation arises by linearization of (2.1) followed by substitution of an exponential function.

For any $c \geq 0$, the function L_c is defined at least on the set

(2.7) $$S := \{\lambda \geq 0 \mid \int_{\mathbb{R}^n} V(x) e^{\lambda x_1} dx < \infty\}.$$

L_c is a convex function of λ (see (2.5) and notice that $\tilde{V}_c$ is nonnegative). For $c = 0$, L_c achieves its infimum for $\lambda = 0$ and consequently $L_0(\lambda) \geq L_0(0) = g'(0) > 1$. By continuity this implies that $L_c(\lambda) > 1$ for $\lambda \in S$ and c sufficiently small. From (2.6) it follows that for fixed $\lambda > 0$, $L_c(\lambda)$ is a monotone decreasing function of c which tends to zero as $c \to \infty$. So the definition

(2.8) $$c_0 := \inf\{c > 0 \mid L_c(\lambda) = 1 \text{ for some } \lambda \in S\}$$

makes sense and $0 < c_0 < \infty$. Moreover, if $c > c_0$ then the set $\{\lambda \in S \mid L_c(\lambda) < 1\}$ is nonempty.

It has been shown that (2.1) has a (unique modulo translation) nondecreasing solution w with $w(-\infty) = 0$ and $w(\infty) = p$ if $c > c_0$ (or even $c=c_0$), whereas no nontrivial solution exists if $0 \le c < c_0$ (see [4] or [10] for the existence proof and [5] for the uniqueness and the non-existence results; we point out that the proofs require some extra technical assumptions concerning A, V and g, so one is advised to consult these references for a precise formulation of the results).

3. c_0 IS THE ASYMPTOTIC SPEED OF PROPAGATION

The assertion that c_0 is the asymptotic speed of propagation for equation (1.1) naturally splits into two parts. Our first theorem deals with the part that admits a straightforward proof.

THEOREM 1. *Suppose that*

(i) $\sup\{f(t,x) \mid t \in \mathbb{R}_+,\ x \in \mathbb{R}^n\} = C < \infty$;

(ii) *there exists* $R > 0$ *such that* $\operatorname{supp} f(t,.) \subset B_R$ *for all* $t \in \mathbb{R}_+$;

(iii) $g(x) \le g'(0)x$ *for all* $x \ge 0$.

Then for any $c > c_0$: $\lim_{t\to\infty} \sup\{u(t,x) \mid |x| \ge ct\} = 0$.

PROOF. If we write (1.1) symbolically as $u = Qu + f$, then u is the limit of the nondecreasing sequence u_n defined by $u_0 = f$, $u_{n+1} = Qu_n + f$ (see [4]). Let $c_1 > c_0$ be arbitrary and choose $c_2 \in (c_0,c_1)$. Because of the assumption (iii) we know that

$$u_1(t,x)e^{\lambda(x_1-c_2t)} \le$$

$$\le g'(0)\int_0^t A(\tau)e^{-\lambda c_2\tau}\int_{\mathbb{R}^n} f(t-\tau,\xi)e^{\lambda(\xi_1-c_2(t-\tau))}\,V(x-\xi)e^{\lambda(x_1-\xi_1)}\,d\xi d\tau$$

$$+ f(t,x)e^{\lambda(x_1-c_2t)} \le Ce^{\lambda R}(1+L_{c_2}(\lambda)),$$

and by induction we find

$$u_n(t,x)e^{\lambda(x_1-c_2t)} \le Ce^{\lambda R}(1+L_{c_2}(\lambda)+\ldots+(L_{c_2}(\lambda))^n). \tag{3.1}$$

Since $c_2 > c_0$ we can choose $\lambda > 0$ such that $L_{c_2}(\lambda) < 1$. For this choice of λ the right-hand side of (3.1) is bounded from above uniformly in n and we obtain

$$u(t,x) \le \frac{Ce^{\lambda R}}{1-L_{c_2}(\lambda)} e^{\lambda(c_2t-x_1)} .$$

If ν is any unit vector, we can rotate the coordinate axes in such a way that with respect to the new basis $\nu = (1,0,\ldots,0)$. Since V is a radial function the estimates given above are not affected by this rotation, and we conclude that we can replace x_1 by $x.\nu$ for any unit vector ν. The choice $\nu = x|x|^{-1}$ leads to

$$u(t,x) \le \frac{Ce^{\lambda R}}{1-L_{c_2}(\lambda)} e^{\lambda(c_2t-|x|)} .$$

Consequently

$$\sup\{u(t,x) \mid |x| \ge c_1t\} \le \frac{Ce^{\lambda R}}{1-L_{c_2}(\lambda)} e^{\lambda(c_2-c_1)t}$$

and since $\lambda > 0$ and $c_1 > c_2$ this proves the theorem. □

REMARK. Actually the result of Theorem 1 is true under less restrictive conditions on f. For instance, suppose that there exists $\lambda_0 > 0$ such that $L_{c_0}(\lambda_0) = 1$, then $L_c(\lambda_0) < 1$ for all $c > c_0$, and one can verify that the proof of Theorem 1 still works if we assume that

$$\sup\{f(t,x)e^{\lambda_0(|x|-c_0t)} \mid t \in \mathbb{R}_+ , x \in \mathbb{R}^n\} < \infty.$$

The second and last theorem establishes that the solution of (1.1) approaches or passes over p on the set $|x| \le ct$ where c is any number between 0 and c_0.

THEOREM 2. *For any* $c \in (0,c_0)$: $\liminf_{t\to\infty} \min\{u(t,x) \mid |x| \le ct\} \ge p$.

4. PROVING THEOREM 2 BIT BY BIT

Since the proof of Theorem 2 is rather involved we shall split it into several steps which are formulated as lemmas. The proof is based upon a comparison principle and the construction of a suitable subsolution. In the construction of this subsolution we mimic ARONSON and WEINBERGER ([1], [2], [3] and [4]) and some of our proofs are merely adaptations of their analysis to the present context.

For any $T > 0$ we define a mapping E_T by

$$E_T[\phi](t,x) := \int_0^T A(\tau) \int_{\mathbb{R}^n} g(\phi(t-\tau,\xi))V(x-\xi)d\xi\, d\tau. \tag{4.1}$$

So E_T maps a function defined on $\mathbb{R}_+ \times \mathbb{R}^n$ onto a function defined on $[T,\infty) \times \mathbb{R}^n$. Roughly speaking E_T is a sort of time-T map for the dynamical system defined by (1.1).

<u>LEMMA 1</u>. (*Comparison Principle*).
Suppose that

$$E_T[\psi](t,.) \succ \psi(t,.) \qquad \text{for all } t \ge T,$$

where $\psi\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *is a nonnegative continuous function such that*

(i) *for any* $t_1 > 0$ *there exists* $R = R(t_1) < \infty$ *such that for any* $t \in [0,t_1]$, $\operatorname{supp} \psi(t,.) \subset B_R$;

(ii) *if* $\{(t_n,x_n)\}_{n=1}^{\infty} \subset \mathbb{R}_+ \times \mathbb{R}^n$ *is a sequence for which* $x_n \in \operatorname{supp} \psi(t_n,.)$ *and* $\lim_{n\to\infty} (t_n,x_n) = (\bar{t},\bar{x})$, *then necessarily* $\bar{x} \in \operatorname{supp} \psi(\bar{t},.)$.

If there exists $t_0 \ge 0$ *such that*

$$u(t_0+t,.) \succ \psi(t,.) \qquad \text{for } 0 \le t \le T,$$

then this relation holds for all $t \ge 0$.

<u>PROOF</u>. Let $\bar{t} := \sup\{t \ge T \mid u(t_0+t,.) \succ \psi(t,.)\}$ and suppose that $\bar{t} < \infty$. Since u is nonnegative it follows that there exists a sequence

$\{(t_n,x_n)\}_{n=1}^{\infty} \subset \mathbb{R}_+ \times \mathbb{R}^n$ such that: (a) $x_n \in \text{supp}\,\psi(t_n,.)$; (b) $u(t_0+t_n,x_n) \le \psi(t_n,x_n)$; (c) $t_n \downarrow \bar{t}$ as $n \to \infty$. From (i) we deduce that the sequence $\{x_n\}_{n=1}^{\infty}$ is contained in a compact subset of $\mathbb{R}^n$ and hence it contains a convergent subsequence. Subsequently (ii) and (b) imply that there exists $\bar{x} \in \text{supp}\,\psi(\bar{t},.)$ such that $u(t_0+\bar{t},\bar{x}) \le \psi(\bar{t},\bar{x})$.

On the other hand, since $\bar{t} \ge T$ and $t_0 \ge 0$, the definition of $\bar{t}$ also implies that

$$
\begin{aligned}
u(t_0+\bar{t},\bar{x}) &\ge \int_0^T A(\tau) \int_{\mathbb{R}^n} g(u(t_0+\bar{t}-\tau,\xi))V(\bar{x}-\xi)d\xi d\tau \\
&\ge \int_0^T A(\tau) \int_{\mathbb{R}^n} g(\psi(\bar{t}-\tau,\xi))V(\bar{x}-\xi)d\xi d\tau = E_T[\psi](\bar{t},\bar{x}) \\
&> \psi(\bar{t},\bar{x}).
\end{aligned}
$$

So we obtain a contradiction and our assumption $\bar{t} < \infty$ must be false. □

Such a function ψ for which for some $T > 0, E_T[\psi](t,.) \succ \psi(t,.)$ for all $t \ge T$ we will call a *subsolution*.

Next our efforts are directed at the construction of a subsolution ψ with the property that $\psi(t,.)$ is bounded away from zero uniformly on the set $|x| \le ct$. This construction is relatively easy if A and V have compact support and if $g(x) \ge g'(0)x$. For the general case we need to cut off the kernels and we use the inequality $g(x) \ge hx$ for $h < g'(0)$ and x sufficiently small. So firstly we have to show that this can be done without changing the characteristic function too much.

Let the function $K_c = K_c(h,T,R,\lambda)$ be defined by

$$
(4.2) \qquad K_c(h,T,R,\lambda) := h \int_0^T e^{-\lambda c\tau}A(\tau)d\tau \int_{B_R} V(x)e^{\lambda x_1}dx,
$$

or, equivalently,

$$
(4.3) \qquad K_c(h,T,R,\lambda) = h \int_{-\infty}^{\infty} e^{\lambda x_1} \Phi_c(R,T,x_1)dx_1,
$$

where

$$(4.5)\qquad \Phi_c(R,T,x_1) := \int_0^T \tilde{V}^R(x_1+c\tau)A(\tau)d\tau,$$

$$(4.6)\qquad \tilde{V}^R(x_1) := \begin{cases} \tilde{V}(x_1) & \text{for } |x_1| \le R, \\ 0 & \text{for } |x_1| > R . \end{cases}$$

Then we have the following result.

LEMMA 2. *Suppose that* $c \in [0,c_0)$. *There exist numbers* $h \in (0,g'(0))$, $T > 0$ *and* $R > 0$ *such that*

$$K_c(h,T,R,\lambda) > 1 \qquad \textit{for all } \lambda \in \mathbb{R} .$$

PROOF. Since $K_c(h,T,R,-\lambda) \ge K_c(h,T,R,\lambda)$ for $\lambda \ge 0$, it suffices to prove the inequality for $\lambda \ge 0$. We split the proof into two steps.

STEP 1. We claim that there exist $\lambda_0 > 0$, $h_0 \in (0,g'(0))$, $T_0 > 0$ and $R_0 > 0$ such that $K_c(h,T,R,\lambda) > 1$ for all $\lambda \ge \lambda_0$, $h \ge h_0$, $T \ge T_0$ and $R \ge R_0$. Indeed, choose R_0 and T_0 such that $\int_0^\infty \Phi_c(R_0,T_0,x_1)dx_1 > 0$ (this can be done since $\int_0^\infty \tilde{V}_c(x_1)dx_1 > 0$ and since Φ_c is a monotone nondecreasing function of R and T converging to $\tilde{V}_c$ pointwise), then $\lim_{\lambda\to\infty} K_c(h,T_0,R_0,\lambda) = \infty$ for all $h > 0$. So we can choose h_0 and λ_0 such that $K_c(h_0,T_0,R_0,\lambda) > 1$ for all $\lambda \ge \lambda_0$ and this proves the claim since K_c is a monotone function of h,T and R.

STEP 2. Suppose that the assertion is not true, then there exist sequences $\{h_n\}$, $\{T_n\}$, $\{R_n\}$ and $\{\lambda_n\}$ such that $h_n \uparrow g'(0)$, $T_n \uparrow \infty$, $R_n \uparrow \infty$ and $\lambda_n \ge 0$ and such that $K_c(h_n,T_n,R_n,\lambda_n) \le 1$. Since $\{\lambda_n\} \subset [0,\lambda_0]$ we can choose a subsequence $\{\lambda_{n_k}\}$ which converges to a limit, say $\bar{\lambda}$. By Fatou's Lemma we have

$$L_c(\bar{\lambda}) \le \liminf_{k\to\infty} K_c(h_{n_k},T_{n_k},R_{n_k},\lambda_{n_k}) \le 1,$$

which is impossible. □

The function $K_c(h,T,R,.)$ is the (two-sided) Laplace transform of the function $\Phi_c(R,T,.)$ which is defined on $\mathbb{R}$. Our next step is to give explicit solutions of a linear one-dimensional convolution-inequality if it is known that the Laplace transform of the kernel is bounded from below by 1. As candidates for solutions we take the members of the two-parameter family of functions

$$(4.7) \qquad q(y;\alpha,\beta) := \begin{cases} e^{-\alpha y}\sin\beta y & \text{for } 0 \le y \le \frac{\pi}{\beta} \\ 0 & \text{for } y \in \mathbb{R}\setminus[0,\frac{\pi}{\beta}]. \end{cases}$$

LEMMA 3. (ARONSON and WEINBERGER [1],[10]).
Let $k \in L_1(\mathbb{R})$ *be a nonnegative function with compact support such that*

$$L(\lambda) := \int_{-\infty}^{\infty} e^{\lambda y}k(y)dy > 1 \quad \textit{for all } \lambda \in \mathbb{R}.$$

Then there exist a positive number β_0, *a continuous function* $\tilde{\alpha} = \tilde{\alpha}(\beta)$ *and a positive function* $\Delta = \Delta(\beta)$ *defined on* $[0,\beta_0]$ *such that for any* $\beta \in [0,\beta_0]$ *and for any* $\delta \in [0,\Delta(\beta))$

$$\phi * k \succ \phi_\delta,$$

where $\phi(y) := q(y;\tilde{\alpha}(\beta),\beta)$ *and* $\phi_\delta(y) := \phi(y-\delta)$.

PROOF. We split the proof into five steps.

STEP 1. Since k is nonnegative we know that L is convex and that $L(\lambda) \to \infty$ as $|\lambda| \to \infty$. So L achieves its infimum, say for $\lambda = \mu$. Then

$$\int_{-\infty}^{\infty} ye^{\mu y}k(y)dy = 0.$$

STEP 2. We define a function $H = H(\alpha,\beta)$ by

$$H(\alpha,\beta) := \beta^{-1}\int_{-\infty}^{\infty} e^{\alpha y}\sin\beta y\, k(y)dy \quad \text{for } \beta \neq 0,$$

$$H(\alpha,0) := \lim_{\beta\to 0} H(\alpha,\beta) = \int_{-\infty}^{\infty} ye^{\alpha y}k(y)dy.$$

Then $H(\mu,0) = 0$ and

$$\frac{\partial H}{\partial \alpha}(\mu,0) = \int_{-\infty}^{\infty} y^2 e^{\mu y} k(y)dy > 0,$$

so the implicit function theorem implies that there exists $\beta_1 > 0$ and a continuous function $\tilde{\alpha} = \tilde{\alpha}(\beta)$ with $\tilde{\alpha}(0) = \mu$ such that for $0 \le \beta \le \beta_1$, $H(\tilde{\alpha}(\beta),\beta) = 0$. Hence for $0 < \beta \le \beta_1$

$$\int_{-\infty}^{\infty} e^{\tilde{\alpha}(\beta)y} \sin\beta y \; k(y)dy = 0.$$

<u>STEP 3.</u> Since $\int_{-\infty}^{\infty} e^{\alpha y}\cos\beta y \; k(y)dy > 1$ for $\alpha = \mu$ and $\beta = 0$ there exists $\beta_2 > 0$ such that this inequality holds for $\alpha = \tilde{\alpha}(\beta)$ and $0 \le \beta \le \beta_2$.

<u>STEP 4.</u>

$$\begin{aligned}
\phi * k(y) &= \int_0^{\pi/\beta} e^{-\tilde{\alpha}(\beta)\eta} \sin\beta\eta \; k(y-\eta)d\eta \\
&\overset{(1)}{\ge} \int_{-\infty}^{\infty} e^{-\tilde{\alpha}(\beta)\eta} \sin\beta y \; k(y-\eta)d\eta \\
&= e^{-\tilde{\alpha}(\beta)y}\sin\beta y \int_{-\infty}^{\infty} e^{\tilde{\alpha}(\beta)\eta}\cos\beta\eta \; k(\eta)d\eta \\
&\quad - e^{-\tilde{\alpha}(\beta)y}\cos\beta y \int_{-\infty}^{\infty} e^{\tilde{\alpha}(\beta)\eta}\sin\beta\eta \; k(\eta)d\eta \\
&\overset{(2)}{\ge} e^{-\tilde{\alpha}(\beta)y}\sin\beta y.
\end{aligned}$$

(1): We restrict our attention to $y \in [0, \frac{\pi}{\beta}]$ (since anyhow $\phi * k(y) \ge 0$). In order to make this inequality valid we have to take care that for $\eta \in \mathbb{R}\backslash[0, \frac{\pi}{\beta}]$ either, $\sin\beta\eta \le 0$ or $k(y-\eta) = 0$. Let $B > 0$ be such that $\text{supp}\; k \subset [-B,B]$. If $y \in [0, \frac{\pi}{\beta}]$ and $|y-\eta| \le B$ then $\eta \in [-B, \frac{\pi}{\beta} + B]$ and this interval is contained in $[-\frac{\pi}{\beta}, \frac{2\pi}{\beta}]$ provided $\beta \le \frac{\pi}{B}$.

If either $y = 0$ or $y = \frac{\pi}{\beta}$, then the part that is added to the integral yields a strictly negative contribution (note that both

$\int_0^B k(y)dy > 0$ and $\int_{-B}^0 k(y)dy > 0$) and hence the inequality is strict for those values of y.

(2): Since $\int_{-\infty}^{\infty} e^{\tilde{\alpha}(\beta)\eta} \cos\beta\eta\, k(\eta)d\eta > 1$ for $\beta \le \beta_2$, this inequality is strict for $y \in (0, \frac{\pi}{\beta})$.

STEP 5. At this point we know that $\phi * k \succ \phi$ for $\beta \in [0,\beta_0]$ where $\beta_0 = \min\{\beta_1, \beta_2, \frac{\pi}{B}\}$. From continuity considerations it follows that also $\phi * k \succ \phi_\delta$ if δ is sufficiently small, notably if $\delta \in [0,\Delta(\beta))$, where

$$\Delta(\beta) := \inf\{\sup\{\varepsilon > 0 \mid \phi * k(y) > \phi(y-\varepsilon)\} \mid 0 \le y \le \tfrac{\pi}{\beta}\}. \qquad \square$$

Starting from q we form a three-parameter family of nonincreasing functions r as follows,

$$r(y;\alpha,\beta,\gamma) := \max_{\eta \ge -\gamma} q(y+\eta;\alpha,\beta) \tag{4.8}$$

or, equivalently,

$$r(y;\alpha,\beta,\gamma) = \begin{cases} M & \text{for } \ y \le \gamma + \rho \\ q(y-\gamma;\alpha,\beta) & \text{for } \ \gamma + \rho \le y \le \gamma + \frac{\pi}{\beta} \\ 0 & \text{for } \ y \ge \gamma + \frac{\pi}{\beta} \end{cases} \tag{4.9}$$

where $M = M(\alpha,\beta) := \max\{q(y;\alpha,\beta) \mid 0 \le y \le \frac{\pi}{\beta}\}$ and $\rho = \rho(\alpha,\beta)$ is the value for which the maximum is achieved. The following lemma completes the construction of subsolutions with the desired property.

LEMMA 4. (ARONSON and WEINBERGER [3],[10]).
Let $c \in (0,c_0)$ *be given. There exist numbers* $T > 0$, $\beta > 0$, $\alpha \in \mathbb{R}$, $D > 0$ *and* $\sigma_0 > 0$ *such that for any* $\sigma \in (0,\sigma_0)$ *and for any* $t \ge T$

$$E_T[\sigma\psi](t,.) \succ \sigma\psi(t,.),$$

where $\psi(t,x) := r(|x|;\alpha,\beta,D+ct)$.

PROOF. Choose $h \in (0,g'(0))$, $T > 0$ and $R > 0$ such that $K_c(h,T,R,\lambda) > 1$

for all $\lambda \in \mathbb{R}$ (see Lemma 2). According to Lemma 3 we can choose $\beta > 0$, $\alpha = \tilde{\alpha}(\beta)$ and $\delta \in (0,\Delta(\beta))$ such that for $\delta \leq x_1 \leq \frac{\pi}{\beta} + \delta$

$$(4.10) \qquad h \int_{-\infty}^{\infty} q(x_1-\xi_1)\Phi_c(R,T,\xi_1)d\xi_1 > q(x_1-\delta).$$

(Here and in the following we suppress the dependence on α and β in the notation.) Let σ_h be the smallest positive root of the equation $g(y) = hy$. Then $g(y) > hy$ for $0 < y < \sigma_h$. Choose $\sigma_0 \in (0,\sigma_h M^{-1})$, where M is the maximum of q. With the exception of D we now have chosen all the parameters and it remains to verify the conclusion of the lemma. Let $\sigma \in (0,\sigma_0)$ and $t \geq T$.

$$E_T[\sigma\psi](t,x) = \int_0^T A(\tau) \int_{\mathbb{R}^n} g(\sigma\psi(t-\tau,x-\xi))V(\xi)d\xi d\tau$$

$$\geq \int_0^T A(\tau) \int_{B_R} g(\sigma\psi(t-\tau,x-\xi))V(\xi)d\xi d\tau.$$

We distinguish two cases.

(i) $|x| \leq D + \rho + c(t-T) - R$.
If $|\xi| \leq R$ and $\tau \in [0,T]$ then

$$|x-\xi| \leq D + \rho + c(t-T) \leq D + \rho + c(t-\tau)$$

and consequently

$$E_T[\sigma\psi](t,x) = g(\sigma M) \int_0^T A(\tau)d\tau \int_{B_R} V(\xi)d\xi > \sigma M K_c(h,T,R,0) > \sigma M.$$

(ii) $D + \rho + c(t-T) - R \leq |x| \leq \frac{\pi}{\beta} + D + ct$.
If $|\xi| \leq R$ and $t \geq T$ then

$$\begin{aligned} |x-\xi| &= (|x|^2-2x.\xi+|\xi|^2)^{\frac{1}{2}} \\ &\leq |x| - \frac{x.\xi}{|x|} + \frac{|\xi|^2}{2|x|} \\ &\leq |x| - \frac{x.\xi}{|x|} + \frac{R^2}{2(D+\rho-R)} \\ &\leq |x| - \frac{x.\xi}{|x|} + \delta \end{aligned}$$

if we choose $D \geq \frac{R^2}{2\delta} - \rho + R$.

Since ψ is a nonincreasing function of $|x|$ this implies

$$E_T[\sigma\psi](t,x) \geq \sigma h \int_0^T A(\tau) \int_{B_R} \max_{\eta \geq -D-c(t-\tau)} q\left(|x| - \frac{x.\xi}{|x|} + \delta + \eta\right) V(\xi) d\xi d\tau$$

$$= \sigma h \max_{\eta \geq -D-ct} \int_{-\infty}^{\infty} q(|x|+\delta+\eta-\xi_1)\Phi_c(R,T,\xi_1)d\xi_1$$

$$> \sigma \max_{\eta \geq -D-ct} q(|x|+\eta) = \sigma\psi(t,x).$$

Here we have used the fact that V is a radial function and the inequality (4.10).

Finally, combination of (i) and (ii) yields that

$$E_T[\sigma\psi](t,.) \succ \sigma\psi(t,.). \quad \square$$

Now that we have a comparison principle and suitable subsolutions $\sigma\psi$, it remains to show that the solution of (1.1) is bounded from below by $\sigma\psi$ on a sufficiently large time interval if σ is chosen sufficiently small. In this connection the next result is useful.

LEMMA 5. *There exists* $t_0 \geq 0$ *such that* $u(t,x) > 0$ *for all* $t \geq t_0$ *and all* $x \in \mathbb{R}^n$.

PROOF. If we define the function w by $w(t,x) := \min\{u(t,x),p\}$ then clearly

$$u(t,x) \geq \int_0^t A(\tau) \int_{\mathbb{R}^n} w(t-\tau,\xi)V(x-\xi)d\xi d\tau.$$

From $w(t,x) \leq p$, $\int_0^\infty A(\tau)d\tau = 1$ and $\int_{\mathbb{R}^n} V(x)dx = 1$ we deduce that also

$$p \geq \int_0^t A(\tau) \int_{\mathbb{R}^n} w(t-\tau,\xi)V(x-\xi)d\xi d\tau.$$

Hence

$$w(t,x) \geq \int_0^t A(\tau) \int_{\mathbb{R}^n} w(t-\tau,\xi)V(x-\xi)d\xi d\tau$$

and by iteration

$$(4.11) \qquad w(t,x) \geq \int_0^t A^{m*}(\tau) \int_{\mathbb{R}^n} w(t-\tau,\xi)V^{m*}(x-\xi)d\xi d\tau$$

for all $m \geq 1$, where A^{m*} and V^{m*} denote the (m-1)-times iterated convolutions of, respectively, A and V with itself. One can show that

$$\mathbb{R}^n = \bigcup_{m=1}^{\infty} \{x \mid V^{m*}(\xi) > 0 \text{ for } \xi \text{ in some neighbourhood of } x\}$$

and that there exists $t_1 \geq 0$ such that

$$[t_1,\infty) \subset \bigcup_{m=1}^{\infty} \{t \mid A^{m*}(\tau) > 0 \text{ for } \tau \text{ in some neighbourhood of } t\}$$

(see [5, Lemma 2.1]; these properties are easily verified if V is positive on some ball and if A is positive on some interval).

Let t_2 be such that $f(t_2,\bar{x}) > 0$ for some $\bar{x} \in \mathbb{R}^n$, then $u(t_2,\bar{x}) \geq f(t_2,\bar{x}) > 0$ and hence also $w(t_2,\bar{x}) > 0$. So if $t \geq t_0 = t_1 + t_2$, then (4.11) shows that $u(t,x) \geq w(t,x) > 0$ for all $x \in \mathbb{R}^n$. □

Although our subsolutions are bounded away from zero on the set $|x| \leq ct$, they do not grow to p on such a set. The idea is now to use estimates on $|x| \leq ct$ and more detailed information about g to get better estimates on a smaller set. Our last lemma is intended to show that we can come arbitrarily close to p in this manner.

<u>LEMMA 6</u>. *Let a sequence* $\{N_n\}$ *of real numbers be defined by* $N_0 = a > 0$,

$$N_{n+1} = \left(\int_0^t A(\tau)d\tau \int_{B_R} V(x)dx\right)g(N_n).$$

For any $\varepsilon > 0$ *there exist positive numbers* $\bar{t}(\varepsilon)$, $\bar{R}(\varepsilon)$ *and* $\bar{n}(\varepsilon)$ *such that for any* $t \geq \bar{t}(\varepsilon)$, $R \geq \bar{R}(\varepsilon)$ *and* $n \geq \bar{n}(\varepsilon)$

$$N_n > p - \varepsilon.$$

<u>PROOF</u>. Let $\varepsilon > 0$ be arbitrary. Since $g(x) > x$ for $0 < x < p$ and $g'(0) > 1$,

we know that $\sup\{x^{-1}g(x) \mid 0 < x \le p - \varepsilon\} > 1$. Hence we can choose $\alpha(\varepsilon) < 1$ such that $\alpha(\varepsilon)g(x) > x$ for $0 < x \le p - \varepsilon$. Let the sequence $\{M_n\}$ be defined by $M_0 = a$, $M_{n+1} = \alpha(\varepsilon)g(M_n)$. We observe that

(i) if $0 < M_n \le p - \varepsilon$ then $M_{n+1} = \alpha(\varepsilon)g(M_n) > M_n$,

(ii) if $M_n > p - \varepsilon$ then $M_{n+1} \ge \alpha(\varepsilon)g(p-\varepsilon) > p - \varepsilon$.

Suppose that $M_n \le p - \varepsilon$ for all n, then (i) shows that M_n converges to a limit $M \le p - \varepsilon$. But then necessarily $M = \alpha(\varepsilon)g(M)$ which is impossible for $M \le p - \varepsilon$. So there exists $\bar{n}(\varepsilon)$ such that $M_{\bar{n}(\varepsilon)} > p - \varepsilon$ and subsequently (ii) implies that $M_n > p - \varepsilon$ for all $n \ge \bar{n}(\varepsilon)$.

Choose $\bar{t}(\varepsilon)$ and $\bar{R}(\varepsilon)$ such that

$$\int_0^{\bar{t}(\varepsilon)} A(\tau)d\tau \int_{B_{\bar{R}(\varepsilon)}} V(x)dx \ge \alpha(\varepsilon).$$

For any $t \ge \bar{t}(\varepsilon)$ and $R \ge \bar{R}(\varepsilon)$ we have $N_1 \ge \alpha(\varepsilon)g(a) = M_1$ and by induction $N_n \ge M_n$. Hence $N_n \ge p - \varepsilon$ for $n \ge \bar{n}(\varepsilon)$. $\square$

Finally, we gather together the pieces in order to give the

<u>PROOF OF THEOREM 2</u>. Let $c_1 \in (0,c_0)$ be arbitrary and choose $c_2 \in (c_1,c_0)$. Let $T > 0$, $\beta > 0$, $\alpha \in \mathbb{R}$, $D > 0$ and $\sigma_0 > 0$ be such that for any $\sigma \in (0,\sigma_0)$ and for any $t \ge T$

$$E_T[\sigma\psi](t,.) \succ \sigma\psi(t,.),$$

where $\psi(t,x) := r(|x|;\alpha,\beta,D+c_2t)$, (see Lemma 4). Let t_0 be such that $u(t,x) > 0$ for all $t \ge t_0$, $x \in \mathbb{R}^n$ (cf. Lemma 5). Then we can choose $\sigma_1 \in (0,\sigma_0)$ such that $u(t_0+t,.) \succ \sigma_1\psi(t,.)$ for $0 \le t \le T$, and we infer from the comparison principle Lemma 1 that this relation holds for all $t \ge 0$ (note that ψ has all the required properties). Hence $u(t_0+t,x) \ge \sigma_1 M$ for $t \ge 0$, $|x| \le \rho + D + c_2t$.

Next we use the inequality

$$u(t_0+t,x) \ge \int_0^t A(\tau) \int_{B_R} g(u(t_0+t-\tau,x-\xi))V(\xi)d\xi d\tau$$

to conclude by induction that

$$u(t_0+t,x) \geq N_n \quad \text{for} \quad t \geq 0,\ |x| \leq \rho + D + c_2t - nR,$$

where N_n is defined as in Lemma 6 with $N_0 = a = \sigma_1 M$. So for any $\varepsilon > 0$ we can find $\bar{t}(\varepsilon)$, $\bar{R}(\varepsilon)$ and $\bar{n}(\varepsilon)$ such that

$$u(t,x) \geq p - \varepsilon \text{ for } t \geq t_0 + \bar{t}(\varepsilon),\ |x| \leq \rho + D - c_2t_0 - \bar{n}(\varepsilon)\bar{R}(\varepsilon) + c_2t.$$

Finally, since $c_2 > c_1$ this implies that

$$u(t,x) \geq p - \varepsilon \text{ for } |x| \leq c_1t$$

$$\text{if } t \geq \max\left\{t_0+\bar{t}(\varepsilon),\ \frac{\bar{n}(\varepsilon)\bar{R}(\varepsilon)+c_2t_0-\rho-D}{c_2-c_1}\right\}. \quad \square$$

5. THE EPIDEMIC MODEL OF KERMACK, McKENDRICK AND KENDALL

In our previous paper [4] we have formulated a model for the geographical spread of a contagious disease. This model is based on a combination of ideas of KERMACK & McKENDRICK [9] and KENDALL [7], [8] and the main assumptions are:

(i) the members of the population can be categorized as either susceptible to or infected by the disease;

(ii) the infectivity of an infected individual as a function of time elapsed since exposure and position relative to the individual's own position is given by $H: \mathbb{R}_+ \to \mathbb{R}$ and $W: \mathbb{R}^n \to \mathbb{R}$, respectively;

(iii) the disease induces permanent immunity, so an individual can pass from the class of susceptibles to the class of infectives, but not vice versa.

If S denotes the density of the susceptibles and u is defined by

$$(5.1) \qquad u(t,x) := -\ln \frac{S(t,x)}{S(0,x)},$$

then the model leads to the following equation for u

$$(5.2)\qquad u(t,x) = \int_0^t H(\tau) \int_{\mathbb{R}^n} \tilde{g}(u(t-\tau,\xi))S(0,\xi)W(x-\xi)d\xi d\tau + f(t,x),$$

where

$$(5.3)\qquad \tilde{g}(y) = 1 - e^{-y},$$

and f is a given nonnegative function describing the history up to t = 0 (see [4] for the details). If $S(0,\xi)$ is constant, say S_0, then clearly (5.2) is of the form (1.1) with $g(y) := \alpha\tilde{g}(y)$ and

$$(5.4)\qquad \alpha := S_0 \int_0^\infty H(\tau)d\tau \int_{\mathbb{R}^n} W(x)dx.$$

For this function g the hypothesis H_g is satisfied if and only if $\alpha > 1$. As we will show, the parameter α has a threshold value 1, i.e., the qualitative behaviour is very different in the two cases $\alpha < 1$ and $\alpha > 1$.

By using $g(y) \le \alpha y$ in the same way as in the proof of Theorem 1 it follows that u is bounded from above by constant. $\sup\{f(t,x) \mid t \in \mathbb{R}_+, x \in \mathbb{R}^n\}$ if $\alpha < 1$. One can interpret this result as stability of the rest state $u \equiv 0$. Moreover, Theorem 1 holds for the case $\alpha \le 1$ with the definition $c_0 = 0$.

In contrast with this, Theorem 2 shows that for $\alpha > 1$, equation (5.2) exhibits the hair-trigger effect, i.e., *every* nontrivial nonnegative perturbation of the rest state $u \equiv 0$ has *everywhere* eventually a large effect. In [4] we proved this result for the case of space dimension n = 1 or n = 2, by using the structure of the forcing function f in the epidemic model and some known results about convolution inequalities.

Moreover, Theorems 1 and 2 show that $c_0 = c_0(\alpha)$ is the asymptotic speed for equation (5.2) if $\alpha > 1$. In biological terms this means that if an epidemic draws near then an observer moving with a fixed speed c will consider the epidemic as severe if and only if c is less than c_0. Although the model does not describe the moving of individuals, one can, by way of speaking, also say that somebody who tries to escape the epidemic will improve his changes considerably if and only if he runs away with a speed that exceeds c_0. For a special case of the model it was shown before by ARONSON [1] that c_0 is the asymptotic speed.

Thus far our discussion was limited to the case that $S(0,\xi)$ is assumed to be constant. A close examination of the proofs of the Theorems 1 and 2 reveals that they are based on inequalities, rather than on equalities. If $S(0,\xi)$ is not constant then one can easily obtain results of the type of Theorem 1 by using an upper bound for $S(0,\xi)$ and results of the type of Theorem 2 by using a lower bound. We do not elaborate this idea any further.

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Erratum TW 176/78

On page 14 Lemma 5 should be:

<u>LEMMA 5</u>. *For any* $T > 0$ *and any* $R > 0$ *there exists* $t_0 = t_0(T,R)$ *such that* $u(t,x) > 0$ *for* $(t,x) \in [t_0,t_0+T] \times B_R$.

The proof remains the same up to the fourth line on page 15 and then it should continue as follows:

One can show that V^{m*} is positive on a ball of radius R_m and that A^{m*} is positive on an interval $[a_m,b_m]$ where both $R_m \to \infty$ and $b_m - a_m \to \infty$ as $m \to \infty$ (see [5, Lemma 2.1]; these properties are easily verified if V is positive on some ball and if A is positive on some interval).

Let $\bar{t}$ be such that $f(\bar{t},\bar{x}) > 0$ for some $\bar{x} \in \mathbb{R}^n$, then $u(\bar{t},\bar{x}) \geq f(\bar{t},\bar{x}) > 0$ and hence also $w(\bar{t},\bar{x}) > 0$. So (4.11) shows that $u(t,x) \geq w(t,x) > 0$ for $t \in [\bar{t}+a_m,\bar{t}+b_m]$ and x in a ball of radius R_m centered at $\bar{x}$. Now for any $T > 0$ and any $R > 0$ this set contains the set $[\bar{t}+a_m,\bar{t}+a_m+T] \times B_R$ if m is sufficiently large and from this observation the conclusion of the lemma follows. □

After this proof one should add the following:

<u>REMARK</u>. Actually one can show that there is a finite speed of propagation of the boundary of supp $u(t,.)$ if the support of A is bounded away from zero and if V has bounded support. This has been pointed out recently by H. Thieme (Asymptotic speed for the spread of populations, preprint).

On page 16 the sentence on line 8 and 7 from below should be:

Let t_0 be such that $u(t,x) > 0$ for $t \in [t_0,t_0+T]$ and $|x| \leq D + c_2T + \frac{\pi}{\beta}$ (cf. Lemma 5).