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NON-UNIQUENESS IN SINGULAR OPTIMAL CONTROL

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Non-uniqueness in singular optimal control *)

by

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ABSTRACT

It is shown that there exists a class of linear singular optimal control problems with a non-unique singular arc. The problems in this class have a controllability subspace in which the trajectories may vary without affecting the performance index. Furthermore, it is investigated how these solutions can be derived from the theory of cheap optimal control.

KEY WORDS & PHRASES: *singular optimal control, cheap control, controllability subspaces*

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*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

1.1 SINGULAR OPTIMAL CONTROL

In this paper we investigate singular optimal control problems of linear time invariant n-dimensional dynamical systems

$$\dot{x} = Ax+Bu, \qquad x(0) = x_0 \qquad (1.1ab)$$

with performance index

$$J = \int_0^T x'Qxdt, \qquad (1.1c)$$

where Q is a symmetric positive semi-definite matrix with K = Ker Q. We denote the finite dimensional linear state space by X and assume the system to be controllable:

$$X = \langle A|B\rangle$$

with

$$\langle A|B\rangle = B + AB +\ldots+ A^{n-1}B, \qquad B = \text{Im } B.$$

The control vector u has its values in a linear m-dimensional space U and u(t) is assumed to be continuous for t > 0. Although the Hamiltonian of (1.1) is singular in its second derivative with respect to u, one is able to construct a feedback $F(\circ,t)$: $X \to U$ that minimizes J. However, there is a restricting condition on x that, in general, is not satisfied by the initial value (1.1b). This problem is solved by introduction of an initial control which is a linear combination of the Dirac δ-function and its formal derivatives up to the (n-1)-th order. Then at the initial point, x jumps instantaneously from the initial value x_0 to a point of the manifold where x satisfies the so-called singular arc condition. We will demonstrate that the problem (1.1) does not necessarily have a unique singular arc for t > 0. The leading idea is that if there exists a subspace in K where we have complete control, we are free to change the solution within this subspace without influencing the performance index. This aspect of the problem is worked out in section 3, while in section 4 we discuss the relationship with cheap optimal control. In section 2 we deal with the method of Moylan and Moore (1) for a certain class of singular optimal control problems and show that the uniqueness of the singular arc of these problems is due to the fact that there exists no controllability subspace in K. In section 1.2 we give a precise definition of controllability subspaces and state some of their characteristic properties. For a more complete treatment of this theory we refer to Wonham (3).

1.2 CONTROLLABILITY SUBSPACES

Before giving a definition of controllability subspaces we introduce the concept of (A,B)-invariant subspaces.

Definition 1.1 A subspace $V \subset X$ is called (A,B)-invariant if for any $x_0 \in V$ there exists a control $u(\circ)$: $\mathbb{R} \to U$ such that x(t) satisfying (1.1ab) remains in V for $t > 0$.

It can be proved that (A,B)-invariant subspaces may be characterized by the property $AV \subset V + B$, or, equivalently, by the existence of a family of feedbacks

$$\underline{F}(V) = \{F: X \to U \,|\, (A+BF)\, V \subset V\},$$

so that the closed loop system starting in V remains in V for $t > 0$. The class of (A,B)-invariant subspaces contained in some subspace of X is closed under addition and, thus, has a supremal element, see (3). In the sequel we denote the supremal (A,B)-invariant subspace contained in K(= Ker Q) by V^*.

Definition 1.2 A subspace $R \subset X$ is called a controllability subspace if for any $x_0, x_1 \in R$ there exists a $T > 0$ and a u(t) such that x(t) given by (1.1ab) satisfies $x(T) = x_1$ and $x(t) \in R$ for $0 < t < T$.

It is clear that a controllability subspace is necessarily (A,B)-invariant. It also can be shown that R is a controllability subspace if and only if

$$R = \langle A+BF | B \cap R\rangle \qquad \text{for } F \in \underline{F}(R).$$

Furthermore, the class of controllability subspaces contained in some subspace of X has a supremal element. The supremal controllability subspace contained in K we denote by R^*. It can be proved that

$$R^* = \langle A+BF | B \cap V^*\rangle \qquad \text{for } F \in \underline{F}(V^*). \tag{1.2}$$

2. UNIQUE SINGULAR ARCS

2.1 TRANSFORMATION TO REGULAR OPTIMAL CONTROL

In this section we consider singular optimal control problems of the type (1.1) satisfying $B'QB > 0$. For this class of problems Moylan and Moore (1) introduced the variables

$$x_1(t) = x(t) - Bu_1(t) \tag{2.1a}$$

$$u_1(t) = u_1(0) + \int_0^t u(\tau)d\tau, \tag{2.1b}$$

so that (1.1) transforms into the regular optimal control problem

$$\dot{x}_1 = Ax_1 + B_1u_1, \tag{2.2a}$$

$$J = \int_0^t \{x_1'Qx_1 + 2x_1'S_1u_1 + u_1'R_1u_1\}dt, \tag{2.2b}$$

where $B_1 = AB$, $S_1 = QB$ and $R_1 = B'QB$. The optimal solution satisfies

$$u_1(t) = K_1'(t)x_1(t), \quad K_1'(t) = -R_1^{-1}\{B_1'P(t)+S_1'\} \tag{2.3}$$

with P(t) satisfying the Riccati differential equation

$$-\dot{P} = P\{A-B_1R_1^{-1}S_1'\} + \{A-B_1R_1^{-1}S_1'\}'P + - PB_1R_1^{-1}B_1'P + \{Q-S_1R_1^{-1}S_1'\}, \qquad P(T) = 0. \tag{2.4}$$

From this equation it follows that $P(t)B \equiv 0$, see (1), and so

$$K_1'(t)B = -I. \tag{2.5}$$

Using this result we conclude from (2.1a) and (2.3) that the transformation to regular optimal control is only possible if

$$K_1'(t)x(t) = 0. \tag{2.6}$$

This so-called singular arc condition is in general not satisfies at $t = 0$ by (1.1b). Since the performance index (1.1c) is independent of u, control pulses can be given without affecting the performance index. In this way x can be transferred instantaneously to the manifold (2.6) by the initial puls

$$u(t) = K_1'(0)x_0\delta(t), \qquad t \geq 0 \tag{2.7}$$

where δ denotes the right-sided δ-function. Using (1.1a), (2.5) and (2.7) we find that indeed $K_1'(+0)x(+0) = 0$. From (2.1) and (2.3) it is derived that on the manifold the optimal feedback satisfies

$$u(t) = K'(t)x(t), \quad K(t) = A'K_1(t) + \dot{K}_1(t). \tag{2.8}$$

The method of Moylan and Moore leads for the class of problems (1.1) with $B'QB > 0$ to a unique singular arc satisfying (2.6) and (2.8). We will check whether this result is consistent with our statement that non-uniqueness only occurs if there exist nontrivial controllability subspaces in K. For the above class of problems $K \cap B = 0$ and so $V^* \cap B = 0$. Using (1.2) we find that indeed $R^* = 0$.

2.2 REPEATED TRANSFORMATIONS

The method of section 2.1 also applies to the more general class (1.1) satisfying for some $k \geq 1$

$$B'(A')^k Q A^k B > 0, \tag{2.9a}$$

$$B'(A')^i Q A^i B = 0, \qquad i = 0,1,\ldots,k-1. \tag{2.9b}$$

Repeated application of transformation (2.1)

$$x_{i+1}(t) = x_i(t) - A^i B u_i(t), \tag{1.10a}$$

$$u_{i+1}(t) = u_{i+1}(0) + \int_0^t u_i(\tau)d\tau, \qquad i = 0,1,\ldots,k \tag{2.10b}$$

with $(x_0,u_0) = (x,u)$, yields the regular optimal control problem

$$\dot{x}_k = Ax_k + B_k u_k, \qquad B_k = A^k B \tag{2.11a}$$

with performance index

$$J = \int_0^T \{x_k' Q x_k + 2x_k' S_k u_k + u_k' R_k u_k\}dt, \tag{2.11b}$$

$$S_k = QB_k, \qquad R_k = B_k' Q B_k.$$

Its optimal solution satisfies

$$u_k(t) = K_k'(t)x_k(t), \; K_k'(t) = -R_k^{-1}\{B_k' P(t) + S_k'\}, \tag{2.12}$$

where $P(t)$ denotes the solution of the corresponding Riccati differential equation. Similar to (2.6) we find the singular arc conditions

$$K_i'(t)x(t) = 0, \qquad i = 1,2,\ldots,k \tag{2.13}$$

with K_i satisfying the recurrent system $K_{i-1} = A'K_i + \dot{K}_i$. By taking an appropriate series of initial pulses of the type $u = u_0\delta(t) + \ldots + u_{k-1}\delta^{(k-1)}(t)$ the system will jump from the initial value x_0 to the manifold (2.13) at $t = 0$. For these problems we also have $R^* = 0$, which is proved as follows. Since $AV^* \subset V^* + B$, we have

$$A^k V^* \subset V^* + A^{k-1}B + \ldots + B$$

and, consequently, by (2.9b)

$$A^k(V^* \cap B) \subset V^* + A^{k-1}B + \ldots + B \subset K.$$

By (2.8a) this implies $V^* \cap B = 0$ and so $R^* = 0$, see (1.2).

3. NON-UNIQUENESS OF SINGULAR ARCS

3.1 NECESSARY AND SUFFICIENT CONDITIONS

As we discussed in section 1.1 we expect a singular optimal control problem to have a family of solutions if $R^* \neq 0$. In the preceding section we proved that under conditions (2.9) $R^* = 0$. We now consider the case $B'QB \geq 0$. Without loss of generality we may assume the system to be of the form

$$\dot{x} = Ax + B_a u_a + B_b u_b \tag{3.1a}$$

with

$$B_a' Q B_a = 0 \qquad \text{and } B_b' Q B_b > 0. \tag{3.1bc}$$

Let us also assume that

$$B_a' A' Q A B_k > 0. \tag{3.1d}$$

In order to have a nontrivial controllability subspace in K the following conditions are necessary

$$A(V^* \cap B_a) \subset V^* + B, \quad V^* \cap B_a \neq 0 \tag{3.2ab}$$

The first condition is understood from the fact that such a subspace needs to be (A,B)-invariant and that $V^* \cap B_a \subset R^* \subset V^*$. From (1.2) we deduce that $R^* = 0$ if the second condition is not satisfied. It is easily verified that the following conditions are sufficient

$$A(V^* \cap B_a) \subset B_b, \quad V^* \cap B_a \neq 0. \tag{3.3ab}$$

As described in (1) we distinguish a hierarchy of cases if (3.1b) is semi-definite. We then have to split up B_a as carried out for B and proceed in this way until there remains a positive definite matrix. The cases (2.9) with (2.9a) semi-definite and $k \geq 1$ can be handled in a same manner.

3.2 SINGULAR SOLUTIONS: THE CASE $B_a \subset V^*$

In this section we construct a family of singular solutions for the problem (3.1) satisfying $B_a \subset V^*$. We introduce the system

$$\dot{x}_a = Ax_a + B_a u_a, \quad x_a(0) = x_{a0} \in R^* \tag{3.4}$$

and denote by $\underline{F}_a(R^*)$ the family of feedbacks for which $x_a(t)$ remains in R^* for $t > 0$. The corresponding family of solutions is parameterized by

$$x_a(t;x_{a0},F_a) \quad \text{with} \quad x_{a0} \in R^*, \; F_a \in \underline{F}_a(R^*). \tag{3.5}$$

Next we consider $x_b(t) = x(t) - x_a(t;x_{a0},F_a)$

satisfying

$$\dot{x}_b = Ax_b + B_b u_b, \quad x_b(0) = x_0 - x_{a0} \tag{3.6a}$$

with performance index

$$J = \int_0^T x_b' Q x_b dt. \tag{3.6b}$$

Since $B_b'QB > 0$ we can solve this problem with the method of section 2.1 under the singular arc condition

$$K_{b1}'(t)\{x(t)-x_a(t)\} = 0, \tag{3.7}$$

which is not automatically satisfied at t = 0. As in (2.7) we therefore give an initial puls

$$u_b(t) = K_{b1}'(t)(x_0-x_{a0})\delta(t). \tag{3.8}$$

On the manifold we have the feedback

$$u_b(t) = K_b'(t)\{x(t)-x_a(t)\}, \quad K_b(t) = A'K_{b1}(t) + \dot{K}_{b1}(t). \tag{3.9}$$

From (3.7)-(3.9) it is understood that the singular arc depends on the choice of x_{a0} and F_a of (3.5).

3.3 SINGULAR SOLUTIONS: THE CASE $\mathcal{B}_a \not\subset \mathcal{V}^*$

The construction of the family of singular solutions becomes slightly more complicated if $\mathcal{B}_a \not\subset \mathcal{V}^*$ for a system (3.1) satisfying (3.3). Partitioning the matrix B_a in an appropriate way we may write

$$\dot{x} = Ax + B_{ap}u_{ap} + B_{aq}u_{aq} + B_b u_b \tag{3.10}$$

with

$$B_{ap}'QB_{ap} = 0, \; B_{aq}'QB_{aq} = 0, \; B_b'QB_b > 0$$

and

$$\mathcal{B}_{ap} \subset \mathcal{V}^*, \; \mathcal{B}_{aq} \cap \mathcal{V}^* = 0, \; A\mathcal{B}_{ap} \subset \mathcal{B}_b, \; A\mathcal{B}_{aq} \cap \mathcal{K} = 0.$$

For the system

$$\dot{x}_{ap} = Ax_{ap} + B_{ap}u_{ap}, \quad x_{ap}(0) = x_{ap0} \tag{3.11}$$

we consider the family of solutions $x_{ap}(t;x_{ap0},F_{ap})$ defined by (3.4)-(3.5). Next we introduce new variables

$$x_c = x - x_{ap}(t;x_{ap0},F_{ap}) - B_{aq}u_{aq1}(t), \tag{3.12a}$$

$$u_{aq1}(t) = u_{aq1}(0) + \int_0^t u_{aq}(\tau)d\tau, \tag{3.12b}$$

so that (3.10) transforms into

$$\dot{x}_c = Ax_c + B_c u_c, \tag{3.13a}$$

with

$$B_c = (AB_{aq}, B_b), \quad u_c' = (u_{aq1}', u_b').$$

The performance index (1.1c) expressed in the new variables reads

$$J = \int_0^T x_c' Q x_c dt, \tag{3.13b}$$

The singular optimal control problem (3.13) is of the type we solved in section 2.1. The singular arc condition and feedback are, respectively,

$$K_{c1}'(t)x_c(t) = 0, \quad u_c(t) = K_c'(t)x_c(t), \quad K_c' = \dot{K}_{c1}' + K_{c1}'A, \tag{3.14abc}$$

see (2.6) and (2.8). Posing $K_{c1} = (K_{aq2}, K_{b1})$ and $K_c = (K_{aq1}, K_b)$ we may write (3.14a) as $K_{aq2}'x_c = 0$ and $K_{b1}'x_c = 0$. Since $K_{aq2}'B_{aq} \equiv 0$, see (1, p.597) and $K_{aq2}'B_b = 0$, see (2.5), we have derive from (3.14ac) that

$$K_{aqi}'(t)\{x(t)-X_{ap}(t;x_{ap0},F_{ap})\} = 0, \quad i = 1,2. \tag{3.15a}$$

On the other hand the equations (3.12a) and (3.14ab) yield

$$K_{b1}'(t)\{I+B_{aq}K_{aq1}'(t)\}^{-1}\{x(t)-x_{ap}(t;x_{ap0},F_{ap})\} = 0. \tag{3.15b}$$

By choosing an appropriate initial control of the type

$$\begin{pmatrix} u_b(t) \\ u_{aq}(t) \end{pmatrix} = \begin{pmatrix} U_{b0} & 0 \\ U_{aq0} & U_{aq1} \end{pmatrix} \begin{pmatrix} \delta(t) \\ \delta'(t) \end{pmatrix}$$

we are in the position to satisfy the singular arc conditions (3.15) for t = +0. Moreover, we are free to choose $x_{ap0} \in \mathcal{R}^*$ and $F_{ap} \in \underline{F}_{ap}(\mathcal{R}^*)$, so that a family of singular arcs is found.

The above results suggest to analyse the singular optimal control problem in the factor space $\mathcal{X}/\mathcal{R}^*$. A general theory, in which this idea is worked out, is under investigation.

4. CHEAP CONTROL

4.1 METHOD OF SINGULAR PERTURBATIONS

For singular optimal control problems the Hamiltonian is singular in its second derivative with respect to u. Instead of using the method of section 1.2 one can avoid this problem by introduction of a small cost of control in the performance index:

$$\dot{x}_\varepsilon = Ax_\varepsilon + Bu_\varepsilon, \qquad x_\varepsilon(0) = x_0, \tag{4.1a}$$

with

$$J = \int_0^T \{x_\varepsilon' Q x_\varepsilon + \varepsilon^2 u_\varepsilon' R u_\varepsilon\} dt, \qquad 0 < \varepsilon << 1 \tag{4.1b}$$

where R is a symmetric positive definite matrix. The class of problems (4.1) satisfying (2.9) has been investigated by O'Malley and Jameson (2). For the solution x_ε and the control u_ε they constructed power series in ε for three time intervals: the initial boundary layer of thickness $O(\varepsilon^{1/(k+1)})$, a terminal boundary layer of the same size and the remaining interval. For $\varepsilon \to 0$ the boundary layer intervals shrink and x_ε as well as u_ε will exhibit rapid change in these intervals.

4.2 RELATIONSHIP WITH SINGULAR OPTIMAL CONTROL

In this section we will discuss the relationship between the cheap optimal control problem and the singular optimal control problem with a non-unique singular arc. The main issue is to find a criterion for the selection of precisely that singular solution that is found as the limit of the solution of the cheap control problem as the cost of control tends to zero.

For $\varepsilon \to 0$ x_ε and u_ε satisfying (4.1) with (2.9) tend to x and u of the corresponding singular optimal control problem. It is remarked that in the initial boundary layer u_ε tends to the initial puls of the singular optimal control and that away from the initial point x_ε tends to the singular arc as $\varepsilon \to 0$. For the system (3.1) satisfying (3.3ab) a family of singular solutions was found. We expect the corresponding cheap control problem to converge to one of these solutions. Before hand it is not clear what value for (x_{a0}, F_a) or (x_{ap0}, F_{ap}) should be taken in order to select the correct singular solution. Moreover, it needs to be investigated whether every singular arc is the limit of a solution of a cheap control problem as $\varepsilon \to 0$. These difficulties are illustrated in the following simple example with n = 2. We consider the system (3.1) with $T = \infty$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.2}$$

For this problem with R^* being the x_1-axis the family of singular solutions satisfies

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \alpha - x_{01} \\ -x_{02} \end{pmatrix} \delta(t) + \begin{pmatrix} \beta & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{4.3}$$

For the corresponding cheap control (4.1) the algebraic Ricatti equation reads

$$Q + P_\varepsilon A' + A' P_\varepsilon - \varepsilon^{-2} P_\varepsilon B^1 R^{-1} B P_\varepsilon = 0. \tag{4.4}$$

Choosing R = I we obtain a set of four solutions for (4.4). Only one of them is positive definite:

$$P_\varepsilon = \begin{pmatrix} \varepsilon^2 & \varepsilon^2 \\ \varepsilon^2 & \varepsilon\sqrt{1+\varepsilon^2} \end{pmatrix}.$$

Since $u_\varepsilon = -\varepsilon^{-2} R^{-1} B' P_\varepsilon x_\varepsilon$, the closed loop system reads

$$\dot{x}_{\varepsilon 1} = -x_{\varepsilon 1}$$
$$\dot{x}_{\varepsilon 2} = -\varepsilon^{-1}\sqrt{1+\varepsilon^2}\, x_{\varepsilon 2}.$$

Consequently, for $\varepsilon \to 0$ the solution converges to the singular solution (4.3) with $(\alpha, \beta) = (x_{01}, -1)$. Similarly, we compute that for

$$R = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a > 0 \tag{4.5}$$

the solution converges to (4.3) with parameters $(\alpha, \beta) = (x_{01}, -1/a)$. It is remarked that for problems with $R^* = 0$ the solution of the cheap control problem converges to the singular solution independently of the choice of the coefficients of the positive definite matrix R. Furthermore, we observe that not all singular solutions (4.3) are approximated by solutions of (4.1) satisfying (4.2) and (4.5). This aspect of the problem needs to be analized in more detail.

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