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T.H. KOORNWINDER

A GLOBAL APPROACH TO THE REPRESENTATION THEORY OF SL(2,ℝ)

2e boerhaavestraat 49 amsterdam

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A global approach to the representation theory of $SL(2,\mathbb{R})^{*)}$

by

T.H. Koornwinder

ABSTRACT

The representation theory of $SL(2,\mathbb{R})$ is developed by the use of global (i.e., non-infinitesimal) methods. The irreducible K-unitary, K-finite representations of $SL(2,\mathbb{R})$ are classified up to equivalence as subrepresentations of the (non-unitary) principal series and the unitarizability of these representations is considered. The results are based on an explicit knowledge of the matrix elements of the principal series with respect to the K-basis.

KEY WORDS & PHRASES: *representation theory of* $SL(2,\mathbb{R})$; *principal series*; *canonical matrix elements*; *Jacobi functions*; K-*unitary*, K-*multiplicity free representations; equivalence of representations*; *unitarizability of representations*.

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*) These notes are an extended version of a lecture given by the author at the MC-University of Leiden seminar "Analysis on Lie groups", 1978/79.

1. INTRODUCTION

The classification of all irreducible unitary representations of SL(2,ℝ) is usually done by means of infinitesimal methods, cf. BARGMANN [1], SUGIURA [12,Ch.V], VAN DIJK [3]. An important motivation for the use of such methods is the fact that they are well-adapted for generalization to the case of an arbitrary noncompact semisimple Lie group. Here we develop an alternative approach to the representation theory of SL(2,ℝ). It is based on an explicit knowledge of the matrix elements of the (non-unitary) principal series representations with respect to the K-basis. Except for one place the methods we use are global. Unfortunately, our approach has not (yet) been generalized to the case of an arbitrary noncompact semisimple Lie group. Maybe an extension is possible to the case of SO(n,1) or SU(n,1). If this would be feasible then the work done here would be fully justified.

The author does not claim that his approach is new. Rather than searching in the literature, I enjoyed it to write down my favourite way of understanding the representation theory of SL(2,ℝ). The paper has become rather lengthy because most results are first formulated in an abstract setting, for a locally compact second countable group, and then applied to SL(2,ℝ). Thus some of the theoretical prerequisites for doing a similar analysis on, say, SO(n,1) are already provided in this paper. These notes are much inspired by VAN DIJK's lecture notes [3], where the traditional approach is presented, and we use his paper as our main reference.

Our program consists of four parts:

(i) Determine all irreducible subrepresentations and subquotient representations of the principle series representations of SL(2,ℝ).

(ii) Determine which equivalences do exist between the representations in (i).

(iii) Prove that each irreducible representation of SL(2,ℝ) is equivalent to some representation in (i).

(iv) Which of the representations in (i) are unitarizable?

We will not only consider unitary representations, but, more generally, strongly continuous representations on a Hilbert space which are K-unitary and K-finite (cf. §3.1 for the definition of these concepts). Accordingly, we need a more general (but still non-infinitesimal) notion of equivalence than the notion of unitary equivalence, cf. §4.

The four parts of the above program will be treated in sections 3, 4, 5 and 6, respectively. We start in section 2 with the computation of the canonical matrix elements of the principal series representations. They can be expressed in terms of hypergeometric functions or, more elegantly, Jacobi functions. These explicit expressions will be used throughout the paper. We hope that this will illuminate the possible usefulness of special functions in the representation theory of specific groups. In the final section 7 we identify some of the irreducible unitary representations we obtained with the familiar models for the complementary and the discrete series; we discuss square integrable representations and we point out possible applications of addition formulas for special functions to representation theory.

2. THE CANONICAL MATRIX ELEMENTS OF THE PRINCIPAL SERIES

2.1. The group SU(1,1)

We shall work with the group $G = SU(1,1)$, which is isomorphic to $SL(2,\mathbb{R})$. The group G consists of all complex 2×2 matrices

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \quad \text{with } \alpha,\ \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1.$$

We consider the two subgroups

$$K := \left\{ u_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\},$$

$$A := \left\{ a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \right\}.$$

We have the decomposition $G = KAK$, that is, each $g \in G$ can be written as

$$g = u_{\theta_1}\, a_t\, u_{\theta_2} \tag{2.1}$$

for some θ_1, θ_2, $t \in \mathbb{R}$.

2.2. The principal series for SU(1,1)

Let U be the unit circle $\{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$. Fix $\lambda \in \mathbb{C}$, $\eta \in \{-1,1\}$. We define the principal series representation $\pi_{\eta,\lambda}$ of G on $L^2(U)$ by (cf. VAN DIJK [3, (8.2)])

$$(2.2) \qquad (\pi_{\eta,\lambda}(g)f)(\zeta) := |\bar{\beta}\zeta+\bar{\alpha}|^{-(\lambda+1)}\left(\frac{\bar{\beta}\zeta+\bar{\alpha}}{|\bar{\beta}\zeta+\bar{\alpha}|}\right)^{\frac{1}{2}(1-\eta)} f\left(\frac{\alpha\zeta+\beta}{\bar{\beta}\zeta+\bar{\alpha}}\right),$$

$$g^{-1} = \begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix} \in SU(1,1),\ f \in L^2(U),\ \zeta \in U.$$

The representation $\pi_{\eta,\lambda}$ is not necessarily unitary, but it is always strongly continuous.

Let

$$(2.3) \qquad \phi_m(e^{i\psi}) := e^{-im\psi}, \qquad m \in \mathbb{Z},\ e^{i\psi} \in U.$$

The functions ϕ_m, $m \in \mathbb{Z}$, form an orthonormal basis for $L^2(U)$. We have

$$(2.4) \qquad \pi_{\eta,\lambda}(u_\theta)\phi_m = e^{i(2m+\frac{1}{2}(1-\eta))\theta}\phi_m, \qquad u_\theta \in K,\ m \in \mathbb{Z}.$$

$\pi_{\eta,\lambda}|_K$ is a unitary representation of K containing each irreducible representation at most once. We say that the representation $\pi_{\eta,\lambda}$ is K-unitary and K-multiplicity free (cf. §3.1). We call the basis $\{\phi_m\}$ a K-basis. Note that $\pi_{\eta,\lambda}|_K$ is the direct sum of all representations $u_\theta \to e^{i\ell\theta}$ with $(-1)^\ell = \eta$.

2.3. The canonical matrix elements of $\pi_{\eta,\lambda}$

For $m,n \in \mathbb{Z}$ let

$$(2.5) \qquad \pi_{\eta,\lambda,m,n}(g) := (\pi_{\eta,\lambda}(g)\phi_n,\phi_m) = \frac{1}{2\pi}\int_0^{2\pi} (\pi_{\eta,\lambda}(g)\phi_n)(e^{i\psi})e^{im\psi}\,d\psi.$$

Then, in terms of the decomposition (2.2):

$$\text{(2.6)} \qquad \pi_{\eta,\lambda,m,n}(u_{\theta_1} a_t u_{\theta_2}) = $$
$$= e^{i(2m+\frac{1}{2}(1-\eta))\theta_1} \, e^{i(2n+\frac{1}{2}(1-\eta))\theta_2} \, \pi_{\eta,\lambda,m,n}(a_t).$$

Hence the representation $\pi_{\eta,\lambda}$ is completely determined by the functions $\pi_{\eta,\lambda,m,n}$ restricted to A. We will find an explicit expression for these functions. First we derive an integral representation. It follows from (2.2), (2.3) and (2.5) that

$$\text{(2.7)} \qquad \pi_{\eta,\lambda,m,n}(a_t) = (\cosh t)^{-(\lambda+1)} \cdot$$
$$\cdot \frac{1}{2\pi} \int_0^{2\pi} (1-\mathrm{tgh}\, t\, e^{i\psi})^{-\frac{1}{2}(\lambda+1)+\frac{1}{4}(1-\eta)+n} \cdot$$
$$\cdot (1-\mathrm{tgh}\, t\, e^{-i\psi})^{-\frac{1}{2}(\lambda+1)-\frac{1}{4}(1-\eta)-n} \, e^{i(m-n)\psi} d\psi.$$

Making the transformation $\psi \to 2\pi - \psi$ in (2.7) we find

$$\text{(2.8)} \qquad \pi_{\eta,\lambda,-m-\frac{1}{2}(1-\eta),-n-\frac{1}{2}(1-\eta)}(a_t) = \pi_{\eta,\lambda,m,n}(a_t).$$

Hence, for further calculations it is sufficient to consider the case $m \geq n$.

2.4. Hypergeometric and Jacobi functions

It is well-known that

$$\text{(2.9)} \qquad (1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k, \qquad |z| < 1, \; a \in \mathbb{C},$$

where $(a)_k$ is the shifted factorial

$$\text{(2.10)} \qquad (a)_k := a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)} .$$

A generalization of (2.9) is the hypergeometric series (cf. ERDÉLYI [4, vol. I, Ch.2])

$$(2.11)\qquad F(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad |z| < 1, \; a,b,c \in \mathbb{C},$$

$$c \notin \{0,-1,-2,\ldots\}.$$

The function $z \to F(a,b;c;z)$ has an analytic continuation to a one-valued function on $\mathbb{C}\backslash[1,\infty)$. The two following transformation formulas are useful:

$$(2.12)\qquad F(a,b;c;z) = (1-z)^{-b}\, F(c-a,b;c;\tfrac{z}{z-1}) =$$

$$= (1-z)^{c-a-b} F(c-a,c-b;c;z)$$

(cf. ERDÉLYI [4, Vol.I, §2.9,(1),(2),(4)]).

Jacobi functions $\phi_\mu^{(\alpha,\beta)}$ $(\mu,\alpha,\beta \in \mathbb{C},\ \alpha \in \{-1,-2,\ldots\})$ are defined on $\mathbb{R}$ by

$$(2.13)\qquad \phi_\mu^{(\alpha,\beta)}(t) := F(\tfrac{1}{2}(\alpha+\beta+1-i\mu),\tfrac{1}{2}(\alpha+\beta+1+i\mu);\ \alpha+1;-(\sinh t)^2).$$

The function $\phi_\mu^{(\alpha,\beta)}$ satisfies the differential equation

$$(2.14)\qquad \left(\frac{d^2}{dt^2} + ((2\alpha+1)\operatorname{cotgh} t+(2\beta+1)\operatorname{tgh} t)\frac{d}{dt}\right)\phi_\mu^{(\alpha,\beta)}(t) =$$

$$= (-\mu^2-(\alpha+\beta+1)^2)\phi_\mu^{(\alpha,\beta)}(t)$$

and it is the unique solution of this differential equation which is regular and equal to 1 at zero. For fixed $\alpha > -1$, $\beta \in \mathbb{R}$, Jacobi functions $\phi_\mu^{(\alpha,\beta)}$ form a continuous orthogonal system with respect to the measure

$$(\sinh t)^{2\alpha+1}\,(\cosh t)^{2\beta+1}dt.$$

The theory of Jacobi functions and the harmonic analysis for Jacobi function expansions was developed in papers by FLENSTED-JENSEN and KOORNWINDER, see [5], [7], [8] and [6, Appendix 1].

2.5. The canonical matrix elements in terms of Jacobi functions

Let $m \geq n$. Replace the first two factors in the integrand of (2.7) by their binomial expansions according to (2.9). By interchanging the order of summation and integration and by integrating with respect to ψ, we obtain a hypergeometric series of the form (2.11):

$$\pi_{\eta,\lambda,m,n}(a_t) = \frac{(\frac{1}{2}(\lambda+1)+\frac{1}{4}(1-\eta)+n)_{m-n}}{(m-n)!} (\cosh t)^{-(\lambda+1)} (\operatorname{tgh} t)^{m-n} \cdot$$

$$\cdot\ {}_2F_1(\tfrac{1}{2}(\lambda+1)-\tfrac{1}{4}(1-\eta)-n, \tfrac{1}{2}(\lambda+1)+\tfrac{1}{4}(1-\eta)+m;\ m-n+1; (\operatorname{tgh} t)^2).$$

By using (2.12) and (2.13) we find

$$\pi_{\eta,\lambda,m,n}(a_t) = \frac{(\frac{1}{2}(\lambda+1)+\frac{1}{4}(1-\eta)+n)_{m-n}}{(m-n)!} \cdot$$

$$\cdot\ (\sinh t)^{m-n} (\cosh t)^{m+n+\frac{1}{2}(1-\eta)} \phi_{i\lambda}^{(m-n,m+n+\frac{1}{2}(1-\eta))}(t).$$

Application of (2.8) and (2.12) yields a similar result in the case $m < n$. Finally we conclude:

THEOREM 2.1. *The canonical matrix elements* $\pi_{\eta,\lambda,m,n}(a_t)$ *can be expressed in terms of Jacobi functions by*

$$\pi_{\eta,\lambda,m,n}(a_t) = \frac{c_{\eta,\lambda,m,n}}{(|m-n|)!} \cdot \tag{2.15}$$

$$\cdot\ (\sinh t)^{|m-n|} (\cosh t)^{m+n+\frac{1}{2}(1-\eta)} \phi_{i\lambda}^{(|m-n|,m+n+\frac{1}{2}(1-\eta))}(t),$$

where

$$c_{\eta,\lambda,m,n} := \begin{cases} (\frac{1}{2}(\lambda+1)+\frac{1}{4}(1-\eta)+n)_{m-n} & \text{if } m \geq n, \\ (\frac{1}{2}(\lambda+1)-\frac{1}{4}(1-\eta)-n)_{n-m} & \text{if } m \leq n. \end{cases} \tag{2.16}$$

See also VILENKIN [13,Ch.VI,§3.3,(2)] for an explicit expression of $\pi_{\eta,\lambda,m,n}(a_t)$.

3. THE IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

3.1. K-unitary, K-multiplicity free representations of G

Let G be a lcsc. (locally compact, second countable) group with compact subgroup K. Let τ be a strongly continuous representation of G on a separable Hilbert space $H(\tau)$. The representation τ is called K-*unitary* if $\tau|_K$ is a unitary representation of K. If τ is K-unitary then τ is called K-*finite* if each $\delta \in \hat{K}$ has finite multiplicity in $\tau|_K$ and τ is called K-*multiplicity free* if each $\delta \in \hat{K}$ has multiplicity 1 or 0 in $\tau|_K$. The representations $\pi_{\eta,\lambda}$ of SU(1,1) are K-unitary and K-multiplicity free.

Let τ be a K-unitary representation of G. Let $\delta \in \hat{K}$ with degree d_δ and character χ_δ. Then

(3.1) $$P_{\tau,\delta}v := d_\delta \int_K \chi_\delta(k^{-1})\tau(k)v\, dk, \quad v \in H(\tau),$$

defines an orthogonal projection $P_{\tau,\delta}$ from $H(\tau)$ onto a closed subspace $H_\delta(\tau)$. Furthermore,

(3.2) $$H(\tau) = \sum_{\delta\in\hat{K}}{}^{\oplus} H_\delta(\tau)$$

and the representation $\tau|_K$ acts on $H_\delta(\tau)$ as a multiple of δ.

Next assume that τ is a K-unitary, K-multiplicity free representation of G. The K-*content* $M(\tau)$ of τ is defined as the set of all $\delta \in \hat{K}$ for which δ has multiplicity 1 in $\tau|_K$. For $\delta,\varepsilon \in M(\tau)$, $g \in G$, we define the *generalized canonical matrix elements* $\tau_{\delta,\varepsilon}(g)$ of τ as linear mappings $\tau_{\delta,\varepsilon}(g) : H_\varepsilon(\tau) \to H_\delta(\tau)$ given by

(3.3) $$\tau_{\delta,\varepsilon}(g)v := P_{\tau,\delta}\, \tau(g)v, \quad v \in H_\varepsilon(\tau).$$

We may consider $\tau(g)$ as a (usually infinite) block matrix with blocks $\tau_{\delta,\varepsilon}(g)$.

3.2. Subquotient representations of K-unitary, K-multiplicity free representations

Let τ be a K-unitary, K-multiplicity free representation of G. Let H_0 be a closed invariant subspace of $H(\tau)$ with respect to τ. Since $\tau|_K$ is unitary and multiplicity free, H_0 must be a direct sum of certain subspaces $H_\delta(\tau)$, $\delta \in M(\tau)$. Hence the subrepresentation σ of G on $H_0 = H(\sigma)$ is again K-unitary and K-multiplicity free and $M(\sigma) \subset M(\tau)$.

Let ρ and σ be subrepresentations of τ with $H(\rho) \subset H(\sigma)$. Let P be the orthogonal projection from $H(\sigma)$ onto $H(\rho)$. We define a new representation π of G on $H(\pi) := H(\sigma) \cap H(\rho)^\perp$ by

$$(3.4) \qquad \pi(g)v := (I-P)\sigma(g)v, \qquad v \in H(\pi).$$

Then π is a strongly continuous representation of G and π is K-unitary and K-multiplicity free with $M(\pi) = M(\sigma)\backslash M(\rho)$. We call π a *subquotient representation* of τ. (Note that any subrepresentation σ of τ is also a subquotient representation of τ since the choice $\rho := 0$ yields $\pi = \sigma$ in (3.4).) For the generalized canonical matrix elements of π we have

$$(3.5) \qquad \pi_{\delta,\varepsilon}(g) = \tau_{\delta,\varepsilon}(g), \qquad \delta,\varepsilon \in M(\pi), \qquad g \in G.$$

(Note that $H_\delta(\pi) = H_\delta(\tau)$ if $\delta \in M(\pi)$.)

3.3. Irreducible subquotient representations

Let again τ be a K-unitary, K-multiplicity free representation of G. For $\delta \in M(\tau)$ let Cycl(δ) be the G-invariant closed linear subspace of $H(\tau)$ which is generated by $H_\delta(\tau)$, that is, Cycl(δ) is the closure of the linear span of the set $\{\tau(g)v \mid g \in G, v \in H_\delta(\tau)\}$. The subrepresentation of τ on Cycl(δ) is a cyclic representation with any nonzero $v \in H_\delta(\tau)$ as a source vector. Clearly, if $\gamma,\delta,\varepsilon \in M(\tau)$ and if $H_\delta(\tau) \subset$ Cycl(γ), $H_\varepsilon(\tau) \subset$ Cycl(δ) then $H_\varepsilon(\tau) \subset$ Cycl(γ).

If $\delta \in M(\tau)$ then let Anticycl(δ) be the closed linear span of all subspaces $H_\gamma(\tau)$ such that $\gamma \in M(\tau)$, $H_\gamma(\tau) \subset$ Cycl(δ) and not $H_\delta(\tau) \subset$ Cycl(γ).

Then Anticycl(δ) is a G-invariant closed subspace of Cycl(δ) which does not include $H_\delta(\tau)$. Let Irr(δ) := Cycl(δ) $\cap$ (Anticycl(δ))$^\perp$. Let τ_δ be the subquotient representation of τ on Irr(δ).

THEOREM 3.1. *The representations* $\tau_\delta, \delta \in M(\tau)$, *are irreducible. Any irreducible subquotient representation of* τ *is equal to some* τ_δ, $\delta \in M(\tau)$. *If* $\delta, \varepsilon \in M(\tau)$ *then* $M(\tau_\delta)$ *and* $M(\tau_\varepsilon)$ *are equal or disjoint.* $M(\tau)$ *is the union of all* $M(\tau_\delta)$, $\delta \in M(\tau)$. *The irreducible subrepresentations of* τ *are just those representations* τ_δ *for which* Anticycl(δ) = $\{0\}$.

PROOF. Let $\delta \in M(\tau)$. If $H_\varepsilon(\tau) \subset$ Irr(δ) then $H_\delta(\tau) \subset$ Cycl(ε). Hence Cycl(ε) = Cycl(δ). This implies the first statement. For the proof of the second statement let π be an irreducible subquotient representation of τ on $H(\pi) = H(\sigma) \cap H(\rho)^\perp$ with ρ and σ subrepresentations of τ such that $H(\rho) \subset H(\sigma)$. Choose $\delta \in M(\pi)$. Since π is irreducible, $H(\pi) \subset$ Cycl(δ). Hence, without loss of generality we may assume that $H(\sigma)$ = Cycl(δ). Clearly, $H(\rho) \subset$ Anticyl(δ). On the other hand, if $H_\varepsilon(\tau) \subset$ Anticycl(δ) then $\varepsilon \in M(\rho)$, for otherwise Cycl(ε) $\cap\, H(\rho)^\perp$ would be a nonzero π-invariant subspace of $H(\pi)$ which does not include $H_\delta(\tau)$, thus contradicting the irreducibility of π. Hence $H(\rho)$ = Anticycl(δ) and $\pi = \tau_\delta$. The other statements are almost obvious. □

By inspection of the generalized matrix elements of τ it can be decided whether or not some $\gamma \in M(\tau)$ is in $M(\tau_\delta)$:

THEOREM 3.2. *Let* τ *be a* K-*unitary,* K-*multiplicity free representation of* G. *Let* $\gamma, \delta \in M(\tau)$. *Then*:

(a) $H_\gamma(\tau) \subset$ Cycl(δ) *iff* $\tau_{\gamma,\delta} \neq 0$.

(b) $\gamma \in M(\tau_\delta)$ *iff both* $\tau_{\gamma,\delta} \neq 0$ *and* $\tau_{\delta,\gamma} \neq 0$.

PROOF. We only have to prove (a). First suppose $\tau_{\gamma,\delta} = 0$. Let $v \in H_\delta(\tau)$, $w \in H_\gamma(\tau)$. Then for all $g \in G$:

$$(\tau(g)v,w) = (\tau_{\gamma,\delta}(g)v,w) = 0.$$

Hence $w \perp$ Cycl(δ). Thus $H_\gamma(\tau) \perp$ Cycl(δ).

Next suppose that $\tau_{\gamma,\delta}(g)v \neq 0$ for some $g \in G$, $v \in H_\delta(\tau)$. Since

$$\tau_{\gamma,\delta}(g)v = P_{\tau,\gamma}\tau(g)v = d_\gamma \int_K \chi_\gamma(k^{-1})\tau(kg)v\, dk,$$

it follows that $\tau_{\gamma,\delta}(g)v \in \mathrm{Cycl}(\delta)$.
Hence $H_\gamma(\tau) \subset \mathrm{Cycl}(\delta)$. □

3.4. Irreducible subquotient representations of $\pi_{\eta,\lambda}$

Fix $\lambda \in \mathbb{C}$, $\eta \in \{1,-1\}$. Then the representation $\pi_{\eta,\lambda}$ of $G = SU(1,1)$ is K-unitary and K-multiplicity free. Let $m \in \mathbb{Z}$ and let S be the irreducible representation of K on $\mathbb{C}.\phi_m$. By abuse of notation we write $\mathrm{Cycl}(\phi_m)$, $\mathrm{Anticycl}(\phi_m)$ and $\mathrm{Irr}(\phi_m)$ instead of $\mathrm{Cycl}(\delta)$, and so on. Now Theorem 3.2 takes the following form:

THEOREM 3.3. *Let* $m, n \in \mathbb{Z}$. *Then*:
(a) $\phi_m \in \mathrm{Cycl}(\phi_n)$ *iff* $c_{\eta,\lambda,m,n} \neq 0$.
(b) $\phi_m \in \mathrm{Irr}(\phi_n)$ *iff* $c_{\eta,\lambda,m,n} \neq 0 \neq c_{\eta,\lambda,n,m}$.

PROOF. We prove (a) using Theorem 3.2 (a). Because of (2.6), $\phi_m \in \mathrm{Irr}(\phi_n)$ iff $\pi_{\eta,\lambda,m,n}(a_t) \neq 0$ for some $t \in \mathbb{R}$. Inspection of (2.15) shows that $\pi_{\eta,\lambda,m,n}(a_t) = c_{\eta,\lambda,m,n}t^{|m-n|}f(t)$ for some continuous function f with $f(0) \neq 0$. Hence, if $c_{\eta,\lambda,m,n} \neq 0$ then $\pi_{\eta,\lambda,m,n}(a_t) \neq 0$ for sufficiently small nonzero t. If $c_{\eta,\lambda,m,n} = 0$ then $\pi_{\eta,\lambda,m,n} = 0$. □

Inspection of (2.16) immediately yields when $c_{\eta,\lambda,m,n} \neq 0$. Then an application of Theorems 3.3 and 3.1 gives:

THEOREM 3.4.*) *Depending on* η *and* λ *the representation* $\pi_{\eta,\lambda}$ *of* SU(1,1) *has the following irreducible subquotient representations and subrepresentations:*

(a) $\eta = 1$, $\lambda \neq$ *odd integer.*
$c_{\eta,\lambda,m,n} \neq 0$ *for all* $m,n \in \mathbb{Z}$.
$\pi_{\eta,\lambda}$ *is irreducible.*

*) An asterix at some place in the diagrams occurring in this theorem means that all coefficients $c_{\eta,\lambda,m,n}$ corresponding to that block are nonzero. The same method of proof was used in BARUT & PHILLIPS [15].

(b) $\underline{\eta = 1,\ \lambda = 2k+1 \textit{ for an integer } k \geq 0}$.

$c_{\eta,\lambda,m,n}$:

$m \downarrow$ $\overrightarrow{n}$	$(-\infty,-k-1]$	$[-k,k]$	$[k+1,\infty)$
$(-\infty,-k-1]$	*	*	0
$[-k,k]$	0	*	0
$[k+1,\infty)$	0	*	*

Irreducible subrepresentations on $\overline{\mathrm{Span}}(\phi_{-k-1},\phi_{-k-2},\ldots)$ *and* $\overline{\mathrm{Span}}(\phi_{k+1},\phi_{k+2},\ldots)$.
Irreducible subquotient but not subrepresentation on $\mathrm{Span}(\phi_{-k},\phi_{-k+1},\ldots,\phi_{k-1},\phi_k)$.

(c) $\underline{\eta = 1,\ \lambda = -2k-1 \textit{ for an integer } k \geq 0}$.

$c_{\eta,\lambda,m,n}$:

$m \downarrow$ $\overrightarrow{n}$	$(-\infty,-k-1]$	$[-k,k]$	$[k+1,\infty)$
$(-\infty,-k-1]$	*	0	0
$[-k,k]$	*	*	*
$[k+1,\infty)$	0	0	*

Irreducible subrepresentation on $\overline{\mathrm{Span}}(\phi_{-k},\phi_{-k+1},\ldots,\phi_{k-1},\phi_k)$.
Irreducible subquotients but not subrepresentations on $\overline{\mathrm{Span}}(\phi_{-k-1},\phi_{-k-2},\ldots)$ *and* $\overline{\mathrm{Span}}(\phi_{k+1},\phi_{k+2},\ldots)$.

(d) $\underline{\eta = -1,\ \lambda \neq \textit{even integer}}$.

$c_{\eta,\lambda,m,n} \neq 0$ *for all* $m,n \in \mathbb{Z}$.
$\pi_{\eta,\lambda}$ *is irreducible.*

(e) $\underline{\eta = -1,\ \lambda = 0}$.

$c_{\eta,\lambda,m,n}$:

$m \downarrow \quad \overrightarrow{n}$	$(-\infty,-1]$	$[0,\infty)$
$(-\infty,-1]$	*	0
$[0,\infty)$	0	*

Irreducible subrepresentations on $\overline{\text{Span}}(\phi_{-1},\phi_{-2},\ldots)$ *and* $\overline{\text{Span}}(\phi_0,\phi_1,\ldots)$.

(f) $\eta = -1$, $\lambda = 2k$ *for an integer* $k > 0$.

$c_{\eta,\lambda,m,n}$:

$m \downarrow \quad \overrightarrow{n}$	$(-\infty,-k-1]$	$[-k,k-1]$	$[k,\infty)$
$(-\infty,-k-1]$	*	*	0
$[-k,k-1]$	0	*	0
$[k,\infty)$	0	*	*

Irreducible subrepresentations on $\overline{\text{Span}}(\phi_{-k-1},\phi_{-k-2},\ldots)$ *and* $\overline{\text{Span}}\ (\phi_k,\phi_{k+1},\ldots)$.
Irreducible subquotient but not subrepresentation on $\text{Span}(\phi_{-k},\phi_{-k+1},\ldots,\phi_{k-2},\phi_{k-1})$.

(g) $\eta = -1$, $\lambda = -2k$ *for an integer* $k > 0$.

$c_{\eta,\lambda,m,n}$:

$m \downarrow \quad \overrightarrow{n}$	$(-\infty,-k-1]$	$[-k,k-1]$	$[k,\infty)$
$(-\infty,-k-1]$	*	0	0
$[-k,k-1]$	*	*	*
$[k,\infty)$	0	0	*

Irreducible subrepresentations on $\text{Span}(\phi_{-k},\ldots,\phi_{k-1})$.
Irreducible subquotients but not subrepresentations on $\overline{\text{Span}}(\phi_{-k-1},\phi_{-k-2},\ldots)$ *and* $\overline{\text{Span}}(\phi_k,\phi_{k+1},\ldots)$.

This completes part (i) of our program formulated in § 1. The same results were obtained in VAN DIJK [3, Theor. 4.1] by infinitesimal methods.

4. EQUIVALENCE BETWEEN IRREDUCIBLE SUBQUOTIENT REPRESENTATIONS OF THE PRINCIPAL SERIES

4.1. The definition of equivalence for K-unitary, K-finite representations

Let G be a lcsc. group with compact subgroup K. Let σ and τ be K-unitary, K-finite representations of G. We want to generalize the concept of unitary equivalence for two unitary representations of G. We say that σ is *equivalent* to τ if there is a closed (possibly unbounded) injective linear operator A with domain $\mathcal{D}(A)$ dense in $H(\sigma)$ and range $\mathcal{R}(A)$ dense in $H(\tau)$ such that $\mathcal{D}(A)$ is invariant with respect to σ and $A\,\sigma(g)v = \tau(g)\,Av$ for all $v \in \mathcal{D}(A)$, $g \in G$. Then we use the notation $\sigma \simeq \tau$ or $\sigma \overset{A}{\cong} \tau$.

It is not difficult to show that this concept of equivalence coincides with the concept of Naimark equivalence as defined in WARNER [14, Vol. I, p. 242]. It is stated in [14, Vol. I, Theorem 4.5.5.2] that, in the case of a connected unimodular Lie group G with compact connected subgroup K, two K-unitary, K-finite representations of G are Naimark equivalent iff they are infinitesimally equivalent.

LEMMA 4.1. *Let* $\sigma \overset{A}{\cong} \tau$. *Then, for all* $\delta \in \hat{K}$, *we have* $H_\delta(\sigma) \subset \mathcal{D}(A)$, $H_\delta(\tau) \subset \mathcal{R}(A)$ *and* $A_\delta := A|_{H_\delta(\sigma)}$ *is a one-to-one K-intertwining operator from* $H_\delta(\sigma)$ *onto* $H_\delta(\tau)$. *Furthermore*

$$(4.1)\qquad \mathcal{D}(A) = \{ \sum_{\delta\in\hat{K}} v_\delta \;\Big|\; v_\delta \in H_\delta(\sigma) \text{ if } \delta \in \hat{K},$$
$$\sum_{\delta\in\hat{K}} \|v_\delta\|^2 < \infty,\ \sum_{\delta\in\hat{K}} \|A_\delta v_\delta\|^2 < \infty\}.$$

PROOF. Let $v \in \mathcal{D}(A)$, $\delta \in \hat{K}$. Using (3.1) and the intertwining property of A we have

$$P_{\sigma,\delta}\, v = d_\delta \int_K \chi_\delta(k^{-1})\sigma(k)v\, dk,$$
$$P_{\tau,\delta}\, Av = d_\delta \int_K \chi_\delta(k^{-1})\tau(k)Av\, dk = d_\delta \int_K \chi_\delta(k^{-1})A\,\sigma(k)v\, dk.$$

Since A is closed and $\mathcal{D}(A)$ is invariant under σ, we conclude that $P_{\sigma,\delta}\, v \in \mathcal{D}(A)$ and $AP_{\sigma,\delta}\, v = P_{\tau,\delta}\, Av$. Now $P_{\sigma,\delta}\mathcal{D}(A)$ is dense in $\mathcal{H}_\delta(\delta)$, so $P_{\sigma,\delta}\mathcal{D}(A) = \mathcal{H}_\delta(\sigma)$ because $\mathcal{H}_\delta(\sigma)$ is finite-dimensional. Thus $\mathcal{H}_\delta(\sigma) \subset \mathcal{D}(A)$. The further statements of the lemma now follow rather easily. For the last statement use the fact that A is closed. □

REMARK 4.2. In the defninition of equivalence for σ and τ we can omit the condition that A is closed, but then we have to require a priori that, for all $\delta \in \hat{K}$, $\mathcal{H}_\delta(\sigma) \subset \mathcal{D}(A)$ and $P_{\tau,\delta}\, Av = AP_{\sigma,\delta}\, v$, $v \in \mathcal{D}(A)$. In that case A can be extended to a closed oparator $\overline{A}$ with domain given by (4.1) and $\overline{A}$ will have all the properties required in the definition of equivalence.

REMARK 4.3. It is rather obvious that $\sigma \overset{A}{\cong} \tau$ implies $\tau \overset{A^{-1}}{\cong} \sigma$. In view of Remark 4.2 the equivalences $\rho \overset{A}{\cong} \sigma$ and $\sigma \overset{B}{\cong} \tau$ imply that $\rho \overset{C}{\cong} \tau$, where C is the closure of BA.

THEOREM 4.4. *If* σ *and* τ *are unitary* K-*finite representations of* G *and if* $\sigma \overset{A}{\cong} \tau$ *then* σ *and* τ *are unitarily equivalent.*

For the proof of this theorem use the polar decomposition $A = U(A^*A)^{\frac{1}{2}}$, where U is a unitary oparator from $\mathcal{H}(\sigma)$ onto $\mathcal{H}(\tau)$. Then U is an intertwining operator for σ and τ.

4.2. Equivalence for K-unitary, K-multiplicity free representations

Let σ and τ be K-unitary, K-multiplicity free representations of G and assume that $\sigma \overset{A}{\cong} \tau$. Then, clearly, $M(\sigma) = M(\tau)$. Furthermore the intertwining property of A implies that

$$A_\delta \sigma_{\delta,\varepsilon}(g) = \tau_{\delta,\varepsilon}(g) A_\varepsilon, \quad \delta,\varepsilon \in M(\sigma).$$

For each $\delta \in M(\sigma)$ we choose a K-intertwining isometry $I_\delta : \mathcal{H}_\delta(\sigma) \to \mathcal{H}_\delta(\tau)$, unique up to a complex scalar factor of absolute value 1. Then $A_\delta = c_\delta I_\delta$ for some complex $c_\delta \neq 0$. It follows that

$$\tau_{\delta,\varepsilon}(g) = \frac{c_\delta}{c_\varepsilon} I_\delta \, \sigma_{\delta,\varepsilon}(g) \, I_\varepsilon^{-1}, \quad \delta,\varepsilon \in M(\sigma), \ g \in G. \tag{4.2}$$

In fact, formula (4.2) can be used for a characterization of equivalence:

THEOREM 4.5. *Let* σ *and* τ *be* K*-unitary,* K*-multiplicity free representations of* G. *Then* $\sigma \simeq \tau$ *iff* $M(\sigma) = M(\tau)$ *and there are nonzero complex numbers* c_δ, $\delta \in M(\sigma)$, *such that* (4.2) *holds.*

PROOF. Assume that $M(\sigma) = M(\tau)$ and that (4.2) holds for certain nonzero c_δ's. Let $A_\delta := c_\delta I_\delta$. Define A on the domain (4.1) by $A(\sum_{\delta\in M(\sigma)} v_\delta) :=$ $:= \sum_{\delta \in M(\sigma)} Av_\delta$. Then A satisfies all the properties which are needed for the equivalence $\sigma \overset{A}{\simeq} \tau$. □

The above theorem would be sufficient in order to check which of the irreducible subquotient representations of the principal series of SU(1,1) are equivalent to each other. However, for irreducible representations σ and τ we need not compare all generalized matrix elements $\sigma_{\delta,\varepsilon}$ and $\tau_{\delta,\varepsilon}$ to each other but just one, as will be shown in the next theorem.

First we mention two useful properties of the generalized matrix elements $\tau_{\delta,\varepsilon}(g)$ of a K-unitary, K-multiplicity free representation of G. Let $\delta,\varepsilon \in M(\tau)$. Then

$$\tau_{\delta,\varepsilon}(k_1 g k_2) = \delta(k_1)\tau_{\delta,\varepsilon}(g)\varepsilon(k_2), \quad g \in G, \ k_1,k_2 \in K, \tag{4.3}$$

$$\tau_{\delta,\varepsilon}(g_1 k g_2) = \sum_{\gamma\in M(\tau)} \tau_{\delta,\gamma}(g_1)\gamma(k)\tau_{\gamma,\varepsilon}(g_2), \quad g_1,g_2 \in G, \ k \in K, \tag{4.4}$$

where the right hand side is an absolutely convergent series in $L(H_\varepsilon(\tau), H_\delta(\tau))$, uniformly for $k \in K$.

THEOREM 4.6. *Let* σ *and* τ *be irreducible,* K*-unitary,* K*-multiplicity free representations. Then* $\sigma \simeq \tau$ *iff there are* $\delta,\varepsilon \in M(\sigma) \cap M(\tau)$ *and a nonzero complex constant* c *such that*

$$\tau_{\delta,\varepsilon}(g) = c \, I_\delta \sigma_{\delta,\varepsilon}(g) I_\varepsilon^{-1}, \quad g \in G. \tag{4.5}$$

PROOF. It follows from (4.5) and (4.4) that

$$(4.6)\qquad \sum_{\gamma\in M(\tau)} \tau_{\delta,\gamma}(g_1)\gamma(k)\tau_{\gamma,\varepsilon}(g_2) = $$

$$= c \sum_{\gamma\in M(\sigma)} I_\delta\sigma_{\delta,\varepsilon}(g_1)\gamma(k)\sigma_{\gamma,\varepsilon}(g_2)I_\varepsilon^{-1},$$

$g_1, g_2 \in G$, $k \in K$. Both sides are absolutely and uniformly convergent Fourier series in $k \in K$. Hence for all $\gamma \in M(\sigma) \cap M(\tau)$ we have

$$(4.7)\qquad \tau_{\delta,\gamma}(g_1)\gamma(k)\tau_{\gamma,\varepsilon}(g_2) = c\, I_\delta\sigma_{\delta,\gamma}(g_1)\gamma(k)\sigma_{\gamma,\varepsilon}(g_2)I_\varepsilon^{-1},$$

$$g_1, g_2 \in G,\ k \in K.$$

By the irreducibility of τ, the functions $\tau_{\delta,\varepsilon}$ and $\tau_{\delta,\gamma}$ are nonzero if $\gamma \in M(\tau)$ and also $\tau_{\delta,\gamma}(g_1)\gamma(k)\tau_{\gamma,\varepsilon}(g_2)$ is not identically zero if $\gamma \in M(\tau)$. A similar statement is valid for σ. By comparing the corresponding terms in (4.6) we conclude that $M(\sigma) = M(\tau)$.

Next we want to prove from (4.7) that $\tau_{\delta,\gamma}(g_1)$ and $\tau_{\gamma,\varepsilon}(g_2)$ are constant multiples of $I_\delta\sigma_{\delta,\gamma}(g_1)I_\gamma^{-1}$ and $I_\gamma\sigma_{\gamma,\varepsilon}(g_2)I_\varepsilon^{-1}$, respectively. Consider (4.7) with g_2 fixed such that $\tau_{\gamma,\varepsilon}(g_2) \neq 0$. Then $\mathrm{tr}(\tau_{\gamma,\varepsilon}(g_2)\tau_{\gamma,\varepsilon}(g_2)^*) \neq 0$. Now multiply to the right both sides of (4.7) with $\tau_{\gamma,\varepsilon}(g_2)^*\gamma(k^{-1})$ and integrate with respect to k. Then

$$\tau_{\delta,\gamma}(g_1)\ \mathrm{tr}(\tau_{\gamma,\varepsilon}(g_2)\tau_{\gamma,\varepsilon}(g_2)^*) =$$

$$= d_\gamma \tau_{\delta,\gamma}(g_1) \int_K \gamma(k)\tau_{\gamma,\varepsilon}(g_2)\tau_{\gamma,\varepsilon}(g_2)^*\gamma(k^{-1})\, dk =$$

$$= c d_\gamma I_\delta\ \sigma_{\delta,\gamma}(g_1) \left(\int_K \gamma(k)\sigma_{\gamma,\varepsilon}(g_2)I_\varepsilon^{-1}\tau_{\gamma,\varepsilon}(g_2)^*\ I_\gamma\gamma(k^{-1})\, dk\right) I_\gamma^{-1} =$$

$$= cI_\delta\sigma_{\delta,\gamma}(g_1)\mathrm{tr}(\sigma_{\gamma,\varepsilon}(g_2)I_\varepsilon^{-1}\tau_{\gamma,\varepsilon}(g_2)^*\ I_\gamma)I_\gamma^{-1}.$$

Thus $\tau_{\delta,\gamma}(g_1) = \text{const.}\ I_\delta\sigma_{\delta,\gamma}(g_1)I_\gamma^{-1}$.

This constant is nonzero, since otherwise the right hand side of (4.7) is identically zero, which cannot be the case. By a similar argument we show that

$$\tau_{\gamma,\varepsilon}(g_2) = \text{const.}\ I_\gamma \sigma_{\gamma,\varepsilon}(g_2) I_\varepsilon^{-1}$$

for some nonzero constant.
Now we repeat the whole reasoning starting from (4.5) with δ replaced by any $\gamma \in M(\sigma)$. We obtain that for all $\delta,\varepsilon \in M(\sigma)$

$$\tau_{\delta,\varepsilon}(g) = c_{\delta,\varepsilon}\ I_\delta \sigma_{\delta,\varepsilon}(g) I_\varepsilon^{-1},\ g \in G,$$

with $c_{\delta,\varepsilon} \neq 0$. Then it follows from (4.7), with c replaced by $c_{\delta,\varepsilon}$, that

$$c_{\delta,\gamma}\ c_{\gamma,\varepsilon} = c_{\delta,\varepsilon}, \qquad \gamma,\delta,\varepsilon \in M(\sigma).$$

It follows from (4.5) with $\delta = \varepsilon$, $g = e$, that $c_{\delta,\delta} = 1$. Hence $(c_{\gamma,\varepsilon})^{-1} = c_{\varepsilon,\gamma}$ and $c_{\delta,\varepsilon} = \dfrac{c_{\delta,\gamma}}{c_{\varepsilon,\gamma}}$. Now apply Theorem 4.5. □

4.3. The case SU(1,1)

Consider irreducible subquotient representations of $\pi_{\eta,\lambda}$ as classified in Theorem 3.4. In view of (2.4) a subquotient representation σ of $\pi_{1,\lambda}$ has a K-content disjoint from the K-content of any subquotient representation τ of $\pi_{-1,\lambda}$. Hence, such σ and τ cannot be equivalent. From Theorem 4.6 we obtain:

COROLLARY 4.7. *Let σ and τ be irreducible subquotient representations of $\pi_{\eta,\lambda}$ and $\pi_{\eta,\mu}$, respectively ($\eta \in \{1,-1\},\lambda,\mu \in \mathbb{C}$). Then $\sigma \simeq \tau$ iff for some $m \in \mathbb{Z}$ we have $\phi_m \in H(\sigma) \cap H(\tau)$ and*

$$\pi_{\eta,\lambda,m,m}(a_t) = \pi_{\eta,\mu,m,m}(a_t),\ t \in \mathbb{R}. \tag{4.8}$$

In view of (2.15) and (2.16), condition (4.8) can be rewritten as

$$(4.9)\qquad \phi_{i\lambda}^{(0,2m+\frac{1}{2}(1-\eta))}(t) = \phi_{i\mu}^{(0,2m+\frac{1}{2}(1-\eta))}(t),\ t \in \mathbb{R}.$$

Formula (4.9) clearly holds if $\lambda = \pm\mu$. Conversely suppose that (4.9) holds. Substitute (2.13) and consider the coefficients of $-(\sinh t)^2$ in the hypergeometric series:

$$\tfrac{1}{4}((2m+\tfrac{1}{2}(1-\eta)+1)^2-\lambda^2) = \tfrac{1}{4}((2m+\tfrac{1}{2}(1-\eta)+1)^2-\mu^2).$$

This yields $\lambda = \pm\mu$. We can formulate our results as follows (cf. the classification in Theorem 3.4).

THEOREM 4.8. *Let σ and τ be distinct irreducible subquotient representations of $\pi_{\eta_1,\lambda}$, $\pi_{\eta_2,\mu}$, respectively. Then $\sigma \simeq \tau$ iff $\eta := \eta_1 = \eta_2$, $\lambda = -\mu$ and $H(\sigma) \cap H(\tau) \neq \{0\}$. In case of equivalence, $H(\sigma) = H(\tau)$ and $\sigma \overset{A}{\simeq} \tau$ with*

$$(4.10)\qquad A\phi_m = \frac{c_{\eta,-\lambda,m,n}}{c_{\eta,\lambda,m,n}}\,\phi_m, \quad \phi_m \in H(\sigma),$$

where $n \in \mathbb{Z}$ is fixed but can be arbitrarily chosen such that $\phi_n \in H(\sigma)$.

Formula (4.10) can be obtained from the observation that

$$\pi_{\eta,-\lambda,m,n}(a_t) = \frac{c_{\eta,-\lambda,m,n}}{c_{\eta,\lambda,m,n}}\,\pi_{\eta,\lambda,m,n}(a_t)$$

and from the proofs of Theorems 4.5 and 4.6.

By Theorem 4.8 all irreducible subquotient representations of $\pi_{\eta,\lambda}$'s (occurring in (b), (c), (f), (g) in Theorem 3.4) are equivalent to irreducible subrepresentations of $\pi_{\eta,\lambda}$'s.

5. EQUIVALENCE OF IRREDUCIBLE REPRESENTATIONS OF SU(1,1) TO SUBREPRESENTATIONS OF THE PRINCIPAL SERIES

In this section we discuss part (iii) of the program formulated in the introduction. We will spend quite a lot of space on general results about spherical functions of type δ. Most of these results are already contained in GODEMENT [9], see also WARNER [14, Vol. II, § 6.1].

The first step is to show that each irreducible K-unitary, K-finite representation of SU(1,1) is K-multiplicity free. In VAN DIJK [3, § 9] this was proved for irreducible unitary representations of SU(1,1). The methods used there also apply to the K-unitary case. Here we follow a slightly different approach, using Gelfand pairs. Essentially we relate Godement's spherical functions of type δ to spherical functions of type 1 on a bigger group. The idea seems to go back to unpublished work of Kostant (personal communication to the author by M. Flensted-Jensen).

5.1. The connection between representations of G and spherical representations of G × K

Let G be a lcsc.group with compact subgroup K. Let $K^* := \{(k,k) \in G \times K \mid k \in K\}$. Then K^* is a compact subgroup of $G \times K$, isomorphic to K. If $\delta \in \hat{K}$ then let δ^* denote the representation of K which is contragredient to δ. If $\delta \in \hat{K}$ and π is a K-unitary representation of G then $\pi \otimes \delta^*$ is a K^*-unitary representation of $G \times K$ on $H(\pi) \otimes H(\delta^*)$.

THEOREM 5.1. *Let* G, K, π *and* δ *be as above. The multiplicity of* δ *in* $\pi|_K$ *is equal to the multiplicity of the representation* 1 *of* K^* *in* $\pi * \delta^*|_{K^*}$. π *is irreducible iff* $\pi \otimes \delta^*$ *is irreducible.* π *is unitary iff* $\pi \otimes \delta^*$ *is unitary.*

PROOF. Let $\pi|_K = \sum^{\oplus}_{\varepsilon\in\hat{K}} n_\varepsilon\, \varepsilon$ $(n_\varepsilon \in \{0,1,2,\ldots,\infty\})$. Then $\pi \otimes \delta^*|_{K\times K} = \sum^{\oplus}_{\varepsilon\in\hat{K}} n_\varepsilon\, \varepsilon \otimes \delta^*$. The representation 1 of K^* has multiplicity 0 in $\varepsilon \otimes \delta^*$ for $\varepsilon \neq \delta$ and 1 for $\varepsilon = \delta$. This proves the first statement. Now suppose that π is irreducible. We will show that $\pi \otimes \delta^*$ is irreducible. Choose an orthonormal basis $e_1,\ldots,e_{d_\delta}$ for $H(\delta^*)$. Pick a nonzero element v of $H(\pi) \otimes H(\delta^*)$. Then $v = \sum_{i=1}^{d_\delta} v_i \otimes e_i$, $v_i \in H(\pi)$, $v_j \neq 0$ for some j. By irreducibility of δ^*, the projection from $H(\delta^*)$ on $\{\lambda e_j\}$ is a linear combination of operators $\delta^*(k)$, $k \in K$. Hence $v_j \otimes e_j$ is in the $\pi \otimes \delta^*$ - invariant closed subspace generated by v. Thus $H(\pi) \otimes e_j$ and $H(\pi) \otimes H(\delta^*)$ are in this subspace, so $\pi \otimes \delta^*$ is irreducible. The other statements are evident. □

5.2. Gelfand pairs

Let G be a unimodular lcsc. group. Let K be a compact subgroup of G. Let $\mathcal{K}(K\backslash G/K)$ be the space of all continuous complex-valued functions on G with compact support which are bi-invariant with respect to K. This space becomes an associative algebra with respect to the convolution product

$$(5.1)\qquad (f_1 * f_2)(x) = \int_G f_1(y)f_2(y^{-1}x)dy$$

DEFINITION 5.2. The pair (G,K) is called a *Gelfand pair* if $\mathcal{K}(K\backslash G/K)$ is a commutative algebra.

THEOREM 5.3. *Let* G *be a unimodular lcsc. group with compact subgroup* K. *Suppose that there is a continuous involutive automorphism* α *on* G *such that* $\alpha(KxK) = \alpha(Kx^{-1}K)$ *for all* $x \in G$. *Then* (G,K) *is a Gelfand pair.*

PROOF. For $f \in \mathcal{K}(K\backslash G/K)$ we have $f(\alpha(x)) = f(x^{-1})$, $x \in G$. Also $d\alpha(x) = dx$, since the automorphism α is involutive. Let $f_1, f_2 \in \mathcal{K}(K\backslash G/K)$. Then

$$(f_1*f_2)(x) = (f_1*f_2)(\alpha(x^{-1})) = \int_G f_1(y)f_2(y^{-1}\alpha(x^{-1}))dy =$$

$$= \int_G f_1((\alpha(y))^{-1})f_2(x\alpha(y))dy = \int_G f_1(y^{-1})f_2(xy)dy =$$

$$= \int_G f_1(y^{-1}x)f_2(y)dy = (f_2*f_1)(x).\ \square$$

THEOREM 5.4. *Let* (G,K) *be a Gelfand pair. Let* π *be a* K*-unitary irreducible representation of* G *and let, in addition,* π *be unitary or* K*-finite. Then the representation* 1 *of* K *has multiplicity* 0 *or* 1 *in* $\pi|_K$.

PROOF. Suppose that $\mathcal{H}_1(\pi)$ has nonzero dimension. The formula

$$(5.2)\qquad f^{\#}(x) := \int_{K_1}\int_{K_2} f(k_1xk_2)dk_1\, dk_2$$

defines a projection from $\mathcal{K}(G)$ (the space of continuous functions on G with compact support) onto $K(K\backslash G/K)$. Let P be the orthogonal projection from $H(\pi)$ onto $H_1(\pi)$:

$$(5.3) \qquad Pv := \int_K \pi(k)v \, dk, \qquad v \in H(\pi).$$

For $f \in K(G)$ define

$$(5.4) \qquad \pi(f)v := \int_G f(x)\pi(x)v \, dx, \qquad v \in H(\pi).$$

Then π is a homomorphism from the algebra $K(G)$ into the algebra $L(H(\pi))$. Since π is an irreducible representation of G, the family of operators $\pi(K(G))$ also acts irreducibly on $H(\pi)$. It follows from (5.2) and (5.3) that that

$$\pi(f^{\#})v = P\pi(f)Pv, \qquad f \in K(G),\ v \in H(\pi).$$

If $v \in H_1(\pi)$ then

$$\pi(f^{\#})v = P\pi(f)v, \qquad f \in K(G).$$

So $\pi(K(K\backslash G/K))v = P\pi(K(G))v$, $v \in H_1(\pi)$. Let $v \in H_1(\pi)$, $v \neq 0$. By the irreducibility of $\pi(K(G))$, $\pi(K(G))v$ is dense in $H(\pi)$, so $P\pi(K(G))v$ is dense in $H_1(\pi)$. We conclude that the algebra $\pi(K(K\backslash G/K))$ acts irreducibly on $H_1(\pi)$. Now $\pi(K(K\backslash G/K))$ is a commutative algebra. Therefore, if $\dim H_1(\pi) < \infty$, the finite-dimensional version of Schur's lemma yields that $\pi(f)|_{H_1(\pi)}$ is a multiple of the identity for each $f \in K(K\backslash G/K)$. Then $\dim H_1(\pi) = 1$. On the other hand, if π is unitary and $\dim H_1(\pi) = \infty$ is admitted, then $\pi(K(K\backslash G/K))$ is a commutative $*$-algebra and the generalization of Schur's lemma again yields the same conclusion. □

5.3. $(G\times K, K^*)$ is a Gelfand pair if $G = SU(2)$

Let G be a unimodular lcsc. group with compact subgroup K. Use the notation of § 5.1. We conclude from Theorems 5.1, 5.3 and 5.4:

COROLLARY 5.5. *Suppose that there exists a continuous involutive automorphism* α *on* G *such that for each* $(g,k) \in G \times K$ *there exist* $k_1, k_2 \in K$ *with the property that* $\alpha(g) = k_1 g^{-1} k_2$, $\alpha(k) = k_1 k^{-1} k_2$.
Then:

(a) $(G \times K, K^*)$ *is a Gelfand-pair.*

(b) *If* π *is an irreducible* K*-unitary representation of* G *which is* K*-finite or unitary then* π *is* K*-multiplicity free.*

THEOREM 5.6. *The conclusions of Corollary* 5.5 *apply to the case* $G = SU(1,1)$.

PROOF. For $g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1,1)$ define

$$\alpha(g) := (g^{-1})^t = \begin{pmatrix} \bar{a} & -\bar{b} \\ -b & a \end{pmatrix}.$$

Then α is a continuous involutive automorphism on G and $\alpha(a_t) = a_{-t}$ on A, $\alpha(u_\Theta) = u_{-\Theta}$ on K. By using (2.1) we conclude that α satisfies the property required in Corollary 5.5. □

5.4. Spherical functions

Let G be a unimodular lcsc. group with compact subgroup K. Let π be a K-unitary representation of G such that $\dim H_1(\pi) = 1$. Choose $v \in H_1(\pi)$ such that $\|v\| = 1$ and define

$$\phi(x) := (\pi(x)v,v), \quad x \in G. \tag{5.5}$$

Then ϕ is called a *spherical function* on G (with respect to K and corresponding to the representation π). ϕ is continuous on G and biinvariant with respect to K and $\phi(e) = 1$. Furthermore, ϕ satisfies the celebrated product formula

$$\phi(x)\phi(y) = \int_K \phi(xky)\, dk, \quad x,y \in G. \tag{5.6}$$

Indeed,

$$\int_K \phi(xky)\,dk = \int_K (\pi(xky)v,v)\,dk = (\int_K \pi(ky)v\,dk,\ \pi(x)^*v) =$$

$$= (P_{\pi,1}\pi(y)v,\ \pi(x)^*v) = (c(y)v,\ \pi(x)^*v) =$$

$$= c(y)(\pi(x)v,\ v) = c(y)\phi(x)$$

for some constant c(y) depending on y. Substitution of x = e in the identity

$$\int_K \phi(xky)\,dk = c(y)\phi(x)$$

yields $\phi(y) = c(y)$.

THEOREM 5.6. *Let G be a unimodular Lie group with compact subgroup K. Let ϕ be a spherical function on G and let D be a differential operator on G which is invariant under left multiplication by elements of G and right multiplication by elements of K. Then ϕ is a C^∞-function and an eigenfunction of D.*

PROOF. Choose $f \in C_c^\infty(K\backslash G/K)$ (C^∞-function on G with compact support, biinvariant with respect to K) such that

$$\int_G \phi(y)f(y)\,dy \neq 0. \text{ (This is always possible.)}$$

Then application of (5.6) yields:

$$\left(\int_G \phi(y)f(y)\,dy\right)\phi(x) = \int_K\left(\int_G \phi(xky)f(y)dy\right)dk =$$

$$= \int_K\left(\int_G \phi(xy)f(k^{-1}y)dy\right)dk = \int_K\left(\int_G \phi(xy)f(y)dy\right)dk = \int_G \phi(y)f(x^{-1}y)dy.$$

Thus ϕ is C^∞. Again using (5.6) we have

$$\phi(x)(D\phi)(y) = \int_K D_y\phi(xky)dk = \int_K (D\phi)(xky)dk.$$

For y = e this becomes

$$\phi(x)(D\phi)(e) = \int_K (D\phi)(xk)dk = \int_K D_x\phi(xk)dk = \int_K D_x\phi(x)dk = (D_\phi)(x).$$

So $(D\phi)(x) = c\ \phi(x)$ with $c = (D\phi)(e)$. □

5.5. Spherical functions of type δ

Let G be a unimodular lcsc. group with compact subgroup K. Let π be a K-unitary representation of G and let $\delta \in \hat{K}$ such that $\dim H_\delta(\pi) = d_\delta$. Then the spherical function on $G \times K$ with respect to K^* corresponding to the representation $\pi \otimes \delta^*$ is well-defined and it has the form $(g,k) \to \phi(gk^{-1})$, where ϕ is its restriction to $G \times \{e\}$. The function ϕ can be expressed in terms of the genralized matrix element $\pi_{\delta,\delta}$:

(5.7) $$\phi(g) = d_\delta^{-1}\operatorname{tr}\pi_{\delta,\delta}(g).$$

Indeed, let $e_1,\ldots,e_{d_\delta}$ be an orthonormal basis for $H_\delta(\pi)$ and let $f_1,\ldots f_{d_\delta}$ be a dual basis for $H(\delta^*)$. Then $d_\delta^{-\frac{1}{2}} \sum_{i=1}^{d_\delta} e_i \otimes f_i$ is a normalized K^*-invariant vector in $H(\pi) \otimes H(\delta^*)$ and

$$\phi(g) = d_\delta^{-1}\Big(\sum_{i=1}^{d_\delta} \pi(g)e_i \otimes f_i, \sum_{j=1}^{d_\delta} e_j \otimes f_j\Big) =$$

$$= d_\delta^{-1} \sum_{i,j=1}^{d_\delta} (\pi(g)e_i,e_j)(f_i,f_j) =$$

$$= d_\delta^{-1} \sum_{i=1}^{d_\delta}(\pi(g)e_i,e_i) = d_\delta^{-1}\operatorname{tr}\pi_{\delta,\delta}(g).$$

Such functions ϕ are called *spherical functions of type* δ on G (cf. GODEMENT [9]). It follows from (5.6) that they satisfy the product formula

(5.8) $$\phi(x)\phi(y) = \int_K \Phi(xkyk^{-1})dk,\quad x,y \in G.$$

A very special example is given by the case that G is compact and $K = G$. Then $(G \times G, G^*)$ is a Gelfand pair, the above functions ϕ are the characters of G (up to the factor d_δ^{-1}) and (5.8) is the well-known product formula for characters.

PROPOSITION 5.7. *Let* G *be a unimodular Lie group with compact subgroup* K. Let π *be a* K*-unitary representation of* G *and let* $\delta \in \hat{K}$ *such that* $\dim H_\delta(\pi) = d_\delta$. *Let* D *be a differential operator on* G *which is left invariant with respect to* G *and right invariant with respect to* K. *Then* $g \to \pi_{\delta,\delta}(g)$ *is* C^∞ *and* $D\pi_{\delta,\delta} = c\pi_{\delta,\delta}$ *for some constant* c.

PROOF. It follows from Theorem (5.6) that $g \to \mathrm{tr}\pi_{\delta,\delta}(g)$ is C^∞ and an eigenfunction of D. Let $e_1,\dots e_{d_\delta}$ be an orthonormal basis of $H_\delta(\pi)$. Then

$$\mathrm{tr}\pi_{\delta,\delta}(gk) = \sum_{i,j=1}^{d_\delta} (\pi_{\delta,\delta}(g))_{i,j}(\delta(k))_{j,i}.$$

Thus

$$(\pi_{\delta,\delta}(g))_{i,j} = d_\delta \int_K \mathrm{tr}\pi_{\delta,\delta}(gk)\, \overline{(\delta(k))_{j,i}}\, dk$$

and the proposition follows. □

THEOREM 5.8. *Let* G *be a unimodular Lie group with compact subgroup* K. *Let* π *be an irreducible,* K*-unitary,* K*-multiplicity free representation of* G. *Let* D *be a differential operator on* G *which is biinvariant with respect to* G. *Then for all* $\delta,\varepsilon \in M(\pi)$ $g \to \pi_{\delta,\varepsilon}(g)$ *is* C^∞ *and* $D\pi_{\delta,\varepsilon} = c\pi_{\delta,\varepsilon}$ *with the constant* c *independent of* δ *and* ε.

PROOF. Let $\gamma,\delta,\varepsilon \in M(\pi)$. It follows from (4.4) that

$$d_\gamma \int_K \gamma(k^{-1})\pi_{\delta,\gamma}(g_1)^* \pi_{\delta,\varepsilon}(g_1 k g_2)\, dk =$$

$$= \mathrm{tr}(\pi_{\delta,\gamma}(g_1)^* \pi_{\delta,\gamma}(g_1))\, \pi_{\gamma,\varepsilon}(g_2)$$

and

$$d_\gamma \int_K \pi_{\delta,\varepsilon}(g_1 k g_2)\pi_{\gamma,\varepsilon}(g_2)^* \gamma(k^{-1})\, dk =$$

$$= \pi_{\delta,\gamma}(g_1)\ \mathrm{tr}(\pi_{\gamma,\varepsilon}(g_2)\pi_{\gamma,\varepsilon}(g_2)^*).$$

Using the fact that $\pi_{\delta,\gamma}$ and $\pi_{\gamma,\varepsilon}$ are not identically zero because of the irreducibility of π, we conclude that, if $\pi_{\delta,\varepsilon}$ is C^∞ and $D\pi_{\delta,\varepsilon} = c\pi_{\delta,\varepsilon}$, then the same holds for $\pi_{\delta,\gamma}$ and $\pi_{\gamma,\varepsilon}$. Now use Prop. 5.7. □

REMARK 5.9. Let G, K and π be as in Theorem 5.8. Let D be a differential operator on G which is left invariant with respect to G and right invariant with respect to K. Then it follows from the proof of Theorem 5.8 that for each $\varepsilon \in M(\pi)$ there is a constant $c(\varepsilon)$ such that $D\pi_{\delta,\varepsilon} = c(\varepsilon)\pi_{\delta,\varepsilon}$ for each $\delta \in M(\pi)$. Let $G \times K$ act on G by $(g,k)g_1 := gg_1k^{-1}$. Then $G = G \times K / K^*$. Since K^* is compact, there is an analytic $G \times K$ - invariant Riemannian metric on G. The corresponding Laplace-Beltrami operator Δ on G is an elliptic differential operator with analytic coefficients on G which is left invariant with respect to G and right invariant with respect to K, and $\Delta\pi_{\delta,\varepsilon} = c(\varepsilon)\pi_{\delta,\varepsilon}$ for all $\delta,\varepsilon \in M(\pi)$. By a theorem of S. Bernstein (cf. JOHN [10, p. 57]) the eigenfunctions of an elliptic differential operator with analytic coefficients are analytic. Consequently, all functions $g \to \pi_{\delta,\varepsilon}(g)$ are analytic. This, in its turn, implies that the vectors belonging to $H_\delta(\pi)$, $\delta \in M(\pi)$, are analytic. (This last result is also contained in VAN DIJK [3, Prop. 10.3], where the proof used Nelson's theorem.)

We will apply Proposition 5.7 to the case of an irreducible K-unitary, K-multiplicity free representation τ of SU(1,1). Let $\delta \in M(\pi)$. Then $\tau_{\delta,\delta}$ is a scalar function on SU(1,1) which is an eigenfunction of, for instance, the Casimir operator on SU(1,1).

5.6. The Casimir operator on SU(1,1)

This is a second order differential operator on SU(1,1), biinvariant with respect to G, which can be given in terms of a basis for the Lie algebra of SU(1,1). Since we want to write the Casimir operator in terms of the variables Θ_1, t, Θ_2 corresponding to the decomposition (2.1), we prefer a different approach, where we consider a biinvariant pseudo-Riemannian metric on SU(1,1) and calculate the corresponding Laplace-Beltrami operator.

Consider the imbedding

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \to (\alpha,\beta) \quad \text{of } SU(1,1) \text{ in } \mathbb{C}^2.$$

Then SU(1,1) is identified with the hyperboloid $\{(\alpha,\beta) \in \mathbb{C}^2 \mid |\alpha|^2 - |\beta|^2 = 1\}$. In this picture left and right multiplications on SU(1,1) are linear transformations of $\mathbb{C}^2$ which leave the quadratic form $|\alpha|^2 - |\beta|^2$ invariant and hence also leave invariant the pseudo-Riemannian metric

$$(ds)^2 = |d\beta|^2 - |d\alpha|^2$$

on $\mathbb{C}^2$. Let us restrict $(ds)^2$ to the hyperboloid. We use coordinates t,ψ:

$$\alpha = \cosh t\; e^{i\psi_1}, \qquad \beta = \sinh t\; e^{i\psi_2}. \tag{5.9}$$

Then

$$(ds)^2 = (dt)^2 - (\cosh t)^2(d\psi_1) + (\sinh t)^2(d\psi_2)^2. \tag{5.10}$$

Remember that the Laplace-Beltrami operator corresponding to a pseudo-Riemannian metric

$$(ds)^2 = \sum_{i,j} g_{ij}(x)\, dx^i dx^j \quad \text{(local coordinates)} \tag{5.11}$$

is given by

$$\Delta = |g(x)|^{-\frac{1}{2}} \sum_{i,j} \frac{\partial}{\partial x^i} \circ |g(x)|^{\frac{1}{2}}\, g^{ij}(x) \frac{\partial}{\partial x^j}\,, \tag{5.12}$$

where $g(x) := \det(g_{ij}(x))$ and

$(g^{ij}(x)) := (g_{ij}(x))^{-1}$ (matrix inversion).

Applying this to (5.10) we obtain

$$(5.13) \qquad \Delta = \frac{\partial^2}{\partial t^2} + 2\mathrm{cotgh}\ 2t \frac{\partial}{\partial t} - (\cosh t)^{-2} \frac{\partial^2}{\partial \psi_1^2} + (\sinh t)^{-2} \frac{\partial^2}{\partial \psi_2^2} .$$

5.7. Each irreducible K-unitary, K-finite representation of SU(1,1) is equivalent to a subrepresentation of the principal series

Let τ be an irreducible, K-unitary, K-multiplicity free representation of $G = SU(1,1)$ and choose $n \in \mathbb{Z}$ such that $\delta : u_\theta \to e^{in\theta}$ is in $M(\tau)$. Consider the function $\tau_{\delta,\delta}$ on G, expressed in terms of the coordinates t, ψ_1, ψ_2 (cf. (5.9)):

$$(5.14) \qquad \tau_{\delta,\delta}\left(\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}\right) = \tau_{\delta,\delta}(u_{\frac{1}{2}(\psi_1+\psi_2)}a_t u_{\frac{1}{2}(\psi_1-\psi_2)}) = e^{in\psi_1}\tau_{\delta,\delta}(a_t).$$

Then $\tau_{\delta,\delta}$ is an eigenfunction of Δ with eigenvalue, say, λ^2-1 ($\lambda \in \mathbb{C}$). Define the function f on $\mathbb{R}$ by

$$(5.15) \qquad \tau_{\delta,\delta}(a_t) = (\cosh t)^n f(t).$$

f is an even C^∞-function and $f(0) = 1$. It follows from (5.13), (5.14) and (5.15) that

$$\frac{d^2}{dt^2} + (\mathrm{cotgh}\ t + (2n+1)\ \mathrm{tgh}\ t) \frac{d}{dt} - \lambda^2+(n+1)^2)f(t) = 0.$$

Comparing with (2.14), (2.15) and (2.6) we find

$$f(t) = \phi_{i\lambda}^{(0,n)}(t),$$

$$\tau_{\delta,\delta}(g) = \pi_{\eta,\lambda,m,n}(g),\ g \in G,$$

where m $[\frac{1}{2}n]$, $\eta = (-1)^n$, so $n = 2m+\frac{1}{2}(1-\eta)$ and $\pi_{\eta,\lambda}(u_\theta)\phi_m = \delta(u_\theta)\phi_m$, cf. (2.4).

We conclude from Theorem 4.6 that τ is equivalent to the irreducible subquotient representation of $\pi_{\eta,\lambda}$ which contains ϕ_m. By combining this result with Theorem 5.6 and the concluding remark in § 4.3 we obtain:

THEOREM 5.10. *Let* τ *be an irreducible* K-*unitary representation of* SU(1,1) *which is* K-*finite or unitary. Then* τ *is equivalent to an irreducible subrepresentation of* $\pi_{\eta,\lambda}$ *for some* η,λ.

6. UNITARIZABILITY OF IRREDUCIBLE SUBREPRESENTATIONS OF THE PRINCIPAL SERIES

In this section we deal with the last part of our program formulated in the introduction.

6.1. The conjugate contragredient to a representation of G

Let G be a lcsc. group and let τ be a strongly continuous representation of G on a separable Hilbert space $H(\tau)$. Then τ is also weakly continuous. Conversely, we will show that weak continuity of τ implies strong continuity.

Assume that τ is a weakly continuous Hilbert representation of G. A twofold application of the Banach-Steinhaus theorem shows that τ is locally bounded, that is, $\sup_{g\in K}\|\tau(g)\| < \infty$ for compact subsets K of G. For each $f \in K(G)$ we define

$$\tau(f) := \int_G f(g)\tau(g)dg, \tag{6.1}$$

where dg is a left Haar measure and the operator-valued integral is considered in the weak sense. Let $\{V_n\}$ a decreasing sequence of open neighbourhoods of e in G such that $\{V_n\}$ is a base for the neighbourhoods of e. Choose a sequence $\{f_n\}$ in $K(G)$ such that $f_n \geq 0$, $\mathrm{supp}(f_n) \subset V_n$ and $\int_G f_n(g)\, dg = 1$. Then $(\tau(f_n)v,w) \to (v,w)$ for all $v,w \in H(\tau)$. We conclude that the linear span of $\{\tau(f)v \mid f \in K(G),\ v \in H(\tau)\}$ is weakly dense in $H(\tau)$. Also observe that

(6.2) $\tau(x)\tau(f) = \tau(\lambda(x)f)$, $f \in K(\tau)$, $x \in G$,

where $(\lambda(x)f)(g) := f(x^{-1}g)$.

THEOREM 6.1 (cf. WARNER, Vol. I, Prop. 4.2.2.1). *A Hilbert representation of a lcsc. group* G *is strongly continuous iff it is weakly continuous.*

PROOF. Assume that τ is a weakly continuous Hilbert representation of G. Let $H_s(\tau)$ be the linear subspace of $H(\tau)$ consisting of all $v \in H(\tau)$ for which $g \to \tau(g)v$ is continuous from G to $H(\tau)$. It is easily seen that $H_s(\tau)$ is a closed subspace of $H(\tau)$. Furthermore it follows from (6.1) and (6.2) that $\tau(f)v \in H_s(\tau)$ for all $f \in K(G)$, $v \in H(\tau)$. Since weak closure and closure in norm coincide for linear subspaces of $H(\tau)$, the weak closure $H(\tau)$ of the linear span of $\{\tau(f)v \mid f \in K(G), v \in H(\tau)\}$ must be included in $H_s(\tau)$. □

Let τ be a weakly continuous Hilbert representation of G. If τ is a unitary representation then $\tau(g^{-1})^* = \tau(g)$. Otherwise, we can still define

(6.3) $\tilde{\tau}(g) := \tau(g^{-1})^*$, $g \in G$.

$\tilde{\tau}$ is again a weakly continuous representation of G on $H(\tau)$. It is called the *conjugate contragredient* to τ.

COROLLARY 6.2. *If* τ *is a strongly continuous Hilbert representation of* G, *then* $\tilde{\tau}$ *is also strongly continuous.*

6.2. A criterium for unitarizability

Let G be a lcsc. group with compact subgroup K. Let σ and τ be K-unitary, K-finite representations of G and let $\sigma \overset{A}{\simeq} \tau$ (cf. § 4.1). Let $A_\delta := A|_{H_\delta(\sigma)}$, $\delta \in \hat{K}$. Then it is easily seen that A^* is an injective closed linear operator from $H(\tau)$ to $H(\sigma)$ with dense domain and range and such that $A^*|_{H_\delta(\tau)} = A_\delta^*$ maps $H_\delta(\tau)$ onto $H_\delta(\sigma)$. Furthermore, since

$$(\tilde{\tau}(g)v,\ Aw) = (v,\tau(g^{-1})Aw) = (v,A\sigma(g^{-1})w) = (A^*v,\ \sigma(g^{-1})w),$$

$$v \in \mathcal{D}(A^*),\ w \in \mathcal{D}(A),$$

we conclude that $\mathcal{D}(A^*)$ is $\tilde{\tau}$ - invariant and that $A^*\tilde{\tau}(g)v = \tilde{\sigma}(g)A^*v$ if $v \in \mathcal{D}(A^*)$, $g \in G$. Thus we have:

LEMMA 6.3. *If* $\sigma \overset{A}{\cong} \tau$ *then* $\tilde{\tau} \overset{A^*}{\cong} \tilde{\sigma}$.

Now we can prove:

THEOREM 6.4. *Let* G *be a lcsc. group with compact subgroup* K. *Let* τ *be a* K*-unitary,* K*-finite representation of* G. *Then* τ *is equivalent to some unitary representation of* G *iff* $\tau \overset{A}{\cong} \tilde{\tau}$ *with* A *self-adjoint and positive definite.*

PROOF. First suppose that $\tau \overset{B}{\cong} \sigma$ with σ unitary. Then $\sigma = \tilde{\sigma}$ and $\tilde{\sigma} \overset{B^*}{\cong} \tilde{\tau}$ (cf. Lemma 6.3), so $\tau \overset{A}{\cong} \tilde{\tau}$, where A is the closure of B^*B (cf. Remark 4.3). Obviously, A is self-adjoint and positive definite.

Next suppose that $\tau \overset{A}{\cong} \tilde{\tau}$ with A self-adjoint and positive definite. Let $(.\ ,\ .)$ be the inner product on $H(\tau)$. We define a new inner product $\langle\ .\ ,\ .\ \rangle$ on $\mathcal{D}(A)$ by

$$(6.4)\qquad \langle v,w\rangle := (Av,w), \qquad v,w \in \mathcal{D}(A).$$

This is indeed a positive definite sesquilinear form on $\mathcal{D}(A)$. For $v,w \in \mathcal{D}(A)$, $g \in G$ we have:

$$\langle\tau(g)v,\ \tau(g)w\rangle = (A\tau(g)v,\ \tau(g)w) = (\tilde{\tau}(g^{-1})A\tau(g)v,w) =$$

$$= (A\tau(g^{-1})\tau(g)v,w) = (Av,w) = \langle v,w\rangle,$$

i.e.

$$(6.5)\qquad \langle\tau(g)v,\ \tau(g)w\rangle = \langle v,w\rangle.$$

Thus τ is a unitary representation of G on $\mathcal{D}(A)$ with respect to the inner product $<.\ ,\ .>$. (Weak continuity is obvious from (6.4) and the weak continuity of the original representation.) Let σ be the extension of this representation to a unitary representation in the Hilbert space completion $\mathcal{H}(\sigma)$ of $\mathcal{D}(A)$ with respect to $<.\ ,\ .>$. Then $\tau \overset{B}{\simeq} \sigma$, where B is the closure of the identity operator on $\mathcal{D}(A)$ (cf. Remark 4.2). □

Next we restrict ourselves to the case that τ is a K-unitary, K-multiplicity free representation of G. Then the same holds for $\tilde{\tau}$. Furthermore $M(\tilde{\tau}) = M(\tau)$ and

$$(6.6) \qquad \tilde{\tau}_{\delta,\varepsilon}(g) = \tau_{\varepsilon,\delta}(g^{-1})^*, \qquad \delta,\varepsilon \in M(\tau),\ g \in G.$$

It follows from Theorem 4.5 that $\tau \overset{A}{\simeq} \tilde{\tau}$ iff there are nonzero complex numbers c_δ, $\delta \in M(\tau)$, such that

$$\tilde{\tau}_{\delta,\varepsilon}(g) = \frac{c_\delta}{c_\varepsilon}\,\tau_{\delta,\varepsilon}(g), \qquad \delta,\varepsilon \in M(\tau),\ g \in G,$$

and $A_\delta = c_\delta I_\delta$, $\delta \in M(\tau)$, where I_δ is the identity operator on $\mathcal{H}_\delta(\tau)$. Now A is self-adjoint and positive definite iff $c_\delta > 0$ for all $\delta \in M(\tau)$. Thus Theorem 6.4 implies:

THEOREM 6.5. *Let τ be a K-unitary, K-multiplicity free representation of G. Then τ is equivalent to some unitary representation iff there are positive numbers c_δ, $\delta \in M(\tau)$, such that*

$$(6.7) \qquad \tau_{\varepsilon,\delta}(g^{-1})^* = \frac{c_\delta}{c_\varepsilon}\,\tau_{\delta,\varepsilon}(g), \quad \delta,\varepsilon \in M(\tau),\ g \in G.$$

In case of unitarizability of τ, the new inner product (6.4) *becomes*

$$(6.8) \qquad <v,w> = \sum_{\delta\in M(\tau)} c_\delta (v_\delta, w_\delta).$$

REMARK 6.6. In the case that τ is an irreducible K-unitary, K-multiplicity free representation of G, unitarizability of τ is already implied if (6.7) holds for some δ and all $\varepsilon \in M(\tau)$ with $c_\varepsilon > 0$ (cf. the proof of Theorem 4.6).

6.3. The case $G = SU(1,1)$

It follows either from (2.2) or from (6.6), (2,15) and (2.16) that

(6.9) $$\tilde{\pi}_{\eta,\lambda} = \pi_{\eta,-\bar{\lambda}}, \qquad \eta \in \{-1,1\},\ \lambda \in \mathbb{C}.$$

In case of the latter proof observe that, essentially, (6.9) is equivalent to the identity

(6.10) $$\overline{c_{\eta,\lambda,n,m}} = (-1)^{m-n} c_{\eta,-\bar{\lambda},m,n}.$$

In § 6.2 we showed that a necessary condition for unitarizability of an irreducible subquotient representation τ of $\pi_{\eta,\lambda}$ is the equivalence of τ and $\tilde{\tau}$. In view of (6.9) and Theorem 4.8 this is only possible if $\bar{\lambda} = \pm\lambda$, that is, if λ is real or imaginary. If λ is imaginary then $\tilde{\pi}_{\eta,\lambda} = \pi_{\eta,\lambda}$, so $\pi_{\eta,\lambda}$ is already unitary. Let us now examine the case that λ is real and nonzero. Then $\tilde{\pi}_{\eta,\lambda} = \pi_{\eta,-\lambda}$. If τ is an irreducible subquotient representation of $\pi_{\eta,\lambda}$ then $\tau \overset{A}{\simeq} \tilde{\tau}$ with (cf. (4.10))

(6.11) $$A_{\phi_m} = \frac{c_{\eta,-\lambda,m,n}}{c_{\eta,\lambda,m,n}}\, \phi_m, \quad \phi_m \in H(\tau),$$

where $\phi_n \in H(\tau)$ is fixed. Now a sufficient condition for the unitarizability of τ is the positivity of the coefficients

$$c_{\eta,-\lambda,m,n}/c_{\eta,\lambda,m,n}, \quad \phi_m \in H(\tau),$$

where $\phi_n \in H(\tau)$ is fixed. Referring to the classification in Theorem 3.4 we will calculate these coefficients. (Because of equivalence, it is not necessary to treat the cases where $\lambda < 0$.)

(a) $\eta = 1,\ \lambda > 0,\ \lambda \neq$ odd integer.

$$\frac{c_{1,-\lambda,m,0}}{c_{1,\lambda,m,0}} = \frac{(\frac{1}{2}(-\lambda+1))_{|m|}}{(\frac{1}{2}(\lambda+1))_{|m|}}, \qquad m \in \mathbb{Z}.$$

If $0 < \lambda < 1$ then the coefficients are positive. If $\lambda > 1$, then the coefficients change sign (consider $m = 1$).

(b) $\eta = 1$, $\lambda = 2k+1$ for an integer $k \geq 0$.

$$\frac{c_{1,-2k-1,m,-k-1}}{c_{1,2k+1,m,-k-1}} = \frac{(-m-k-1)!}{(2k+2)_{-m-k-1}}, \quad m \leq -k-1.$$

The coefficients are positive.

$$\frac{c_{1,-2k-1,m,-k}}{c_{1,2k+1,m,k}} = \frac{(-2k)_{m+k}}{(m+k)!}, \quad -k \leq m \leq k.$$

The coefficients change sign if $k \geq 1$ (consider $m = -k+1$) and are positive if $k = 0$. In the latter case we have the one-dimensional identity representation (cf. (2.15) and (2.13)).

$$\frac{c_{1,-2k-1,m,k+1}}{c_{1,2k+1,m,k+1}} = \frac{(m-k-1)!}{(2k+2)_{m-k-1}}, \quad m \geq k+1.$$

The coefficients are positive.

(d) $\eta = -1$, $\lambda > 0$, $\lambda \neq$ even integer.

$$\frac{c_{-1,-\lambda,m,0}}{c_{-1,\lambda,m,0}} = \begin{cases} \dfrac{(1-\frac{1}{2}\lambda)_m}{(1+\frac{1}{2}\lambda)_m} & \text{if } m \geq 0, \\[2ex] \dfrac{(-\frac{1}{2}\lambda)_{-m}}{(\frac{1}{2}\lambda)_{-m}} & \text{if } m < 0. \end{cases}$$

The coefficients change sign (consider $m = -1$).

(f) $\eta = -1$, $\lambda = 2k$ for an integer $k \geq 1$.

$$\frac{c_{-1,-2k,m,-k-1}}{c_{-1,2k,m,-k-1}} = \frac{(-m-k-1)!}{(2k+1)_{-m-k-1}}, \quad m \leq -k-1.$$

The coefficients are positive.

$$\frac{c_{-1,-2k,m,-k}}{c_{-1,2k,m,-k}} = \frac{(-2k+1)_{m+k}}{(m+k)!}, \qquad -k \le m \le k-1.$$

The coefficients change sign (consider m = -k+1).

$$\frac{c_{-1,-2k,m,k}}{c_{-1,2k,m,k}} = \frac{(m-k)!}{(2k+1)_{m-k}}, \qquad m \ge k.$$

The coefficients are positive.

Combining these results with Theorems 5.10, 4.4 and 4.8 we reobtain Bargmann's classification of irreducible unitary representations of SU(1,1) (cf. VAN DIJK [3, Theor. 12.5]):

<u>THEOREM 6.7</u>. *Any irreducible unitary representation of* SU(1,1) *is unitarily equivalent to one and only one of the following representations:*

1) $\pi_{1,i\nu}$, $\nu \ge 0$.

2) $\pi_{-1,i\nu}$, $\nu > 0$.

3) *The subrepresentation of* $\pi_{-1,0}$ *on* $\overline{\mathrm{Span}}(\phi_{-1},\phi_{-2},\ldots)$.

4) *The subrepresentation of* $\pi_{-1,0}$ *on* $\overline{\mathrm{Span}}(\phi_0,\phi_1,\ldots)$.

5) *The subrepresentation of* $\pi_{1,2k+1}$ (k = 0,1,2,...) *on the closure of* $\mathrm{Span}(\phi_{-k-1},\phi_{-k-2},\ldots)$ *with respect to the inner product*

$$< \phi_m,\phi_n > = \frac{(-m-k-1)!}{(2k+2)_{-m-k-1}}\,\delta_{m,n}, \qquad m,n \le k-1.$$

6) *The subrepresentation of* $\pi_{1,2k+1}$ (k = 0,1,2,...) *on the closure of* $\mathrm{Span}(\phi_{k+1},\phi_{k+2},\ldots)$ *with respect to the inner product*

$$< \phi_m,\phi_n > := \frac{(m-k-1)!}{(2k+2)_{m-k-1}}\,\delta_{m,n}, \qquad m,n \ge k+1.$$

7) *The subrepresentation of* $\pi_{-1,2k}$ (k = 1,2,...) *on the closure of* $\mathrm{Span}(\phi_{-k-1},\phi_{-k-2},\ldots)$ *with respect to the inner product*

$$< \phi_m,\phi_n > := \frac{(-m-k-1)!}{(2k+1)_{-m-k-1}}\,\delta_{m,n}, \qquad m,n \le -k-1.$$

8) *The subrepresentation of* $\pi_{-1,2k}$ $(k = 1,2,\ldots)$ *on the closure of* $\mathrm{Span}(\phi_k,\phi_{k+1},\ldots)$ *with respect to the inner product*

$$<\phi_m,\phi_n> := \frac{(m-k)!}{(2k+1)_{m-k}}\,\delta_{m,n},\qquad m,n \geq k.$$

9) *The representation* $\pi_{1,\lambda}$ $(0 < \lambda < 1)$ *on the closure of* $\mathrm{Span}(\ldots,\phi_{-1},\phi_0,\phi_1,\ldots)$ *with respect to the inner product*

$$<\phi_m,\phi_n> := \frac{(\frac{1}{2}(-\lambda+1))_{|m|}}{(\frac{1}{2}(\lambda+1))_{|m|}}\,\delta_{m,n},\qquad m,n \in \mathbb{Z}.$$

10) *The subrepresentation of* $\pi_{1,-1}$ *on* $\mathrm{Span}(\phi_0)$. *This is the identity representation of* $SU(1,1)$.

7. SOME FURTHER RESULTS AND OBSERVATIONS

7.1. Identification with the complementary series

In case 9) of Theorem 6.7 we obtained a class of irreducible unitary representation of $SU(1,1)$ by restricting $\pi_{1,\lambda}$ $(0 < \lambda < 1)$ to $\mathcal{D}(A) \subset L^2(U)$, where

$$A\phi_m := \frac{(-\frac{1}{2}\lambda+\frac{1}{2})_{|m|}}{(\frac{1}{2}\lambda+\frac{1}{2})_{|m|}}\,\phi_m,\qquad m \in \mathbb{Z}, \tag{7.1}$$

and then extending this to the Hilbert space completion of $\mathcal{D}(A)$ with respect to the inner product

$$<f_1,f_2> := (Af_1,f_2),\quad f_1,f_2 \in \mathcal{D}(A). \tag{7.2}$$

It follows from (7.1) that

$$Af = \alpha * f \text{ (convolution on } U), \tag{7.3}$$

where α is a function or distribution on U with Fourier series

$$\alpha \sim \sum_{m=-\infty}^{\infty} \frac{(-\frac{1}{2}\lambda+\frac{1}{2})_{|m|}}{(\frac{1}{2}\lambda+\frac{1}{2})_{|m|}}\,\phi_m, \tag{7.4}$$

We will show that

$$(7.5) \qquad \alpha(\psi) = \text{const.}(1-\cos\psi)^{\frac{1}{2}(\lambda-1)}.$$

This is a L^1-function, so $f \to \alpha * f$ is a bounded linear operator on $L^2(U)$, i.e., $\mathcal{D}(A) = L^2(U)$. Let us prove (7.5):

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-im\psi}(1-\cos\psi)^{\frac{1}{2}(\lambda-1)}\,d\psi =$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos m\psi(1-\cos\psi)^{\frac{1}{2}(\lambda-1)}\,d\psi =$$

$$= \frac{1}{\pi}\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}}\, T_{|m|}(x)(1-x)^{\frac{1}{2}(\lambda-1)}\,dx =$$

$$= \frac{(-1)^m}{2^{|m|}(\frac{1}{2})_{|m|}}\int_{-1}^{1}\{(\frac{d}{dx})^{|m|}(1-x^2)^{|m|-\frac{1}{2}}\}(1-x)^{\frac{1}{2}(\lambda-1)}dx,$$

where $T_{|m|}(x)$ is a Chebyshev polynomial and where we substituted Rodrigues' formula for these polynomials (cf. ERDÉLYI [4, Ch. II, § 10. 11, (2), (14)]). Integration by parts and substitution of the integral formula for the beta function yields

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-im\psi}(1-\cos\psi)^{\frac{1}{2}(\lambda-1)}\,d\psi =$$

$$\frac{(-\frac{1}{2}\lambda+\frac{1}{2})_m}{(\frac{1}{2}\lambda+\frac{1}{2})_m} \cdot \frac{2^{\frac{1}{2}\lambda-\frac{1}{2}}\Gamma(\frac{1}{2}\lambda)}{\pi^{\frac{1}{2}}\Gamma(\frac{1}{2}\lambda+\frac{1}{2})}.$$

Thus (7.2) becomes

$$(7.6) \qquad \langle f_1,f_2\rangle = \text{const.}\int_0^{2\pi}\int_0^{2\pi}(1-\cos(\psi_1-\psi_2))^{\frac{1}{2}(\lambda-1)} f_1(\psi_1)\,\overline{f_2(\psi_2)}\,d\psi_1 d\psi_2,$$

$$f_1,f_2 \in L^2(U).$$

We have identified the family of representations in case 9) of Theorem 6.7 with the complementary series of representations π_λ of SU(1,1) (cf. VAN DIJK [3, § 8.2]).

7.2. Identification with the discrete series

Let D be the unit disk in the complex plane and let $dm(\zeta)$ be the Lebesgue measure on D. The holomorphic discrete series of representations π_n $(n = 1,\frac{3}{2},2,\ldots)$ of SU(1,1) is defined on the Hilbert space H_n of all holomorphic functions in $L^2(D, (2n-1)\,\pi^{-1}(1-|\zeta|^2)^{2n-2}\,dm(\zeta))$ as follows:

$$(7.7)\qquad (\pi_n(g)f)(\zeta) := (\bar{\beta}\zeta+\bar{\alpha})^{-2n} f\left(\frac{\alpha\zeta+\beta}{\bar{\beta}\zeta+\bar{\alpha}}\right),$$

$$g^{-1} = \begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix} \in SU(1,1), \quad f \in H_n, \quad \zeta \in D.$$

The antiholomorphic discrete series of representations π_{-n} $(n = 1,\frac{3}{2},2,\ldots)$ of SU(1,1) is defined on the Hilbert space H_{-n} of all antiholomorphic functions in the above L^2-space as follows:

$$(7.8)\qquad (\pi_{-n}(g)f)(\zeta) := \overline{(\pi_n(g)\bar{f})(\zeta)},\quad g \in SU(1,1),\ f \in H_{-n},\ \zeta \in D.$$

It was shown in VAN DIJK [3, § 7] that the representations π_n $(n \in \frac{1}{2}\mathbb{Z},\ |n| \geq 1)$ are unitary and irreducible.

For $n = 1,\frac{3}{2},2,\ldots$ the functions

$$(7.9)\qquad \phi^n_m(\zeta) := \left(\frac{(2n-1)!\,m!}{(2n+m-1)!}\right)^{\frac{1}{2}} \zeta^m,\quad m = 0,1,2,\ldots,$$

form an orthonormal basis of H_n and, similarly, the functions

$$(7.10)\qquad \phi^{-n}_m(\zeta) := \overline{\phi^n_m(\zeta)},\quad m = 0,1,2,\ldots,$$

form an orthonormal basis of H_{-n}. We have

$$(7.11)\quad \begin{cases} \pi_n(u_\theta)\,\phi_m^n = e^{i(-2n-2m)\theta}\phi_m^n, \\ \pi_{-n}(u_\theta)\phi_m^{-n} = e^{i(2n+2m)\theta}\phi_m^n. \end{cases}$$

The representations π_n and π_{-n} must be unitarily equivalent to certain representations listed in Theorem 6.7. On comparing K-contents we conclude:

PROPOSITION 7.1.

(a) *The subrepresentation of* $\pi_{1,2k+1}$ *considered in case* 5) *is equivalent to* π_{k+1}.

(b) *The subrepresentation of* $\pi_{1,2k+1}$ *considered in case* 6) *is equivalent to* π_{-k-1}.

(c) *The subrepresentation of* $\pi_{-1,2k}$ *considered in case* 7) *is equivalent to* $\pi_{k+\frac{1}{2}}$.

(d) *The subrepresentation of* $\pi_{-1,2k}$ *considered in case* 8) *is equivalent to* $\pi_{-k-\frac{1}{2}}$.

7.3. Identification of square integrable representations

A unitary representation τ of a unimodular lcsc. group is called *square integrable* if for each $v,w \in H(t)$ the function $x \to (\tau(x)v,w)$ is in $L^2(G)$. If τ is unitary and irreducible and if $x \to (\tau(x)v,w)$ is in $L^2(G)$ for some nonzero $v,w \in H(\tau)$ then it can be shown that τ is square integrable (cf. BOREL [2, Théorème 5.15]). Thus we can examine square integrability of the irreducible unitary representations of SU(1,1) listed in Theorem 6.7 by considering whether the functions $\pi_{\eta,\lambda,m,m}$ are in $L^2(G)$. From (2.6) and (2.15) we have

$$(7.12)\quad \pi_{\eta,\lambda,m,m}(u_{\frac{1}{2}(\psi_1+\psi_2)}\, a_t\, u_{\frac{1}{2}(\psi_1-\psi_2)}) = $$
$$= e^{i(2m+\frac{1}{2}(1-\eta))\psi_1}\,(\cosh t)^{2m+\frac{1}{2}(1-\eta)}\phi_{i\lambda}^{(0,2m+\frac{1}{2}(1-\eta))}(t).$$

Thus it is convenient to express the Haar measure on G in terms of the coordinates t, ψ_1, ψ_2. This can be done by using (5.10). Remember that, corresponding to a pseudo-Riemannian metric (5.11), there is a measure

$$dm(x^1,\ldots,x^n) := |g(x)|^{\frac{1}{2}}\, dx^1 \ldots dx^n \tag{7.13}$$

(in terms of local coordinates) which is invariant under all smooth transformations which leave the metric invariant. Thus the measure corresponding to (5.10) equals

$$d(u_{\frac{1}{2}(\psi_1+\psi_2)}\, a_t\, u_{\frac{1}{2}(\psi_1-\psi_2)}) = \sinh t \cosh t\, dt\, d\psi_1\, d\psi_2, \tag{7.14}$$

and it is invariant under left and right multiplication on SU(1,1), i.e., it is a Haar measure. So an unitarizable irreducible subquotient representation τ of $\pi_{\eta,\lambda}$ with $\phi_m \in H(\tau)$ is square integrable iff

$$\int_0^\infty |\phi_{i\lambda}^{(0,2m+\frac{1}{2}(1-\eta))}(t)|^2 \sinh t\, (\cosh t)^{4m+2-\eta} dt < \infty. \tag{7.15}$$

To decide whether (7.15) holds is a purely analytic exercise in the theory of Jacobi functions. The solution is given by FLENSTED-JENSEN [6, Lemma A.3]. Applying this result we obtain that precisely the cases 5), 6), 7) and 8) (i.e. the discrete series representations) in Theorem 6.7 yield square integrable representations.

7.4. An addition formula approach

Let G be a lcsc. group with compact subgroup K and let τ be a K-unitary, K-multiplicity free representation of G. In this paper we developed machinery to obtain irreducible sub(quotient) representations of τ and to decide about their unitarizability. This machinery works quite well if we have an explicit knowledge of the generalized matrix elements $\tau_{\delta,\varepsilon}(g)$, $\delta,\varepsilon \in M(\tau)$. This happened to be the case in our simple example $G = SU(1,1)$, $\tau = \pi_{\eta,\lambda}$, but for more general G these matrix elements are usually unknown. Now it may happen that we may have an explicit expression for $\tau_{\delta,\delta}(g)$ for some $\delta \in M(\tau)$ (for instance, for $\delta = 1$) and that we are able to calculate explicitly the Fourier expansion

$$(7.16) \qquad \tau_{\delta,\delta}(g_1 k g_2^{-1}) = \sum_{\gamma\in\hat{K}} \tau_{\delta,\delta;\gamma}(g_1,g_2,k),$$

where

$$(7.17) \qquad \tau_{\delta,\delta;\gamma}(g_1,g_2,k) := d_\gamma \int_K \tau_{\delta,\delta}(g_1 k_1 g_2^{-1})\chi_\gamma(k_1^{-1}k)dk_1.$$

Formula (7.16) is called the *addition formula* for $\tau_{\delta,\delta}$. Let τ_δ be the irreducible subquotient representation of τ which was defined in § 3.3. Now a knowledge of the functions $\tau_{\delta,\delta;\gamma}$ gives us important information about the representation τ_δ:

THEOREM 7.2.
(a) $M(\tau_\delta) = \{\gamma \in \hat{K} | \; \tau_{\delta,\delta;\gamma} \not\equiv 0\}$.
(b) τ_δ *is unitarizable iff*

$$(7.18) \qquad \tau_{\delta,\delta}(g^{-1}) = \tau_{\delta,\delta}(g)^*, \quad g \in G,$$

and the matrices $\tau_{\delta,\delta;\gamma}(g,g,e)$ *are positive (semi-) definite for all* $\gamma \in \hat{K}$, $g \in G$.

PROOF.
(a) On comparing (7.16) with (4.4) we have

$$(7.19) \qquad \tau_{\delta,\delta;\gamma}(g_1,g_2,k) = \begin{cases} \tau_{\delta,\gamma}(g_1)\gamma(k)\tau_{\gamma,\delta}(g_2^{-1}) & \text{if } \gamma \in M(\tau_\delta), \\ 0 & \text{if } \gamma \notin M(\tau_\delta). \end{cases}$$

By irreducibility of τ_δ, $\tau_{\delta,\delta;\gamma}$ cannot be identically zero if $\gamma \in M(\tau_\delta)$.
(b) Suppose that τ_δ unitarizable. Then Theorem 6.5 implies (7.18) and substitution of (6.7) in (7.19) yields

$$(7.20) \qquad \tau_{\delta,\delta;\gamma}(g,g,e) = \frac{c_\delta}{c_\gamma} \tau_{\delta,\gamma}(g)\tau_{\delta,\gamma}(g^*), \; \gamma \in M(\tau_\delta),$$

where the c_γ's are positive numbers. Thus $\tau_{\delta,\delta;\gamma}(g,g,e)$ is positive semi-definite.

Conversely, assume (7.18) and the positive semidefiniteness of the matrices $\tau_{\delta,\delta;\gamma}(g,g,e)$. Formula (7.18) together with Theorem 4.6 implies that there are complex constants c_ε, $\varepsilon \in M(\tau_\delta)$, such that (6.7) holds. Substitution in (7.19) again gives (7.20). By irreducibility of τ_δ, the function $g \to \tau_{\delta,\delta;\gamma}(g,g,e)$ is not identically zero for $\gamma \in M(\tau_\delta)$. Thus the positive semidefiniteness of $\tau_{\delta,\delta;\gamma}(g,g,e)$ implies that $c_\delta/c_\gamma > 0$. Now the unitarizability of τ_δ follows from Remark 6.6. □

In FLENSTED-JENSEN & KOORNWINDER [8] Theorem 6.9 (b) was used in order to find all irreducible unitary spherical representations of noncompact semisimple Lie groups G of rank one. Theorem 6.9 (b) was proved there by using that a spherical function on G corresponds to a unitary representation iff it is a positive definite function on G. In a forthcoming paper by Flensted-Jensen and the author the same approach will be used for representations of G = SU(n,1)(or its simply connected covering group) which are spherical with respect to the (non-maximal!) compact subgroup K = SU(n). The spherical functions on this homogeneous space G/K, considered as special functions, are generalizations of the functions (7.12), cf. FLENSTED-JENSEN [6].

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